Week-6-theorem of Z.T

DAWAH-2025-2026

Z-Transform

The Z-transform is a mathematical tool which is used to convert the difference equations in discrete time domain into the algebraic equations in z-domain. Mathematically, if (x(n))is a discrete time function, then its Z-transform is defined as,

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

Final Value Theorem of Z-Transform

The final value theorem of Z-transform enables us to calculate the steady state value of a sequence, x(n),ie,,, x(∞) directly from its Z-transform, without the need for finding its inverse Z-transform.

Statement - If is a x(n) causal sequence, then the final value theorem of Z-transform states that if,

$$x(n) \overset{ZT}{\leftrightarrow} X(z)$$

And if the Z-transform X(z) has no poles outside the unit circle, and it has no Higher poles on the unit circle centred at the origin of the z-plane, then,

$$x(\infty) = \lim_{n\to\infty} x(n) = \lim_{z\to 1} (\mathrm{z}-1)X(z)$$

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Initial Value Theorem

If x(n) is a causal sequence, which has its Z-transformation as X(z), then the initial value theorem can be written as

$$X(n)(at \quad n=0) = \lim_{z\to\infty} X(z)$$

Proof − We know that,

$$X(Z) = \sum_{n=0}^{\infty} x(n)Z^{-n}$$

Expanding the above series, we get;

$$= X(0)Z^0 + X(1)Z^{-1} + X(2)Z^{-2} + \dots$$

$$= X(0) \times 1 + X(1)Z^{-1} + X(2)Z^{-2} + \dots$$

In the above case if Z → ∞ then $Z^{-n} \to 0$ (Because n>0)

Therefore, we can say;

$$\lim_{z\to\infty} X(z) = X(0)$$ (

Example

Let us find the Initial and Final value of x(n) whose signal is given by

$$X(Z) = 2 + 3Z^{-1} + 4Z^{-2}$$

Solution − Let us first, find the initial value of the signal by applying the theorem

$$x(0) = \lim_{z\to\infty} X(Z)$$

$$= \lim_{z\to\infty} [2 + 3Z^{-1} + 4Z^{-2}]$$

$$= 2 + (\tfrac{3}{\infty}) + (\tfrac{4}{\infty}) = 2$$

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Now let us find the Final value of signal applying the theorem

$$x(\infty) = \lim_{z\to\infty}[(1 - Z^{-1})X(Z)]$$

$$= \lim_{z\to\infty}[(1 - Z^{-1})(2 + 3Z^{-1} + 4Z^{-2})]$$

$$= \lim_{z\to\infty}[2 + Z^{-1} + Z^{-2} - 4Z^{-3}]$$

$$= 2 + 1 + 1 - 4 = 0$$

Numerical Example

Find $x(\infty)$ if X(z) is given by,

$$X(z) = \frac{z^2}{(z-1)(z-0.3)}$$

Solution

The given Z-transform of the sequence is, using the final value theorem for Z-transform

$$\left[\text{i. e}, x(\infty) = \lim_{z\to 1}[(z-1)X(z)]\right]$$

$$x(\infty) = \lim_{z\to 1}(z-1)\left[\frac{z^2}{(z-1)(z-0.3)}\right] = \lim_{z\to 1}\left[\frac{z^2}{(z-0.3)}\right]$$

$$\therefore x(\infty) = \left[\frac{1}{(1-0.3)}\right] = 1.43$$

Numerical Example

Find $x(\infty)$ if X(z) is given by,

$$X(z) = \frac{z+1}{3\,(z-1)(z+0.4)}$$

$$X(z) = \frac{z+1}{3\,(z-1)(z+0.4)}$$

$$\therefore\ (z-1)X(z) = \frac{z+1}{3(z+0.4)}$$

As we can see, $(z-1)X(z)$ has no poles on or outside the unit circle. Therefore, using the final value theorem for Z-transform, we have,

$$x(\infty) = \lim_{z\to 1}\left[\frac{z+1}{3(z+0.4)}\right] = \left[\frac{1+1}{3(1+0.4)}\right]$$

$$\therefore\ x(\infty) = \left[\frac{2}{3\times 1.4}\right] = 0.48$$

Example:

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Let's say we have the z-transform:

X(z) = z / (z - 1)

To find the initial value, x(0), using the initial value theorem:

x(0) = lim (z->∞) z / (z - 1)

Dividing both numerator and denominator by z:

x(0) = lim (z->∞) 1 / (1 - 1/z)

As z approaches infinity, 1/z approaches 0:

x(0) = 1 / (1 - 0) = 1

❖ **Example:** Calculate the initial value of $X(z) = \dfrac{z(1-e^{-5T})}{(z-1)(z-e^{-5T})}$

Rewrite $X(z)$ as $X(z) = \dfrac{z^{-1}(1-e^{-5T})}{(1-z^{-1})(1-e^{-5T}z^{-1})}$

Dividing both nume and den BY Z^2

Then, we have: $x(0) = \lim_{z\to\infty} \dfrac{z^{-1}(1-e^{-5T})}{(1-z^{-1})(1-e^{-5T}z^{-1})} \to \dfrac{0}{1\cdot 1} = 0$

Example/Obtain the z-transform of $f(n)=1-a^n$, $0<a<1$

Verify the initial value theorem for the z-transform pair you obtain.

Using standard z-transforms we obtain

$$Z\{f_n\} = F(z) = \frac{z}{z-1} - \frac{z}{z-a}$$

$$= \frac{1}{1-z^{-1}} - \frac{1}{1-az^{-1}}$$

hence, as $z \to \infty : F(z) \to 1 - 1 = 0$

Similarly, as $n \to 0$

$f_n \to 1 - 1 = 0$

so the initial value theorem is verified for this case.