Week- 8-Numerical computations

Week- 9 DAWAH-2025-2026

Numerical computation involves using numerical methods to approximate solutions to mathematical problems, especially when analytical solutions are difficult or impossible to obtain. These methods rely on algorithms, computations, and iterative processes to get numerical approximations

Numerical Computation

In principle, one can use any number β as the base. Writing such a number with decimal point, we have

$$\Big(\overbrace{a_n a_{n-1}\cdots a_1 a_0}^{\text{integer part}}.\overbrace{b_1 b_2 b_3 \cdots}^{\text{fractional part}}\Big)_\beta$$

$$\begin{aligned} = \quad & a_n\beta^n + a_{n-1}\beta^{n-1} + \cdots + a_1\beta + a_0 \qquad &\text{(integer part)} \\ & +b_1\beta^{-1} + b_2\beta^{-2} + b_3\beta^{-3} + \cdots \qquad &\text{(fractonal part)} \end{aligned}$$

Thinking of $\beta = 10$ above, we will understand the decimal representation.

The above formula allows us to convert a number in any base β into decimal base.

One can convert the numbers between different bases. We now go through some examples.

Example 1. octal → decimal

$$(45.12)_8 = 4 \times 8^1 + 5 \times 8^0 + 1 \times 8^{-1} + 2 \times 8^{-2} = (37.15625)_{10}$$

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Bisection Method Algorithm

This is a very simple method. Identify two points $x = a$ and $x = b$ such that $f(a)$ and $f(b)$ are having opposite signs. Let $f(a)$ be negative and $f(b)$ be positive. Then there will be a root of

[$f(x) = 0$] in between a and b.

Let the first approximation be the mid point of the interval (a, b). i.e.

$$x_1 = \frac{(a+b)}{2}$$

If $f(x1) = 0$, then $x1$ is a root, other wise root lies between a and $x1$ or $x1$ and b according as $f(x1)$ is positive or negative.

Then again we bisect the interval and continue the process until the root is found to desired accuracy. Let $f(x_1)$ is positive, then root lies in between a and x_1

(see fig). The second approximation to the root is given by,

$$x_2 = \frac{(a+x_1)}{2}$$

If $f(x_2)$ is negative, then next approximation is given by

$$x_3 = \frac{(x_2+x_1)}{2}$$

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Follow the below procedure to get the solution for the continuous function: For any continuous function f(x),

- **Find two points, say a and b such that a < b and f(a)* f(b) < 0**
- **Find the midpoint of a and b, say "t"**
- **t is the root of the given function if f(t) = 0; else follow the next step**
- **Divide the interval [a, b] – If f(t)*f(a) <0, there exist a root between t and a**
  – else if f(t) *f (b) < 0, there exist a root between t and b
- **Repeat above three steps until f(t) = 0.**

The bisection method is an approximation method to find the roots of the given equation by repeatedly dividing the interval. This method will divide the interval until the resulting interval is found, which is extremely small.

Bisection Method

Example: Determine the root of the given equation $x^2-3 = 0$ for $x \in [1, 2]$

Solution: Given: $x^2-3 = 0$,,,, Let $f(x) = x^2-3$

Now, find the value of f(x) at a= 1 and b=2.

$f(x=1) = 1^2-3 = 1 – 3 = -2 < 0$

$f(x=2) = 2^2-3 = 4 – 3 = 1 > 0$

The given function is continuous, and the root lies in the interval [1, 2].

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Let "t" be the midpoint of the interval.

I.e., t = (1+2)/2 ,,,t =3 / 2 ,,,(t = 1.5)

Therefore, the value of the function at "t" is

$f(t) = f(1.5) = (1.5)^2 - 3 = 2.25 - 3 = -0.75 < 0$

If f(t)<0, assume a = t. and

If f(t)>0, assume b = t.

f(t) is negative, so a is replaced with t = 1.5 for the next iterations.

The iterations for the given functions are:

Iterations	a	b	t	f(a)	f(b)	f(t)
1	1	2	1.5	-2	1	-0.75
2	1.5	2	1.75	-0.75	1	0.062
3	1.5	1.75	1.625	-0.75	0.0625	-0.359
4	1.625	1.75	1.6875	-0.3594	0.0625	-0.1523
5	1.6875	1.75	1.7188	-01523	0.0625	-0.0457
6	1.7188	1.75	1.7344	-0.0457	0.0625	0.0081
7	1.7188	1.7344	1.7266	-0.0457	0.0081	-0.0189

Example

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Use the bisection method to find the root of the nonlinear equation

$x^3=20$

Use initial lower and upper guesses of 1 and 4, respectively.

Conduct three iterations to estimate the root of the equation.

Find the absolute relative approximate error at the end of each iteration.

Find the number of significant digits that are at least correct at the end of each iteration

Solution

Rewrite the equation $x^3=20$ in the form $f(x)=0$ that gives $f(x)=x^3-20=0$

Check if the function changes the sign between the two initial guesses, x_l and x_u. The initial guesses are given as $x_l=1$ and $x_u=4$

$$\begin{aligned} f(x_l) &= f(1) \\ &= 1^3 - 20 \\ &= -19 \end{aligned}$$

$$\begin{aligned} f(x_u) &= f(4) \\ &= 4^3 - 20 \\ &= 44 \end{aligned}$$

$$\begin{aligned} f(x_l)f(x_u) &= f(1)f(4) \\ &= (-19)(44) < 0 \end{aligned}$$

This change in sign tells us that the initial bracket of [1,4] given to us is valid. Iteration 1

$$x_l = 1,\ x_u = 4$$

The estimate of the root is

$$\begin{aligned} x_m &= \frac{x_l + x_u}{2} \\ &= \frac{1+4}{2} \\ &= 2.5 \end{aligned}$$

EXAMPLE Find a real root of the equation $f(x) = x^3 - x - 1 = 0$, using Bisection method.

SOLUTION

First find the interval in which the root lies, by trail and error method.

$f(1) = 13 - 1 - 1 = -1$, which is negative

$f(2) = 23 - 2 - 1 = 5$, which is positive

A root of $f(x) = x^3 - x - 1 = 0$ lies in between 1 and 2.

$$x_1 = \frac{(1+2)}{2} = \frac{3}{2} = 1.5$$

$f(x_1) = f(1.5) = (1.5)^3 - 1.5 - 1 = 0.875$, which is positive.

Hence, the root lies in between 1 and 1.5

$$x_2 = \frac{(1+1.5)}{2} = 1.25$$

$f(x_2) = f(1.25) = (1.25)^3 - 1.25 - 1 = -0.29$, which is negative.

Hence, the root lies in between 1.25 and 1.5

$$\therefore \quad x_3 = \frac{(1.25+1.5)}{2} = 1.375$$

Similarly, we get $x_4 = 1.3125$, $x_5 = 1.34375$, $x_6 = 1.328125$ etc.

EXAMPLE

Find a root of $f(x) = xe^x - 1 = 0$, using Bisection method, correct to three decimal places.

SOLUTION

$f(0) = 0.e^0 - 1 = -1 < 0$

$f(1) = 1.e^1 - 1 = 1.7183 > 0$

Hence a root of $f(x) = 0$ lies in between 0 and 1.

$$x_1 = \frac{(0+1)}{2} = 0.5$$

$$f(0.5) = 0.5\, e^{0.5} - 1 = -0.1756$$

Hence the root lies in between 0.5 and 1

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$$x_2 = \frac{(0.5+1)}{2} = 0.75$$

Proceeding like this, we get the sequence of approximations as follows.

$x^3 = 0.625$

$x^4 = 0.5625$

$x^5 = 0.59375$

$x^6 = 0.5781$

$x^7 = 0.5703$

$x^8 = 0.5664$

$x^9 = 0.5684$

$x^{10} = 0.5674$

$x^{11} = 0.5669$

$x^{12} = 0.5672,$

$x^{13} = 0.5671,$

Hence, the required root correct to three decimal places is, $x = 0.567$.

we shall deal with the methods for solving the equations. Sometimes, a rough approximation of a root can be found by graph and more accurate results by the following methods :

(i) Newton Raphson method or successive substitution method.

(ii) Rule of false position (*Regula falsi*).

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(*iii*) Iteration method.

SOLUTION OF THE EQUATIONS GRAPHICALLY

Step 1. Find a small interval (*a*, *b*) between which the root of the equation lies.

Let $f(x) = 0$ (1)

and $f(a) =$ **– ve** and $f(b) =$ **+ve**

then the root of the equation (1) lies between *a* and *b*.

For example $f = 2x^2 + x - 15 = 0$

$f(2) = 8 + 2 – 15 = – 5 =$ **– ve**

$f(3) = 18 + 3 – 15 = + 6 =$ **+ve**

∴ The root of the equation lies between 2 and **3**.

Step 2. Write the equation f (x) = 0 as .. $\phi(x) = \psi(x)$

For example $2x^2 + x - 15 = \mathbf{0}$ OR $2x^2 = 15 - \mathbf{x}$

Step 3. Prepare two tables for y = $\phi(x)$ and y = $\psi(x)$ taking values of **X** between a and **b.**

Step 4. Plot these points and join them to get smooth curves.

Step 5. Note down the abscissa of the point of intersection of the curves $y = \phi(x)$ and $y = \psi(x)$. This is the required root of the equation $f(x) = 0$.

Note. Sometimes we do not write $f(x) = 0$ as $\phi(x) = \psi(x)$

We adopt the following method :

(*i*) Find a small interval (*a*, *b*) between which the root lies. *f* (*a*) and *f* (*b*) are of opposite sign.

(*ii*) Prepare a table of the different values of *x* between *a* and *b*, for $y = f(x)$.

(*iii*) Plot these points and join them to get smooth curve

(*iv*) The real root of the equation $f(x) = 0$ is the abscissa where the curve cuts the *x*-axis.

Example . Find graphically the positive root of the equation.

$x^3 - 6x - 13 = 0$

Solution.

$$f(x) = x^3 - 6x - 13 = 0 \quad ....(1)$$

f (3) = 27 - 18 - 13 = - 4 = - ve

f (4) = 64 - 24 - 13 = 27 = + ve

The root of (1) lies between 3 and 4 as f (3) and f (4) are opposite in sign. (1) is written as

$$f(x) = x^3 - 6x - 13 = 0$$

$$y = x^3$$

$$y = 6x + 13$$

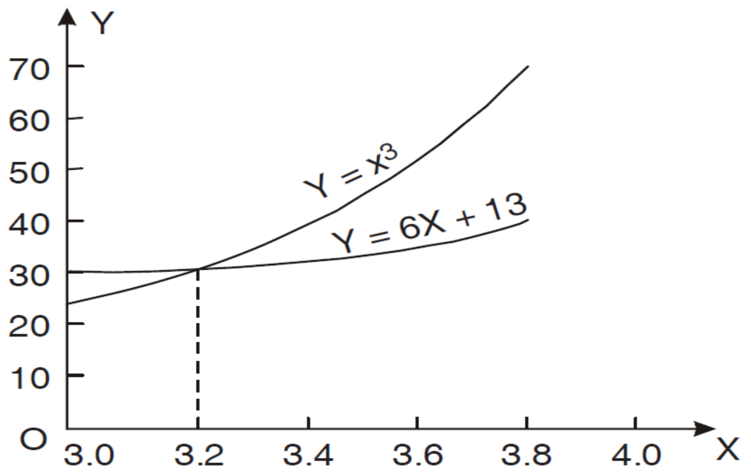
Let us draw two curves for $y = x^3$ and $y = 6x + 13$.

$y = x^3$

x	3	3.2	3.4	3.6	3.8	4.0
y	27	32.8	39.3	46.7	54.9	64

$y = 6x + 13$

x	3	3.2	3.4	3.6	3.8	4
y	31	32.2	33.4	34.6	35.8	37



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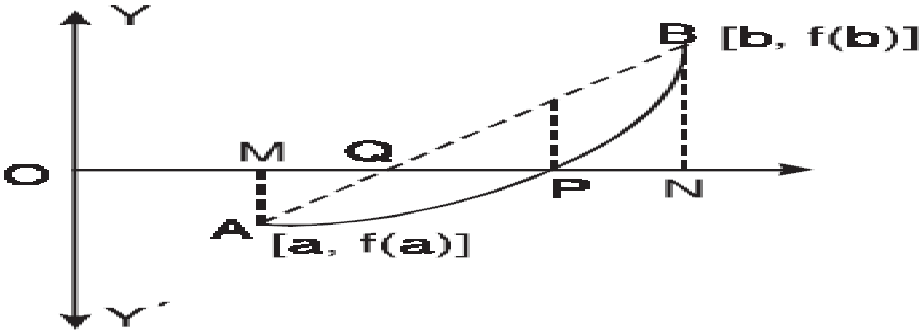
Let the origin be (3, 0).
The graphs of $y = x3$ and $y = 6x + 13$ are sketched in the figure. The abscissa of the point of intersection of two curves is 3.2.
The root of the given equation is [3.2].

ULE OF FALSE POSITION (REGULA FALSI)

Let f (x) = 0 .
Let y = f (x) be represented by the curve AB
The curve AB cuts the x-axis at P
The real root of (1) is OP
The false position of the curve AB is taken as the chord AB
The chord AB cuts the x-axis at Q. The approximate root of f (x) = 0 is OQ
By this method, we find OQ
A [a, f(a)] , B [b, f (b)] be the extremities of the chord AB The equation of the chord AB is:



$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

To find OQ, put $y = 0$, $-f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$

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$$(x-a) = \frac{-(b-a)f(a)}{f(b)-f(a)} \quad \text{or} \quad x = a + \frac{(a-b)f(a)}{f(b)-f(a)}$$

$$x = \frac{af(b)-bf(a)}{f(b)-f(a)}$$

Example . Find an approximate value of the root of the equation x^3+ x - 1 = 0 near x = 1, using the method of false position (regula falsi) two times ?

Solution. f (x) =x^3+ x - 1 = 0

f (1) = 1 + 1 – 1 = + 1

f (.5) = $(0.5)^3$+ (0.5) – 1 = – 0.375

The root lies between 0.5 and 1

Let x_1 = 0.5 and x_2 = 1

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} \quad \text{or} \quad x_3 = \frac{0.5 f(1) - 1 f(0.5)}{f(1) - f(0.5)}$$

$$= \frac{0.5\,(1) - 1\,(-0.375)}{1 + 0.375} = 0.6363$$

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Now $f(0.6363) = -0.1061$ and $f(1) = 1$

$\therefore$ Root lies between .6363 and 1.

$$x_3 = 0.6363, \quad x_2 = 1$$

$$x_4 = \frac{0.6363 f(1) - 1 f(0.6363)}{f(1) - f(0.6363)} = \frac{0.6363 - 1(-0.1061)}{1 + 0.1061} = 0.6712$$

New, $f(0.6712) = -0.0264$ and $f(1) = 1$

$$x_5 = \frac{0.6712 f(1) - 1 f(0.6712)}{f(1) - f(0.6712)} = \frac{0.6712 - (-0.0264)}{1 - (-0.0264)}$$

$$= 0.6797$$ **Ans.**

Example . *Find by the method of Regula Falsi a root of the equation*

$x^3 + x^2 - 3x - 3 = 0$ lying between 1 and 2.

Solution. $f(x) = x^3 + x^2 - 3x - 3 = 0$

$$f(1) = 1 + 1 - 3 - 3 = -4 = -ve$$

$$f(2) = 8 + 4 - 6 - 3 = +3 = +ve$$

The root lies between 1 and 2 as $f(1)$ is –ve and $f(2)$ is +ve.

By Regula Falsi method:

$$x_1 = \frac{1 f(2) - 2 f(1)}{f(2) - f(1)} = \frac{1 \times 3 - 2 \times -4}{3 - (-4)} = \frac{11}{7} = 1.571$$

$$f(1.571) = (1.571)^3 + (1.571)^2 - 3(1.571) - 3$$

$$= 3.877 + 2.468 - 4.713 - 3 = -1.368 = -ve$$

The root lies between 1.571 and 2 as $f(1.571)$ is –ve and $f(2)$ is + ve.

$$x_2 = \frac{1.571\, f(2) - 2\, f(1.571)}{f(2) - f(1.571)}$$

$$= \frac{1.571 \times 3 - 2 \times (-1.368)}{3 - (-1.368)} = \frac{4.713 + 2.736}{4.368} = 1.705$$

$$f(1.705) = (1.705)^3 + (1.705)^2 - 3\,(1.705) - 3 = 4.956 + 2.907 - 5.115 - 3$$

$$= -0.252 = -ve.$$

The root lies between 1.705 and 2 as f (1.705) is –ve and f (2) is +ve.

$$x_3 = \frac{1.705\, f(2) - 2\, f(1.705)}{f(2) - f(1.705)} = \frac{1.705 \times 3 - 2 \times (-0.252)}{3 - (-0.252)} = 1.728$$

Ans.

EXAMPLE

Find a real root of the equation $f(x) = x^3 - 2x - 5 = 0$ by method of **False position.**

SOLUTION

f (2) = – 1 and f (3) = 16

Hence the root lies in between 2 and 3. Take $a = 2$, $b = 3$.

$$x_1 = \frac{a\, f(b) - b\, f(a)}{f(b) - f(a)}$$

$$= \frac{2(16) - 3(-1)}{16 - (-1)} = \frac{35}{17} = 2.058823529.$$

$f(x_1) = f(2.058823529) = -0.390799917 < 0.$

Therefore the root lies between 0.058823529 and 3. Again, using the formula, we get the second approximation as,

$$x_2 = \frac{2.058823529(16) - 3(-0.390799917)}{16 - (-0.390799917)} = 2.08126366$$

Proceeding like this, we get the next approximation as,

$x_3 = 2.089639211,$

$x_4 = 2.092739575,$

$x_5 = 2.09388371$

$x_6 = 2.094305452,$

$x_7 = 2.094460846$

EXAMPLE

Determine the root of the equation $\cos x - x e^x = 0$ by the method of False position.

SOLUTION

$f(0) = 1$ **and** $f(1) = -2.177979523$

$a = 0$ **and** $b = 1$. **The root lies in between 0 and 1**

$$x_1 = \frac{0(-2.177979523) - 1(1)}{-2.177979523 - 1} = 0.3146653378$$

$$f(x_1) = f(0.314653378) = 0.51986.$$

The root lies in between 0.314653378 and 1.

$$x_2 = \frac{0.3146653378(-2.177979523) - 1(0.51986)}{-2.177979523 - 0.51986} = 0.44673$$

Proceeding like this, we get

$x_3 = 0.49402$,

$x_4 = 0.50995$,

$x_5 = 0.51520$,

$x_6 = 0.51692$,