Week-2- properties, theorem of L.T

DAWAH-2025-2026

In all of the following, $F(s) = L\{f(t)\}$

1. Linearity:

$L\{af1(t) + bf2(t)\} = aF1(s) + F2(s)$

where a and b are constants.

2. First translation (or Shift rule):

$$L\{e^{at}f(t)\} = F(s - a)$$

3. Second translation:

$L\{f(t - a)u(t - a)\} = e^{-as}F(s).$

Table Common Laplace transforms

$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$	
$u(t)$	$1/s$	$\text{Re}(s) > 0$
$\delta(t)$	1	$\text{Re}(s) > 0$
$\dfrac{t^{n-1}}{(n-1)!}$	$\dfrac{1}{s^n}$	$\text{Re}(s) > 0$
e^{-at}	$\dfrac{1}{s+a}$	$\text{Re}(s) > -a$
$\dfrac{1}{a}\sin(at)$	$\dfrac{1}{s^2+a^2}$	$\text{Re}(s) > 0$

$\cos(at)$	$\frac{s}{s^2+a^2}$	$\text{Re}(s) > 0$
$\frac{1}{a}\sinh(at)$	$\frac{1}{s^2-a^2}$	$\text{Re}(s) > \|a\|$
$\cosh(at)$	$\frac{s}{s^2-a^2}$	$\text{Re}(s) > \|a\|$

4. Change of scale: $L\{f(at)\} = \frac{1}{a} F \frac{s}{a}$

5. Laplace transforms of derivatives:

$L\{f'(t)\} = sF(s) - f(0)$

$L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$

and

$L\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$

6. Integrals:

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{F(s)}{s}.$$

7. Convolution:

$$\mathcal{L}\{f * g\} = \mathcal{L}\left\{\int_0^t f(\tau)g(t-\tau)\,d\tau\right\} = F(s)G(s).$$

8. Derivatives of the transform:

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)$$

Where

$$F^n(s) = \frac{\mathrm{d}^n F^{(n)}}{\mathrm{d}s^n}(s).$$

Example (***Linearity***) Find L{$3t^2$ + sin$(2t)$}.

Solution From Table above.

$$\mathcal{L}\left\{\frac{t^2}{2}\right\} = \frac{1}{s^3} \text{ and } \mathcal{L}\left\{\frac{\sin(2t)}{2}\right\} = \frac{1}{(s^2+4)}.$$

$$\mathcal{L}\{3t^2 + \sin(2t)\} = 6\mathcal{L}\left\{\frac{t^2}{2}\right\} + 2\mathcal{L}\left\{\frac{\sin(2t)}{2}\right\} = \frac{6}{s^3} + \frac{2}{s^2+4}$$

Example (*Linearity and the inverse transform*) Find

$$\mathcal{L}^{-1}\left\{\frac{2}{s+4} + \frac{4s}{s^2+9}\right\}$$

Solution From Table above

$$\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} = e^{-4t}$$ and $$\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} = \cos(3t)$$

$$\mathcal{L}^{-1}\left\{\frac{2}{s+4} + \frac{4s}{s^2+9}\right\} = 2e^{-4t} + 4\cos(3t)$$

Example (*First translation*) Find $L\{t^2 e^{-3t}\}$.

Solution **As**

$$\mathcal{L}\{t^2\} = \frac{2}{s^3}$$ Using $\mathcal{L}\{e^{-at} f(t)\} = F(s-a)$. **Then multiplying by e^{-3t} in the *t* domain will shift the function *F(s)* by 3:** $L[t^2 e^{-3t}] = \frac{2}{(s+3)^3}$

Example (*First translation – inverse transform*) Find

$$\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+9}\right\}.$$ *Solution* As

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} = \cos(3t)$$

is s/(s2 + 9) translated by 2, using the first translation rule this will multiply in the *t* domain by e − 2*t*, so

$$\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+9}\right\} = \cos(3t)\mathrm{e}^{-2t}$$

Example (*Second translation*) Find the Laplace transform of:

$$f(t) = \begin{cases} 0 & t < \frac{2}{3} \\ \sin(3t-2) & t \geqslant \frac{2}{3} \end{cases}$$

Solution f (t) can be expressed using the unit step function

$$f(t) = \sin(3t-2)u\left(t-\frac{2}{3}\right) = \sin\left(3\left(t-\frac{2}{3}\right)\right)u\left(t-\frac{2}{3}\right)$$

Using L{*f (t − a)u(t − a)*} = $e^{-as}F(s)$ and as

$$\mathcal{L}\{\sin(3t)\} = \frac{3}{s^2+9}$$

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$$\sin\left(3\left(t-\frac{2}{3}\right)\right)u\left(t-\frac{2}{3}\right)=\frac{3e^{-2s/3}}{s^2+9}$$

Example (*Change of scale*) Given

$$\mathcal{L}\{\cos(t)\}=\frac{s}{s^2+1}$$

find L{cos*(*3*t)*} using the change of scale property of the Laplace transform

$$\mathcal{L}\{f(at)\}=\frac{1}{a}F\frac{s}{a}. \qquad \mathcal{L}\{\cos(t)\}=\frac{s}{s^2+1}=F(s)$$

..

$$\mathcal{L}\{\cos(3t)\}=\frac{1}{3}\frac{s/3}{(s/3)^2+1}=\frac{s}{s^2+9}$$

Example (*Derivatives*) Given

$$\mathcal{L}\{\cos(2t)\}=\frac{s}{s^2+4}$$

Solution If *f (t)* = cos*(*2*t)* then *f'(t)* = − 2 sin*(*2*t)*:

and f (0) = cos(0) = 1. From the derivative rule

L{f'(t)} = sF(s) − f (0),

$$\mathcal{L}\{-2\sin(2t)\} = s\left(\frac{s}{s^2+4}\right) - 1 = \frac{s^2}{s^2+4} - 1 = \frac{s^2-s^2-4}{s^2+4}$$
$$= \frac{-4}{s^2+4}.$$
$$\mathcal{L}\{\sin(2t)\} = \frac{1}{-2}\left(\frac{-4}{s^2+4}\right) = \frac{2}{s^2+4}.$$

Example (*Integrals*) Using

$\mathcal{L}\left\{\int_0^t f(\tau)\,\mathrm{d}\tau\right\} = \frac{F(s)}{s}$	and $\mathcal{L}\{t^2\} = \frac{2}{s^3}$	find $\mathcal{L}\left\{\frac{t^3}{3}\right\}$

$$\mathcal{L}\left\{\int_0^t \tau^2\,\mathrm{d}\tau\right\} = \frac{2/s^3}{s} = \frac{2}{s^4} \quad \Rightarrow \quad \mathcal{L}\left\{\frac{t^3}{3}\right\} = \frac{1}{s}\left(\frac{2}{s^3}\right) = \frac{2}{s^4}.$$

Example (*Convolution*) Find

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-2)(s-3)}\right\}$$ using the convolution property:

$$\mathcal{L}\left\{\int_0^1 f(t)g(t-\tau)\mathrm{d}\tau\right\} = F(s)G(s)$$

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$\dfrac{1}{(s-2)(s-3)} = \dfrac{1}{(s-2)}\dfrac{1}{(s-3)}$	$F(s) = \dfrac{1}{s-2}, \qquad G(s) = \dfrac{1}{s-3}$

$f(t) = \mathcal{L}^{-1}\left\{\dfrac{1}{s-2}\right\} = e^{2t}$	$g(t) = \mathcal{L}^{-1}\left\{\dfrac{1}{s-3}\right\} = e^{3t}$

Then by the convolution rule

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-2)(s-3)}\right\} = \int_0^t e^{2\tau} e^{3(t-\tau)}\, d\tau.$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-2)(s-3)}\right\} = e^{3t}\int_0^t e^{-\tau}\, d\tau = e^{3t}\left[\frac{e^{-\tau}}{-1}\right]_0^t$$

$$= e^{3t}(-e^{-t} + 1)$$

$$= -e^{2t} + e^{3t}.$$

Example (*Derivatives of the transform*) Find L{*t* sin*(3t)*}.

Solution Using the derivatives of the transform property

$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)$	$\mathcal{L}\{\sin(3t)\} = \dfrac{3}{s^2+9} = F(s)$

$$\mathcal{L}\{t\sin(3t)\} = -\frac{d}{ds}\left(\frac{3}{s^2+9}\right) = \frac{6s}{(s^2+9)^2}.$$