Week-7- Applications of Z.T

DAWAH-2025-2026

Applications of Z-Transform

The Z-transform is a powerful tool used in signal processing and control systems to analyse and design discrete-time systems. Here are some of the applications of Z-transform:

1. Digital Signal Processing

Filter Design*: Z-transform is used to design digital filters, such as low-pass, high-pass, band-pass, and band-stop filters.

Signal Analysis*: Z-transform is used to analyse discrete-time signals and systems.

2. Control Systems

- *Discrete-Time Systems*: Z-transform is used to analyse and design discrete-time control systems, such as digital control systems.
- *Stability Analysis*: Z-transform is used to analyse the stability of discrete-time systems.

3. Communication Systems

- *Digital Communication*: Z-transform is used in digital communication systems, such as pulse code modulation (PCM) and delta modulation (DM).
- *Error Detection and Correction*: Z-transform is used in error detection and correction techniques, such as cyclic redundancy check (CRC).

4. Image Processing

- *Image Filtering*: Z-transform is used in image filtering techniques, such as image smoothing and image sharpening.
- *Image Compression*: Z-transform is used in image compression techniques, such as JPEG compression.

5. Other Applications

- *Audio Processing*: Z-transform is used in audio processing techniques, such as audio filtering and audio compression.
- *Biomedical Signal Processing*: Z-transform is used in biomedical signal processing techniques, such as ECG and EEG signal analysis.

Benefits of Z-Transform

- *Efficient Analysis*: Z-transform provides an efficient way to analyse discrete-time systems and signals.
- *Design of Digital Filters*: Z-transform enables the design of digital filters with specific frequency response characteristics.
- *Stability Analysis*: Z-transform enables the analysis of stability of discrete-time systems.

Conclusion

The Z-transform is a powerful tool with a wide range of applications in signal processing, control systems, communication systems, image processing, and other fields. Its ability to analyze and design discrete-time systems makes it an essential tool in many areas of engineering and science.

Example /in DSP

Find the response of the system s(n+2)−3s(n+1)+2s(n)=δ(n), when all the initial conditions are zero.

Solution − Taking Z-transform on both the sides of the above equation, we get

$$S(z)Z^2 - 3S(z)Z^1 + 2S(z) = 1$$

$$\Rightarrow S(z)\{Z^2 - 3Z + 2\} = 1$$

$$\Rightarrow S(z) = \frac{1}{\{z^2-3z+2\}} = \frac{1}{(z-2)(z-1)} = \frac{\alpha_1}{z-2} + \frac{\alpha_2}{z-1}$$

$$\Rightarrow S(z) = \frac{1}{z-2} - \frac{1}{z-1}$$

Taking the inverse Z-transform of the above equation, we get

$$S(n) = Z^{-1}[\frac{1}{Z-2}] - Z^{-1}[\frac{1}{Z-1}]$$

$$= 2^{n-1} - 1^{n-1} = -1 + 2^{n-1}$$

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Example

Find the system function H(z) and unit sample response h(n) of the system whose difference equation is described as under

$$y(n) = \frac{1}{2}y(n-1) + 2x(n)$$

where, y(n) and x(n) are the output and input of the system, respectively.

Solution − Taking the Z-transform of the above difference equation, we get

$$y(z) = \frac{1}{2}Z^{-1}Y(Z) + 2X(z)$$

$$= Y(Z)[1 - \frac{1}{2}Z^{-1}] = 2X(Z)$$

$$= H(Z) = \frac{Y(Z)}{X(Z)} = \frac{2}{[1-\frac{1}{2}Z^{-1}]}$$

This system has a pole at $Z = \frac{1}{2}$ and $Z = 0$ and $H(Z) = \frac{2}{[1-\frac{1}{2}Z^{-1}]}$

Hence, taking the inverse Z-transform of the above, we get

$$h(n) = 2(\frac{1}{2})^n U(n)$$

Example

Determine Y(z),n≥0 in the following case –

$$y(n) + \frac{1}{2}y(n-1) - \frac{1}{4}y(n-2) = 0 \quad given \quad y(-1) = y(-2) = 1$$

Solution − Applying the Z-transform to the above equation, we get

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$$Y(Z)+\frac{1}{2}[Z^{-1}Y(Z)+Y(-1)]-\frac{1}{4}[Z^{-2}Y(Z)+Z^{-1}Y(-1) + 4(-2)]=0$$

$$\Rightarrow Y(Z)+\frac{1}{2Z}Y(Z)+\frac{1}{2}-\frac{1}{4Z^2}Y(Z)-\frac{1}{4Z}-\frac{1}{4}=0$$

$$\Rightarrow Y(Z)[1+\frac{1}{2Z}-\frac{1}{4Z^2}]=\frac{1}{4Z}-\frac{1}{2}$$

$$\Rightarrow Y(Z)[\frac{4Z^2+2Z-1}{4Z^2}]=\frac{1-2Z}{4Z}$$

$$\Rightarrow Y(Z)=\frac{Z(1-2Z)}{4Z^2+2Z-1}$$

1 Applications of z-transforms

1.1 Transfer (or system) function

Consider a first order linear constant coefficient difference equation

y_n+a … $y_{n-1}=bx_n$ ….n=0,1,2,… (1)

where $\{x_n\}$ is a given sequence.

Assume an initial condition y_{-1} is given.

Task! Take the z-transform of (1), insert the initial condition and obtain Y(z) in terms of X(z)

Answer/ Using the right shift theorem

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$$Y(z) + a\left(z^{-1}Y(z) + y_{-1}\right) = b \quad X(z)$$

where X(z) is the z-transform of the given or input sequence $\{x_n\}$ and Y(z) is the z-transform of the response or output sequence $\{y_n\}$.

Solving for Y(z)

$$Y(z)\left(1 + az^{-1}\right) = bX(z) - ay_{-1}$$

so

$$Y(z) = \frac{bX(z)}{1+az^{-1}} - \frac{ay_{-1}}{1+az^{-1}} \quad (2)$$

The form of (2) shows us clearly that Y(z) is made up of two components, Y1(z) and Y2(z) say, where

1. $Y_1(z) = \frac{bX(z)}{1+az^{-1}}$ which depends on the input $X(z)$
2. $Y_2(z) = \frac{-ay_{-1}}{1+az^{-1}}$ which depends on the initial condition y_{-1} .

Clearly, from (2), if $y_{-1} = 0$ (zero initial condition) then

$$Y(z) = Y_1(z)$$

and hence the term **zero-state response** is sometimes used for $Y_1(z)$.

Similarly if $\{x_n\}$ and hence $X(z) = 0$ (zero input)

$$Y(z) = Y_2(z)$$

and hence the term **zero-input** response can be used for $Y_2(z)$.

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$$Y(z) = \frac{bX(z)}{1+az^{-1}}$$

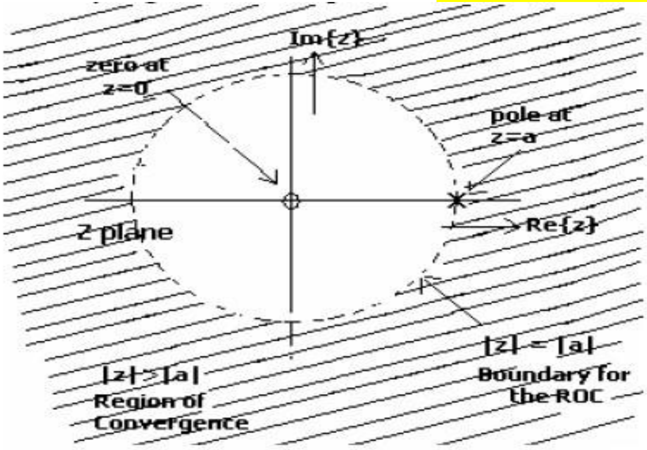
so

$$\frac{Y(z)}{X(z)} = \frac{b}{1+az^{-1}} \quad (3)$$

Poles and zeros of the Z transform

Values of z for which X(z) = 0 are called the zeros of X(z). A zero is indicated by a 'O' in the z plane. Values of z for which X(z) = ∞ are called the poles of X(z). A pole is indicated by a 'X' in the z plane.
For example, in the previous example, where
X(z) = z / (z-a)

The z transform has one zero at z=0 and one pole at z=a.



ROC for the Z transform of $a^n u(n)$

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If $x(n) \overset{Z}{\longleftrightarrow} X(z)$ & $h(n) \overset{Z}{\longleftrightarrow} H(z)$

then $x(n) * h(n) \overset{Z}{\longleftrightarrow} X(z) \cdot H(z)$

ie., convolution in the time domain is transformed into multiplication in the z-domain.

(where * denotes convolution)

Z- TRANSFORMS OF SOME USEFUL SEQUENCES:

1) A) Unit impulse δ(n):

$x(n) = \delta(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = \sum_{n=-\infty}^{\infty} \delta(n)z^{-n} = \delta(0)z^{-0} = 1$$ with ROC : the entire z-plane.

ie., $\delta(n) \overset{Z}{\longleftrightarrow} 1$ with ROC : the entire z-plane

B) $x(n) = \delta(n-n_0)$, where n_0 is positive.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = \sum_{n=-\infty}^{\infty} \delta(n-n_0)z^{-n} = 1.z^{-n}{}_0 = z^{-n}{}_0$$ (because $\delta(n-n_0) = 1$ at $n = n_0$)

with ROC : the entire z-plane except z =0. ie., ROC : $|z| > 0$

ie., $\delta(n-n_0) \overset{Z}{\longleftrightarrow} z^{-n}{}_0$ with ROC : $|z| > 0$

C) $x(n) = \delta(n+n_0)$, where n_0 is positive.

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$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = \sum_{n=-\infty}^{\infty} \delta(n+n_0)z^{-n} = 1.z^{n}_{0} = z^{n}_{0} \quad \text{(because } \delta(n+n_0) = 1 \text{ at } n = -n_0)$$

with ROC : the entire z-plane except z =. ∝ ie., ROC : $|z| < \infty$

$$\text{ie., } \delta(n+n_0) \overset{Z}{\longleftrightarrow} z^{n}_{0} \quad \text{with ROC : } |z| < \infty$$

Example. *Find the Z-transform of* **UNIT IMPULSE**

$$\delta(k) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

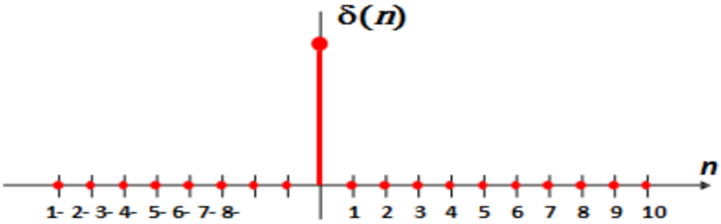
SOLUTION

$$Z[\{f(k)\}] = \sum_{k=-\infty}^{\infty} \delta(k) z^{-k}$$

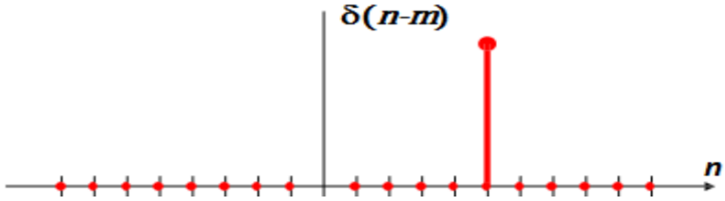
$$= [\ldots + 0 + 0 + 0 + 1 + 0 + 0 + \ldots\ldots]$$

- **delta function or unit-impulse (sample) sequence $\delta(n)$**

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$



$$\delta(n-m) = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$



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Example . Find the Z-transform of discrete UNIT STEP

$$U(k) = \begin{Bmatrix} 0 & k < 0 \\ 1 & k \geq 0 \end{Bmatrix}$$

SOLUTION

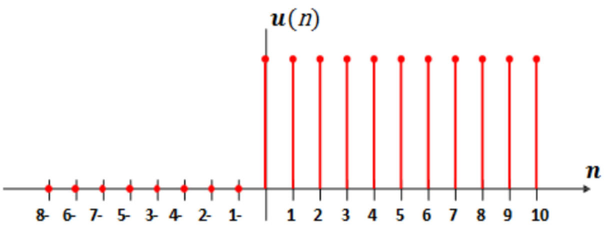
$$Z\,[\{\,U(k)\,\}] = \sum_{k=0}^{\infty} U(k)\,z^{-k} = [\,1 + z^{-1} + z^{-2} + z^{-3} + \ldots\,]$$

G.P. its sum is $\frac{a}{1-r}$.

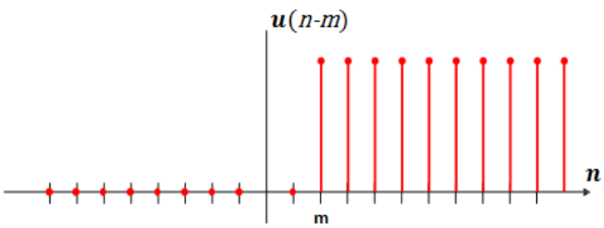
$$= \frac{1}{1-z^{-1}} = \frac{1}{1-\frac{1}{z}} = \frac{z}{z-1}$$

- **unit-step sequence** ***U(n)***

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



$$u(n\text{-}m) = \begin{cases} 1 & n \geq \mathrm{m} \\ 0 & n < \mathrm{m} \end{cases}$$



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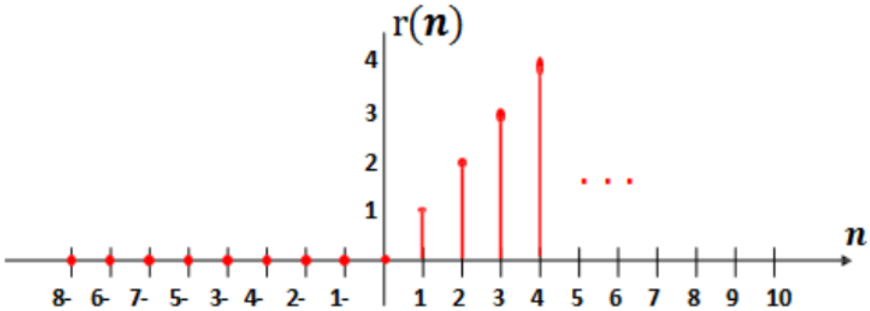
$$u(n) = \sum_{m=0}^{\infty} \delta(n-m)$$

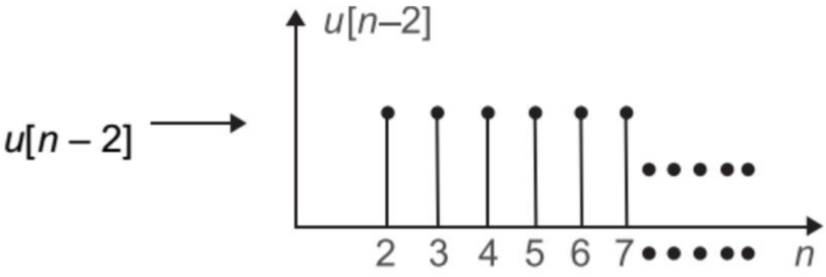
$$\delta(n) = u(n) - u(n-1)$$

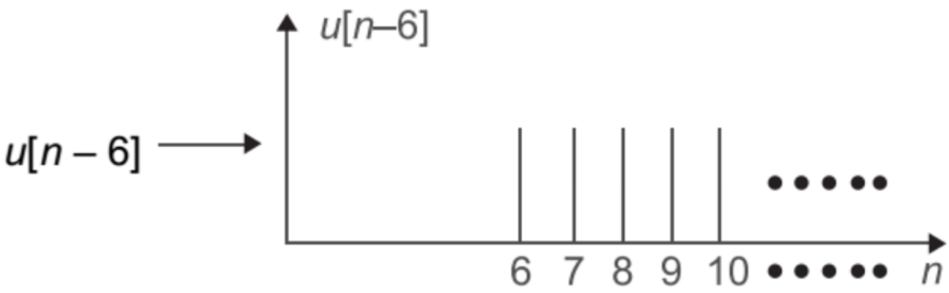
- **unit-ramp sequence** ***r(n)***

$$r(n) = n\,u(n) = \begin{cases} n & n \geq 0 \\ 0 & n < 0 \end{cases}$$



Example *Find*





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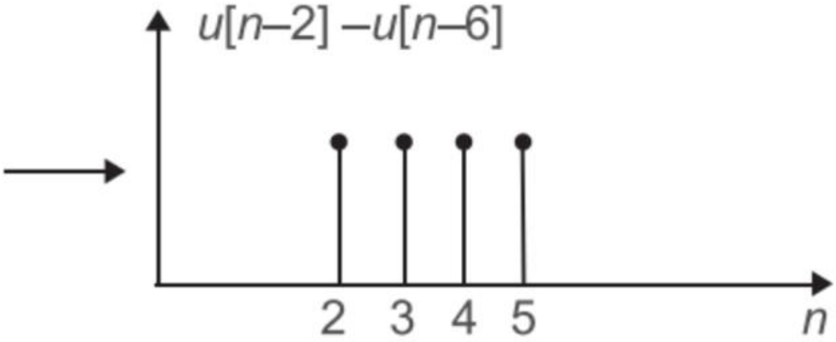
$u[n-2]-u[n-6]$



Example. **Find the Z-transform of** $\frac{a^k}{k!}$

SOLUTION

$$Z\left[\left\{\frac{a^k}{k\,!}\right\}\right]=\sum_{k=0}^{\infty}\frac{a^k}{k\,!}\,z^{-k}$$

$$=\sum_{k=0}^{\infty}\frac{(a\,z^{-1})^k}{k\,!}=1+\frac{a\,z^{-1}}{1!}+\frac{(a\,z^{-2})^2}{2!}+\frac{(a\,z^{-1})^3}{3!}+\ldots$$

CLASSIFICATION OF SYSTEM CAUSALAND NON CAUSAL

INPUT → system → **OUTPUT**

1/ IF THE OUTPUT ≥ INPUT → **CAUSAL**

2/ IF THE OUTPUT < INPUT → **NONCAUSAL**

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3/ IN ANY STEP OF THE SOLUTION INPUT > OUTPUT

STOP THE SOLUTION AND THE SYSTEM IS NON CAUSAL

EX-y(t)=x(t) y(0)=x(0) y(1)=x(1) y(-1)=x(-1) causal	EX-y(t)=x(2t) y(0)=x(0) y(1)=x(2) y(-1)=x(-2) NONcausal
EX-y(t)=x(t)+x(t-2) y(0)=x(0)+x(-2) y(1)=x(1)+x(-1) y(-1)=x(-1)+x (-3) causal	EX-y(t)=x(t-4) (t+4) y(0)=x(-4) (4) y(1)=x(3) (5) y(-1)=x(-5) (3) non causal

For casual sequence

$$Z[\{f(k-1)\}] = z^{-1}F(z) \text{ as } f(-1) = 0$$

$$Z[\{f(k+1)\}] = zF(z) - zf(0)$$

$$Z[\{f(k+2)\}] = z^2F(z) - z^2f(0) - zf(1)$$

Difference Equations

$Dy_n = y_{n+1} - y_n,$

$D^2y_n = y_{n+2} - 2y_{n+1} + y_n.$

$D^3y_n = y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n$ and so on

Example Form the difference equation for the Fibonacci sequence .

The integers 0,1,1,2,3,5,8,13,21, . . . are said to form a Fibonacci sequence.

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0+1=1+1=2+1=3+2=5+3=8+5=13+8=21……..

DAWAH Fibonacci sequence

If y_n be the n^{th} term of this sequence, then

$$y_n = y_{n-1} + y_{n-2} \text{ for } n > 2$$

$$\text{or } y_{n+2} - y_{n+1} - y_n = 0 \text{ for } n > 0$$

Fibonacci numbers, commonly denoted F_n. The sequence commonly starts from 0 and 1, although some authors start the sequence from 1 and 1 or sometimes (as did Fibonacci) from 1 and 2. Starting from 0 and 1, the sequence begins0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144

THEOREM

If $\{f(k)\} = F(z)$, $\{g(k)\} = G(z)$, and a and b are constant,

Example/ Find the Z–transform of

1. $n(n-1)$
2. $n^2 + 7n + 4$
3. $(1/2)(n+1)(n+2)$

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(i) $Z\{n(n-1)\} = Z\{n^2\} - Z\{n\}$

$$= \frac{z(z+1)}{(z-1)^3} - \frac{z}{(z-1)^2}$$

$$= \frac{z(z+1) - z(z-1)}{(z-1)^3}$$

$$= \frac{2z}{(z-1)^3}$$

: $Z\{n^2 + 7n + 4\} = Z\{n^2\} + 7Z\{n\} + 4Z\{1\}$

$$= \frac{z(z+1)}{(z-1)^3} + 7\frac{z}{(z-1)^2} + 4\frac{z}{z-1}$$

$$= \frac{z\{(z+1) + 7(z-1) + 4(z-1)^2\}}{(z-1)^3} = \frac{2z(z^2-2)}{(z-1)^3}$$

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(iii) $Z\left\{\dfrac{(n+1)(n+2)}{2}\right\} = \dfrac{1}{2}\{ Z\{n^2\} + 3Z\{n\}+2Z\{1\}\}$

$$= \frac{1}{2}\left\{\frac{z(z+1)}{(z-1)^3} + \frac{3z}{(z-1)^2} + \frac{2z}{(z-1)}\right\} \text{ if } |z| > 1$$

$$= \frac{z^3}{(z-1)^3}$$

Example

Show that $Z\{1/ n!\} = e^{1/z}$ and hence find $Z\{1/ (n+1)!\}$ and $Z\{1/ (n+2)!\}$

$$Z\left\{\frac{1}{n!}\right\} = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} = \sum_{n=0}^{\infty} \frac{(z^{-1})^n}{n!}$$

$$= 1 + \frac{z^{-1}}{1!} + \frac{(z^{-1})^2}{2!} + \ldots = e^{z^{-1}} = e^{1/z}$$

To find $Z\left\{\dfrac{1}{(n+1)!}\right\}$

We know that $Z\{f_{n+1}\} = z\{ F(z) - f_0\}$ Therefore,

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$$Z\left\{\frac{1}{(n+1)!}\right\} = z\left\{Z\left\{\frac{1}{n!}\right\} - 1\right\}$$

$$= z\,\{ e^{1/z} - 1\}$$

Similarly,

$$Z\left\{\frac{1}{(n+2)!}\right\} = z^2\,\{ e^{1/z} - 1 - (1/z)\}.$$