Week-11- solution of linear simultaneous equations

1)Direct methods: a)Gauss elimination B)Gauss Jordan

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Gauss method

By this method elimination of unknown is done more systematically and we have a check to detect the errors. The method is explained in the following example

Example *Solve the following simultaneous equations*

$$2x + 3y + z = 13$$
$$x - y - 2z = -1$$
$$3x + y + 4z = 15$$

Step 1. We write the equation, first, which has unity as coefficient of x, otherwise divide the equation by the coefficient of X to make it unity. Thus

$$x - y - 2z = -1$$
$$2x + 3y + z = 13$$
$$3x + y + 4z = 15$$

Step 2. To eliminate X, subtract suitable multiples of first equation from the remaining equations, and we get

$$x - y - 2z = -1$$
$$5y + 5z = 15 \qquad \text{....R2 - 2R1=R2}$$
$$4y + 10z = 18 \qquad \text{....R3 – 3R1=R3}$$

Step 3. The coefficient of y is made unity in none of the resulting equations and we have

$$x - y - 2z = -1$$
$$y + z = 3 \qquad \text{....R2/5 = R2}$$
$$4y + 10z = 18$$

Step 4. To eliminate *y*, subtract suitable multiple of second equation from the third. Thus we have

$$x - y - 2z = -1$$
$$y + z = 3$$
$$6z = 6$$

Step 5. Start from bottom and substitute.

6 z = 6 or Z = 1

y + z = 3 or y + 1 = 3 or y = 2

x - y - 2 z = - 1 or x - 2 - 2 = - 1 or X = 3

EXAMPLE . Solve the following system of equations:

$$\begin{aligned} 2x + 4y + 6z &= 22 \\ 3x + 8y + 5z &= 27 \\ -x + y + 2z &= 2 \end{aligned}$$

Solution First, we transform this system into an equivalent system in which the coefficient of *x* in the first equation is 1:

$$\begin{aligned} 2x + 4y + 6z &= 22 \\ 3x + 8y + 5z &= 27 \\ -x + y + 2z &= 2 \end{aligned} \qquad (4a)$$

$$\begin{aligned} x + 2y + 3z &= 11 \\ 3x + 8y + 5z &= 27 \\ -x + y + 2z &= 2 \end{aligned} \qquad \text{(4b)}$$

Multiply the first equation in (4a) by $\frac{1}{2}$.

Next, we eliminate the variable x from all equations except the first:

$$\begin{aligned} x + 2y + 3z &= 11 \\ 2y - 4z &= -6 \\ -x + y + 2z &= 2 \end{aligned} \qquad \text{(4c)}$$

Replace the second equation in (4b) by the sum of $-3 \times$ the first equation + the second equation:

$$\begin{aligned} -3x - 6y - 9z &= -33 \\ 3x + 8y + 5z &= 27 \\ \hline 2y - 4z &= -6 \end{aligned}$$

$$\begin{aligned} x + 2y + 3z &= 11 \\ 2y - 4z &= -6 \\ 3y + 5z &= 13 \end{aligned} \qquad \text{(4d)}$$

Replace the third equation in (4c) by the sum of the first equation + the third equation:

$$\begin{aligned} x + 2y + 3z &= 11 \\ -x + y + 2z &= 2 \\ \hline 3y + 5z &= 13 \end{aligned}$$

Then we transform System (4d) into yet another equivalent system, in which the coefficient of y in the second equation is 1:

$$\begin{aligned} x + 2y + 3z &= 11 \\ y - 2z &= -3 \\ 3y + 5z &= 13 \end{aligned} \qquad \text{(4e)}$$

Multiply the second equation in (4d) by $\frac{1}{2}$.

We now eliminate y from all equations except the second, using operation 3 of the elimination method:

$$\begin{aligned} x \qquad + 7z &= 17 \\ y - 2z &= -3 \\ 3y + 5z &= 13 \end{aligned} \qquad (4f)$$

Replace the first equation in (4e) by the sum of the first equation + $(-2) \times$ the second equation:

$$\begin{array}{r} x + 2y + 3z = 11 \\ -2y + 4z = 6 \\ \hline x \qquad + 7z = 17 \end{array}$$

$$\begin{aligned} x \qquad + 7z &= 17 \\ y - 2z &= -3 \\ 11z &= 22 \end{aligned} \qquad (4g)$$

Replace the third equation in (4f) by the sum of $(-3) \times$ the second equation + the third equation:

$$\begin{array}{r} -3y + 6z = 9 \\ 3y + 5z = 13 \\ \hline 11z = 22 \end{array}$$

Multiplying the third equation by $\frac{1}{11}$ in (4g) leads to the system

$$\begin{aligned} x \qquad + 7z &= 17 \\ y - 2z &= -3 \\ z &= 2 \end{aligned}$$

Eliminating z from all equations except the third (try it!) then leads to the system

$$\begin{aligned} x \qquad &= 3 \\ y \qquad &= 1 \\ z &= 2 \end{aligned} \qquad (4h)$$

In its final form, the solution to the given system of equations can be easily read off! We have $x = 3$, $y = 1$, and $z = 2$. Geometrically, the point (3, 1, 2) is the intersection of the three planes described by the three equations comprising the given system.

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Augmented Matrices

Observe from the preceding example that the variables x, y, and z play no significant role in each step of the reduction process, except as a reminder of the position of each coefficient in the system. With the aid of matrices, which are rectangular arrays of numbers, we can eliminate writing the variables at each step of the reduction and thus save ourselves a great deal of work. For example, the system

$$\begin{aligned} 2x + 4y + 6z &= 22 \\ 3x + 8y + 5z &= 27 \\ -x + y + 2z &= 2 \end{aligned} \qquad \textbf{(4a)}$$

may be represented by the matrix

EXAMPLE Write the augmented matrix corresponding to each equivalent system given in (4a) through (4h).

Solution The required sequence of augmented matrices follows.

Equivalent System | **Augmented Matrix**

a. $$\begin{aligned} 2x + 4y + 6z &= 22 \\ 3x + 8y + 5z &= 27 \\ -x + y + 2z &= 2 \end{aligned} \qquad \left[\begin{array}{rrr|r} 2 & 4 & 6 & 22 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{array}\right] \qquad \textbf{(7a)}$$

b. $$\begin{aligned} x + 2y + 3z &= 11 \\ 3x + 8y + 5z &= 27 \\ -x + y + 2z &= 2 \end{aligned} \qquad \left[\begin{array}{rrr|r} 1 & 2 & 3 & 11 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{array}\right] \qquad \textbf{(7b)}$$

c. $$\begin{aligned} x + 2y + 3z &= 11 \\ 2y - 4z &= -6 \\ -x + y + 2z &= 2 \end{aligned} \qquad \left[\begin{array}{rrr|r} 1 & 2 & 3 & 11 \\ 0 & 2 & -4 & -6 \\ -1 & 1 & 2 & 2 \end{array}\right] \qquad \textbf{(7c)}$$

d. $$\begin{aligned} x + 2y + 3z &= 11 \\ 2y - 4z &= -6 \\ 3y + 5z &= 13 \end{aligned} \qquad \left[\begin{array}{rrr|r} 1 & 2 & 3 & 11 \\ 0 & 2 & -4 & -6 \\ 0 & 3 & 5 & 13 \end{array}\right] \qquad \textbf{(7d)}$$

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e. $x + 2y + 3z = 11$, $y - 2z = -3$, $3y + 5z = 13$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 11 \\ 0 & 1 & -2 & -3 \\ 0 & 3 & 5 & 13 \end{array}\right] \quad \text{(7e)}$$

f. $x + 7z = 17$, $y - 2z = -3$, $3y + 5z = 13$

$$\left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -2 & -3 \\ 0 & 3 & 5 & 13 \end{array}\right] \quad \text{(7f)}$$

g. $x + 7z = 17$, $y - 2z = -3$, $11z = 22$

$$\left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 11 & 22 \end{array}\right] \quad \text{(7g)}$$

h. $x = 3$, $y = 1$, $z = 2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \quad \text{(7h)}$$

The augmented matrix in (7h) is an example of a matrix in row-reduced form. In general, an augmented matrix with *m* rows and *n* columns (called an *m x n* matrix) is in **row-reduced form** if it satisfies the following conditions.

Row-Reduced Form of a Matrix

1-Each row consisting entirely of zeros lies below all rows having nonzero entries

.2-The first nonzero entry in each (nonzero) row is 1 (called a leading 1)

3-In any two successive (nonzero) rows, the leading 1 in the lower row lies to the right of the leading 1 in the upper row

4-If a column in the coefficient matrix contains a leading 1, then the other entries in that column are zeros.

EXAMPLE Solve the system of linear equations given by

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$$\begin{aligned} 2y + 3z &= 7 \\ 3x + 6y - 12z &= -3 \\ 5x - 2y + 2z &= -7 \end{aligned}$$

Solution Using the Gauss–Jordan elimination method, we obtain the following sequence of equivalent augmented matrices:

$$\left[\begin{array}{ccc|c} 0 & 2 & 3 & 7 \\ 3 & 6 & -12 & -3 \\ 5 & -2 & 2 & -7 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 3 & 6 & -12 & -3 \\ 0 & 2 & 3 & 7 \\ 5 & -2 & 2 & -7 \end{array}\right]$$

$$\xrightarrow{\frac{1}{3}R_1} \left[\begin{array}{ccc|c} 1 & 2 & -4 & -1 \\ 0 & 2 & 3 & 7 \\ 5 & -2 & 2 & -7 \end{array}\right]$$

$$\xrightarrow{R_3 - 5R_1} \left[\begin{array}{ccc|c} 1 & 2 & -4 & -1 \\ 0 & 2 & 3 & 7 \\ 0 & -12 & 22 & -2 \end{array}\right]$$

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$$\xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 2 & -4 & -1 \\ 0 & 1 & \frac{3}{2} & \frac{7}{2} \\ 0 & -12 & 22 & -2 \end{array}\right]$$

$$\xrightarrow[R_3 + 12R_2]{R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -7 & -8 \\ 0 & 1 & \frac{3}{2} & \frac{7}{2} \\ 0 & 0 & 40 & 40 \end{array}\right]$$

$$\xrightarrow{\frac{1}{40}R_3} \left[\begin{array}{ccc|c} 1 & 0 & -7 & -8 \\ 0 & 1 & \frac{3}{2} & \frac{7}{2} \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[R_2 - \frac{3}{2}R_3]{R_1 + 7R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

The solution to the system is given by $x = -1$, $y = 2$, and $z = 1$; this may be verified by substitution into the system.

Problem 6. A d.c. circuit comprises three closed loops. Applying Kirchhoff's laws to the closed loops gives the following equations for current flow in milliamperes:

$$2I_1 + 3I_2 - 4I_3 = 26$$

$$I_1 - 5I_2 - 3I_3 = -87$$

$$-7I_1 + 2I_2 + 6I_3 = 12$$

Use determinants to solve for I_1, I_2 and I_3.

(i) Writing the equations in the $a_1x + b_1y + c_1z + d_1 = 0$ form gives:

$$2I_1 + 3I_2 - 4I_3 - 26 = 0$$

$$I_1 - 5I_2 - 3I_3 + 87 = 0$$

$$-7I_1 + 2I_2 + 6I_3 - 12 = 0$$

(ii) The solution is given by

$$\frac{I_1}{D_{I_1}} = \frac{-I_2}{D_{I_2}} = \frac{I_3}{D_{I_3}} = \frac{-1}{D},$$

where D_{I_1} is the determinant of coefficients

$$D_{I_1} = \begin{vmatrix} 3 & -4 & -26 \\ -5 & -3 & 87 \\ 2 & 6 & -12 \end{vmatrix}$$

$$= (3)\begin{vmatrix} -3 & 87 \\ 6 & -12 \end{vmatrix} - (-4)\begin{vmatrix} -5 & 87 \\ 2 & -12 \end{vmatrix}$$

$$+ (-26)\begin{vmatrix} -5 & -3 \\ 2 & 6 \end{vmatrix}$$

$$= 3(-486) + 4(-114) - 26(-24)$$

$$= \mathbf{-1290}$$

$$D_{I_2} = \begin{vmatrix} 2 & -4 & -26 \\ 1 & -3 & 87 \\ -7 & 6 & -12 \end{vmatrix}$$

$$= (2)(36 - 522) - (-4)(-12 + 609)$$

$$+ (-26)(6 - 21)$$

$$= -972 + 2388 + 390$$

$$= \mathbf{1806}$$

$$D_{I_3} = \begin{vmatrix} 2 & 3 & -26 \\ 1 & -5 & 87 \\ -7 & 2 & -12 \end{vmatrix}$$

$$= (2)(60 - 174) -$$
$$(3)(-12 + 609) + (-26)(2 - 35)$$
$$= -228 - 1791 + 858 = \mathbf{-1161}$$

and $$D = \begin{vmatrix} 2 & 3 & -4 \\ 1 & -5 & -3 \\ -7 & 2 & 6 \end{vmatrix}$$

$$= (2)(-30 + 6) - (3)(6 - 21)$$
$$+ (-4)(2 - 35)$$
$$= -48 + 45 + 132 = \mathbf{129}$$

Thus

$$\frac{I_1}{-1290} = \frac{-I_2}{1806} = \frac{I_3}{-1161} = \frac{-1}{129}$$

giving

$$I_1 = \frac{-1290}{-129} = 10\,\text{mA},$$

$$I_2 = \frac{1806}{129} = 14\,\text{mA}$$

and $$I_3 = \frac{1161}{129} = 9\,\text{mA}$$

Example : Solve the following systems by (Gauss- Elimination) method:

$$x_1 + x_2 + x_3 = 6$$
$$2x_1 + 4x_2 + 2x_3 = 16$$
$$-x_1 + 5x_2 - 4x_3 = -3$$

It is convenient to rewrite this in the form $Ax = b$ where $A \in R3\times3$ and x and b are column vectors of size 3; thus,

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & 5 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 16 \\ -3 \end{pmatrix}$$

We begin by adding the first row, multiplied by -2, to the second row, and adding the first row to the third row, giving the new system

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$$\begin{pmatrix} 1 & 1 & 1 \\ \mathit{0} & 2 & 0 \\ \mathit{0} & 6 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 3 \end{pmatrix}$$

The newly created 0 entries in the first column have been typeset in italics. Now adding the new second row, multiplied by −3, to the third row, we find

$$\begin{pmatrix} 1 & 1 & 1 \\ \mathit{0} & 2 & 0 \\ \mathit{0} & \mathit{0} & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ -9 \end{pmatrix}$$

$X_3=\frac{-9}{-3}=3$,,,,$X_2=\frac{4}{2}=2$,,,,, from $x_1+x_2+x_3=6$

$X_1+2+3=6$ there for x1=1

E X A M P L E // *Least Squares Regression Analysis*

The world population in billions for the years between 1965 and 2000, is shown in Table?

Year	1965	1970	1975	1980	1985	1990	1995	2000
Population	3.36	3.72	4.10	4.46	4.86	5.28	5.69	6.08

Find the second-degree least squares regression polynomial for these data and use the resulting model to predict the world population for 2005 and 2010.

Begin by letting X=-4 represent 1965, X=-3 represent 1970, and so on. So the collection of points is given by[(-4, 3.36), (-3, 3.72), (-2, 4.10), (-1, 4.46), (0, 4.86),(1, 5.28), (2 , 5.69),(3 , 6.08)

which yields

$$n = 8, \quad \sum_{i=1}^{8} x_i = -4, \quad \sum_{i=1}^{8} x_i^2 = 44, \quad \sum_{i=1}^{8} x_i^3 = -64,$$

$$\sum_{i=1}^{8} x_i^4 = 452, \quad \sum_{i=1}^{8} y_i = 37.55, \quad \sum_{i=1}^{8} x_i y_i = -2.36, \quad \sum_{i=1}^{8} x_i^2 y_i = 190.86.$$

So the system of linear equations giving the coefficients of the quadratic model

$y = a_2x^2 + a_1x + a_0$ is

$$\begin{aligned} 8a_0 - 4a_1 + 44a_2 &= 37.55 \\ -4a_0 + 44a_1 - 64a_2 &= -2.36 \\ 44a_0 - 64a_1 + 452a_2 &= 190.86. \end{aligned}$$

Gaussian elimination with pivoting on the matrix

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$$\begin{bmatrix} 8 & -4 & 44 & 37.55 \\ -4 & 44 & -64 & -2.36 \\ 44 & -64 & 452 & 190.86 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1.4545 & 10.2727 & 4.3377 \\ 0 & 1 & -0.6000 & 0.3926 \\ 0 & 0 & 1 & 0.0045 \end{bmatrix}$$

$a_2 = 0.0045$, $a_1 = 0.3953$, $a_0 = 4.8667$,

Systems of Linear Equations and the Gauss-Jordan Method

In this section, we learn to solve systems of linear equations using a process called the Gauss-Jordan method. The process begins by first expressing the system as a matrix, and then reducing it to an equivalent system by simple row operations. The process is continued until the solution is obvious from the matrix. The matrix that represents the system is called the augmented matrix, and the arithmetic manipulation that is used to move from a system to a reduced equivalent system is called a row operation

Example/ Write the following system as an augmented matrix.

$$\begin{aligned} 2x + 3y - 4z &= 5 \\ 3x + 4y - 5z &= -6 \\ 4x + 5y - 6z &= 7 \end{aligned}$$

Solution/ We express the above information in matrix form

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$$\left[\begin{array}{ccc|c} 2 & 3 & -4 & 5 \\ 3 & 4 & -5 & -6 \\ 4 & 5 & -6 & 7 \end{array}\right]$$