Week-15-Matrices solution-Cayley-Hamilton

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What is Cayley Hamilton Theorem?

$$p(A) = A^n + a_{n-1}A^{n-1} + \ldots + a_1A + a_0I_n = 0$$

(OR)

$p(A) = 0$,where A is an n x n square matrix

The Cayley Hamilton theorem states that the characteristic polynomial expression of a real or complex square matrix will be equal to the zero matrix. The characteristic polynomial

$p(\lambda) = \det(\lambda I_n - A)$ can be decomposed as

$p(\lambda) = a_n\lambda_n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0\lambda^0$. This is a monic polynomial where the leading coefficient, i.e, the coefficient of the highest degree variable, will be equal to 1. Thus, $a_n = 1$.

Here, $a_{n-1}, \ldots, a_1$, a_0 are coefficients of the variables $\lambda^{n-1}, \ldots$, λ1, λ0 respectively.

We have $p(\lambda) =$

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0\lambda^0$$

$$p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0$$

On replacing λ with the matrix, A, the polynomial can be written as follows:

$$p(A) = A^n + a_{n-1}A^{n-1} + ... + a_1A + a_0I_n$$

Now according to the Cayley Hamilton Theorem, this polynomial will be 0

Thus, $p(A) = A^n + a_{n-1}A^{n-1} + ... + a_1A + a_0I_n = 0$ or $p(A) = 0$

Cayley Hamilton Theorem Example

Suppose a matrix is given as $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. The characteristic polynomial is $\lambda^2 - 5\lambda - 2$. Use the matrix A in place of the variable to get $p(A) = A^2 - 5A - 2I = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^2 - 5\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Now perform the required computations to get the final value as 0.

Cayley Hamilton Theorem Formula

calculations with speed and accuracy. It can also be used to determine the inverse of a matrix. The formula is given as follows:

Cayley Hamilton Theorem 2 × 2

To apply the Cayley Hamilton Theorem to a 2 × 2 square matrix, the first step is to determine the characteristic polynomial expression.

General form of the characteristic polynomial

$$p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0$$ **,,As n = 2, thus,**

$$p(\lambda) = \lambda^2 + a_1\lambda + a_0$$

For a 2 × 2 square matrix this polynomial is written as

$$p(\lambda) = \lambda^2 - S_1\lambda + S_0$$

where, S_1 = sum of the diagonal elements and S_0 = determinant of the 2 × 2 square matrix.

Now according to the Cayley Hamilton theorem, if λ is substituted with a square matrix then the characteristic polynomial will be 0. The formula can be written as

$$B^2 - S_1B + S_0I = 0$$

here, B is a 2 × 2 square matrix

Cayley Hamilton Theorem 3 × 3

For a 3 × 3 square matrix the characteristic polynomial is given as $p(\lambda) = \lambda^3 - T_2\lambda^2 + T_1\lambda - T_0$

where, T_2 = sum of the main diagonal elements, T_1 = sum of the minors of the main diagonal elements, T_0 = determinant of the 3 × 3 square matrix.

On applying the Cayley Hamilton theorem, the formula is given as

$$C^3 - T_2C^2 + T_1C - T_0I = 0$$

here, C is a 3 × 3 square matrix.

Cayley Hamilton Theorem Example 3 × 3 Matrix Verify the Cayley Hamilton Theorem for the

$$\text{matrix } C = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}.$$

Solution: The characteristic polynomial is given

as $C^3 - T_2C^2 + T_1C - T_0$.

$$T_2 = 2 + 2 + 2 = 6$$

$$T_1 = 3 + 2 + 3 = 8$$

$$T_0 = |C| = 3$$

Now $p(C) = C^3 - 6C^2 + 8C - 3I$

$$C^3 = \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix}$$

$$6C^2 = \begin{bmatrix} 42 & -36 & 54 \\ -30 & 36 & -36 \\ 30 & -30 & 42 \end{bmatrix}$$

$$8C = \begin{bmatrix} 16 & -8 & 16 \\ -8 & 16 & -8 \\ 8 & -8 & 16 \end{bmatrix}$$

$$3I = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Now substituting these values in p(C) we get

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, the Cayley Hamilton Theorem has been verified.