Week-4-Fundamental of Z-transform (Z.T), properties of Z.T

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INTRODUCTION

Z-transform plays an important role in discrete analysis. Its role in discrete analysis is the same as that of Laplace and Fourier transforms in continuous system. Communication is one of the field whose development is based on discrete analysis. Difference equations are also based on discrete system and their solutions and analysis are carried out by Z- transform .

The Z transform is a powerful mathematical tool used in **digital signal processing** and **control systems analysis**. It allows us to **transform signals from the time domain to the frequency domain**, simplifying the analysis and design of digital systems

SEQUENCE

Sequence **{ *f* (*k*) }** is an ordered list of real or complex numbers.

REPRESENTATION OF A SEQUENCE

FIRST METHOD

The elementary way is to list all the members of the sequence such as

$$\{ f(k) \} = \{\underset{\uparrow}{15}, 10, 7, 4, 1, -1, 0, 3, 6\}$$

The symbol ↑ is used to denote the term in zero position i.e., k = 0,

k is an index of position of a term in the sequence.

$$\{g(k)\} = \{15, 10, \underset{\uparrow}{7}, 4, 1, -1, 0, 3, 6\}$$

Two sequences {*f* (*k*)} and {*g* (*k*)} have the same terms but these sequences are not identically the same as the zeros term of those sequences are different.

In case the symbol ↑ is not given then left hand end term is considered as the term corresponding to $K = 0$.

SECOND METHOD **The second way of specifying the sequence is to define the general term of the sequence {$f\ (k)$}as function of k..**

For example IF, $f\ (k) = \frac{1}{3^k}$

This sequence represents $\{\ldots \frac{1}{3^{-3}}, \frac{1}{3^{-2}}, \frac{1}{3^{-1}}, 1, \frac{1}{3^{1}}, \frac{1}{3^{2}} \ldots \ldots\}$

$\uparrow$ K=0

Definition.The Z- transform of a sequence {f (k)} is denoted as Z [{f (k)}] . It is defined as

$$Z\ [\{f\,(k)\}] = F\,(z) = \sum_{k=-\infty}^{\infty} f\,(k)\, z^{-k} = \sum_{k=-\infty}^{\infty} \frac{f\,(k)}{z^{k}}$$

Where

1. **Z is a complex number.**
2. **Z is an operator of Z-transform**
3. **F (z) is the Z transform of {f (k)}.**

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$z = Ae^{j\phi} = A \cdot (\cos\phi + j \sin\phi)$$

where A is the magnitude of z, j is the imaginary unit, and ϕ is the complex argument (also referred to as *angle* or *phase*) in radians.

Z – Transform of some time sequences

1) Right side sequences As an example, let us find the z-transform and ROC of the right sided sequence **ROC Region of Convergence**

$$x(n) = (\underset{\uparrow}{1}, 2, 2, 1)$$

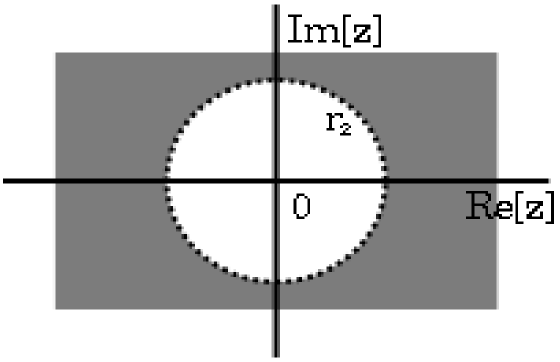
$$X(z) = Z\{x(n)\} = \sum_{n=0}^{3} x(n)z^{-n} = x(0)z^{-0} + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3}$$

$$= 1z^{-0} + 2z^{-1} + 2z^{-2} + 1z^{-3}$$

$$= 1 + 2z^{-1} + 2z^{-2} + z^{-3}$$

We see that X(z) becomes infinity at z =0. Except at z = 0, X(z) is finite for all values of z. Therefore we can say that the ROC of this z transform is the entire z-plane except z = 0. ie.,

ROC : |z| > 0. **The ROC of a right-sided sequence**



2) Left sided sequences:

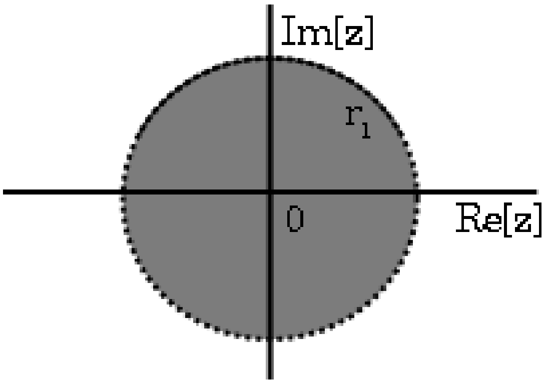
Let us find the z-transform and ROC of the left sided sequence

$$x(n) = (1, 1, 2, \underset{\uparrow}{2})$$

$$X(z) = Z\{x(n)\} = \sum_{n=-3}^{0} x(n)z^{-n} = x(-3)z^{3} + x(-2)z^{2} + x(-1)z^{1} + x(0)z^{0}$$

$$= z^{3} + z^{2} + 2z + 2$$

We see that X(z) becomes infinity at z = ∞.. Except at z = ∞, X(z) is finite for all values of z. Therefore we can say that the ROC of this z transform is the entire z-plane except z = ∞ ie., ROC : |z| < ∞



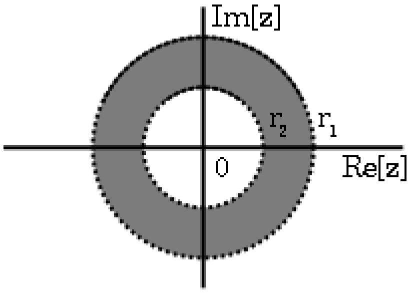
The ROC of a left-sided sequence.

3) Double sided sequences:

A sequence x(n) is said to be double sided if x(n) has both right and left sides. For example, x(n) = (2, 1, 1, 2) is a double sided sequence because x(n) exists in the range -2 ≤ n ≤ 1. Z transform of this sequence is given by

$$X(z) = Z\{x(n)\} = \sum_{n=-2}^{1} x(n)z^{-n} = x(-2)z^{2} + x(-1)z^{1} + x(0)z^{0} + x(1)z^{-1}$$

$$= 2z^{2} + 1z^{1} + 1z^{0} + 2z^{-1}$$



The ROC of a two-sided sequence

Example $\quad$ If $g(k) = \{15,\ 10,\ 7,\ 4,\ 1,\ -1,\ 0,\ \underset{\uparrow}{3},\ 6\}$

$$Z[\{g(k)\}] = F(z) = 15z^{7} + 10z^{6} + 7z^{5} + 4z^{4} + z^{3} - z^{2} + 0 + 3 + \frac{6}{z}.$$

Example $\quad$ If $f(k) = \dfrac{1}{3^{k}}, \quad -4 \le k \le 3,$ then

$$Z[\{f(k)\} = 81z^{4} + 27z^{3} + 9z^{2} + 3z + 1 + \frac{1}{3z} + \frac{1}{9z^{2}} + \frac{1}{27z^{3}}$$

Example . Find Z-transform of the sequence $\{a^{k}\},\ k \ge 0.$

Solution. $\quad F(z) = \displaystyle\sum_{k=0}^{\infty} a^{k} z^{-k} = 1 + \frac{a}{z} + \frac{a^{2}}{z^{2}} + \frac{a^{3}}{z^{3}} + \ldots.$

This is a Geometrical series whose sum $= \dfrac{a}{1-r}$

$$= \frac{1}{1-\frac{a}{z}} = \frac{z}{z-a}$$

Table *z transforms*

f_n	$F(z)$	
u_n	$\dfrac{z}{z-1}$	$\lvert z\rvert > 1$
δ_n	1	
n	$\dfrac{z}{(z-1)^2}$	$\lvert z\rvert > 1$
r^n	$\dfrac{z}{z-r}$	$\lvert z\rvert > \lvert r$
$\cos(\theta n)$	$\dfrac{z(z-\cos(\theta))}{z^2-2z\cos(\theta)+1}$	$\lvert z\rvert > 1$
$\sin(\theta n)$	$\dfrac{z\sin(\theta)}{z^2-2z\cos(\theta)+1}$	$\lvert z\rvert > 1$
$e^{j\theta n}$	$\dfrac{z}{z-e^{j\theta}}$	$\lvert z\rvert > 1$

Example

Let us find the Z-transform and the ROC of a signal given as $x(n) = \{7, 3, 4, 9, 5\}$, where origin of the series is at 3.

Solution – Applying the formula we have –

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)Z^{-n}$$

$$= \sum_{n=-1}^{3} x(n)Z^{-n}$$

$$= x(-1)Z + x(0) + x(1)Z^{-1} + x(2)Z^{-2} + x(3)Z^{-3}$$

$$= 7Z + 3 + 4Z^{-1} + 9Z^{-2} + 5Z^{-3}$$

ROC is the entire Z-plane excluding Z = 0, ∞, -∞

Properties of ROC

- ROC does not include any pole.
- For right-sided signal, ROC will be outside the circle in Z-plane.
- For left sided signal, ROC will be inside the circle in Z-plane.
- For stability, ROC includes unit circle in Z-plane.
- For Both sided signal, ROC is a ring in Z-plane.

- **For finite-duration signal, ROC is entire Z-plane**

Properties of the *z* transform

For the following

$$\mathcal{Z}\{f_n\} = \sum_{n=0}^{\infty} f_n z^{-n} = F(z), \qquad \mathcal{Z}\{g_n\} = \sum_{n=0}^{\infty} g_n z^{-n} = G(z).$$

1. Linearity:

$$\mathcal{Z}\{af_n + bg_n\} = aF(z) + bG(z).$$

Example 15.24 (*Linearity*) Find the *z* transform of 3*n* + 2 × 3^n.

Solution **From the linearity property**

Z{3*n* + 2 × 3^n} = 3Z{*n*} + 2Z{3^n}

and from the Table above

$$\mathcal{Z}\{n\} = \frac{z}{(z-1)^2} \quad \text{and} \quad \mathcal{Z}\{3^n\} = \frac{z}{z-3}$$

(r^n with *r* = 3). Therefore

$$\mathcal{Z}\{3n + 2 \times 3^n\} = \frac{3z}{(z-1)^2} + \frac{2z}{z-3}$$

Example (*Linearity and the inverse transform*) Find the inverse *Z-T*

$$\frac{2z}{z-1}+\frac{3z}{z-2}.$$

Solution **From Table**

$\mathcal{Z}^{-1}\left\{\frac{z}{z-1}\right\}=u_n$	$\mathcal{Z}^{-1}\left\{\frac{z}{z-2}\right\}=2^n \quad (r=2)$

$$\mathcal{Z}^{-1}\left\{\frac{2z}{z-1}+\frac{3z}{z-2}\right\}=2u_n+3\times 2^n$$

2. Left shifting property:

$$\mathcal{Z}\{f_{n+k}\}=z^k F(z)-\sum_{i=0}^{k} z^{k-i} f_i .$$

3. Right shifting property (although usually we assume *fn* = 0 for *(n <* 0)we use $f-1$, $f-2$ for the initial conditions when solving difference equations using *z* transforms):

$$\mathcal{Z}\{f_{n-1}\} = z^{-1}\mathcal{Z}\{f_n\} + f_{-1}$$

$$\mathcal{Z}\{f_{n-2}\} = z^{-2}\mathcal{Z}\{f_n\} + f_{-2} + z^{-1}f_{-1}$$

$$\mathcal{Z}\{f_{n-k}\} = z^{-k}\mathcal{L}\{f_n\} + \sum_{i=0}^{k-1} f_{i-k}z^{-i}.$$

4. Change of scale:

$$\mathcal{Z}\{a^n f_n\} = F\left(\frac{z}{a}\right)$$

where *a* is a constant.

Example (*Change of scale*) Find the inverse *z* transform of

$\dfrac{z}{(z-2)^2}$	$\dfrac{z}{(z-2)^2} = \dfrac{\frac{1}{2}(z/2)}{((z/2)-1)^2}.$	From Table $\mathcal{Z}^{-1}\left\{\dfrac{z}{(z-1)^2}\right\} = n$

$$\mathcal{Z}^{-1}\left\{\frac{\frac{1}{2}(z/2)}{((z/2)-1)^2}\right\} = \frac{1}{2}n(2)^n = n2^{n-1}.$$

5. Convolution:

$$\mathcal{Z}\left\{\sum_{k=0}^{n} g_k f_{n-k}\right\} = G(z)F(z).$$

The convolution of *f* and *g* can be written as

$$g * f = \sum_{k=0}^{n} g_k f_{n-k}.$$

where g_n and f_n are sequences defined for $n \geqslant 0$.

Example (*Convolution*) Find the inverse *z* transform of

$\frac{z}{z-1}\frac{z}{z-4}$	Note that	$\mathcal{Z}^{-1}\left\{\frac{z}{z-1}\right\} = u_n$	$\left\{\frac{z}{z-4}\right\} = 4^n$

Hence, using convolution

$$\mathcal{Z}^{-1}\left\{\frac{z}{z-1}\frac{z}{z-4}\right\} = u_n * 4^n = \sum_{k=0}^{n} u_k 4^{n-k}.$$

Writing out this sequence for $n = 0, 1, 2, 3, \ldots$

$$\begin{array}{cccc} 1, & (1+4), & 1+4+16, & 1+4+16+64, \ldots \\ (n=0) & (n=1) & (n=2) & (n=3) \end{array}$$

We see that the nth term is a geometric series with $n+1$ terms and first term 1 and common ratio 4. From the formula for the sum for n terms of a geometric progression, $S_n = a(r^n - 1)/(r - 1)$ where a is the first term, r is the common ratio and n is the number of terms. Therefore, for the n th term of the above sequence, we get:

$\frac{4^{n+1}-1}{4-1} = \frac{4^{n+1}-1}{3}.$	So we have found	$\mathcal{Z}^{-1}\left\{\frac{z}{z-1}\frac{z}{z-4}\right\} = \frac{4^{n+1}-1}{3}$

6. Derivatives of the transform:

$$\mathcal{Z}\{nf_n\} = -z\frac{dF}{dz}(z).$$

Example / (Derivatives of the transform) Using

$\mathcal{Z}\{n\} = \dfrac{z}{(z-1)^2}$	find $\mathcal{Z}\{n^2\}$.

Solution Using the derivative of the transform property

$\mathcal{Z}\{n^2\} = \mathcal{Z}\{nn\} = -z\dfrac{\mathrm{d}}{\mathrm{d}z}\mathcal{Z}\{n\}$	$= -z\dfrac{d}{dz}\left(\dfrac{z}{(z-1)^2}\right)$

$$\frac{d}{dz}\left(\frac{z}{(z-1)^2}\right) = \frac{(z-1)^2 - 2z(z-1)}{(z-1)^4} = \frac{z-1-2z}{(z-1)^3}$$

$$= \frac{-z-1}{(z-1)^3}$$

$$\mathcal{Z}\{n^2\} = -z\left(\frac{-z-1}{(z-1)^3}\right) = \frac{z(z+1)}{(z-1)^3}$$