Week-10- Newton Raphson's method, solution of algebraic and transcendental equations

DAWAH-2025-2026

NEWTON-RAPHSON METHOD
OR SUCCESSIVE SUBSTITUTION METHOD

By this method, we get closer approximation of the root of an equation if we already know its approximate root.

$$a_1 = a_0 - \frac{f(a_0)}{f'(a_0)}$$

$$a_2 = a_1 - \frac{f(a_1)}{f'(a_1)} \quad , \quad a_3 = a_2 - \frac{f(a_2)}{f'(a_2)}$$,and so on

Example. Starting with $x_0 = 3$, *find a root of* $x^3 - 3x - 5 = 0$, correct to three decimal *places. Use* **Newton-Raphson** *method.*

SOLUTION

$$f(x) = x^3 - 3x - 5 = 0, \quad f'(x) = 3x^2 - 3$$

$$f(3) = 27 - 9 - 5 = 13, \quad f'(3) = 27 - 3 = 24$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3 - \frac{f(3)}{f'(3)} = 3 - \frac{13}{24} = 3 - 0.5417 = 2.4583$$

$$x_2 = 2.4583 - \frac{f(2.4583)}{f'(2.4583)} = 2.4583 - \frac{2.4812}{15.1297} = 2.4583 - 0.1640 = 2.2943$$

$$x_3 = 2.2943 - \frac{f(2.2943)}{f'(2.2943)} = 2.2943 - \frac{0.1939}{12.7914} = 2.2791$$

$$f(2.2791) = 0.0010$$

the required root $= 2.2791$ **Ans.**

EXAMPLE/ Find the smallest positive root of $x - e^{-x} = 0$, using Newton –Raphson method?

SOLUTION

Here $f(x) = x - e^{-x}$

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f 1 (x) = 1 + e^{-x}

f (0) = – 1 and f (1) = 0.63212.

The smallest positive root of f (x) = 0 lies in between 0 and 1.

Let x_0 = 1… The Newton – Raphson method formula is,

$$x_{n+1} = x_n - \frac{x_n - e^{-x}n}{1+e^{-x}n}, \ n = 0, 1, 2, .$$

f (0) = f (1) = 0.63212

$$f'(0) = f'(1) = 1.3679$$

$$x_1 = x_0 - \frac{x_0 - e^{-x_0}}{1+e^{-x_0}} = 1 - \frac{0.63212}{1.3679} = 0.5379.$$

f (0.5379) = – 0.0461

f ' (0.5379) = 1.584.

$$x_2 = 0.5379 + \frac{0.0461}{1.584} = 0.567$$

Similarly, $x3$ = 0.56714

x = 0.567 can be taken as the smallest positive root of $x - e^{-x}$ = 0.,

Example . Find the real root of the following equation, correct three decimal places using Newton-Raphson method.

SOLUTION :

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$$x^3 - 2x - 5 = 0$$

$$x^3 - 2x - 5 = 0$$

$$f(x) = x^3 - 2x - 5$$

$$f(2) = 8 - 4 - 5 = -1$$

$$f(2.5) = (2.5)^3 - 2(2.5) - 5 = +5.625$$

Since $f(2)$ and $f(2.5)$ are, of opposite sign, the root of (1) lies between 2 and 2.5 ; $f(2)$ is near to zero than $f(2.5)$, so 2 is better appropriate root than 2.5.

$$f'(x) = 3x^2 - 2 \qquad f'(2) = 12 - 2 = 10$$

Let 2 be an approximate root of (1). By Newton-Raphson method

$$a_1 = a - \frac{f(a)}{f'(a)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{-1}{10} = 2.1$$

$$f(2.1) = (2.1)^3 - 2(2.1) - 5 = 9.261 - 4.2 - 5 = 0.061$$

$$f'(2.1) = 3(2.1)^2 - 2 = 11.23$$

$$a_2 = 2.1 - \frac{f(2.1)}{f'(2.1)} = 2.1 - \frac{0.061}{11.23} = 2.1 - 0.00543 = 2.0945$$

$$f(2.09457) = (2.09457)^3 - 2(2.09457) - 5$$

$$= 9.1893 - 4.18914 - 5 = 0.00016$$

$$f'(2.09457) = 3(2.09457)^2 - 2 = 13.16167 - 2 = 11.16167$$

$$a_3 = 2.09457 - \frac{f(2.09457)}{f'(2.09457)} = 2.09457 - \frac{0.00016}{11.16167} = 2.09457 + 0.000014 = 2.09456$$

As $a_3 = a_2$ correct upto four places of decimal, hence the root of (1) is 2.0945.

Ans.

Example . Find an interval of length 1, in which the root of
f (x) = 3 x^3 - 4 x^2 - 4 x - 7 = 0 lies. Take the middle point of this interval as th
starting approximation
and iterate two times, using the **Newton- Raphson** method.

Solution.

$$f(x) = 3x^3 - 4x^2 - 4x - 7 = 0$$
$$f(2) = 24 - 16 - 8 - 7 = -7$$
$$f(3) = 81 - 36 - 12 - 7 = +26$$

The root of (1) lies between 2 and 3 as $f(2)$ and $f(3)$ are of opposite sign. The middle point of this interval is 2.5.

$$f(2.5) = 46.875 - 25 - 10 - 7 = 4.875$$
$$f'(x) = 9x^2 - 8x - 4$$

$$f'(2.5) = 56.25 - 20 - 4 = 32.25$$

Newton-Raphson

$$a_1 = a - \frac{f(a)}{f'(a)} \quad \text{and}$$

$$a_1 = 2.5 - \frac{f(2.5)}{f'(2.5)} = 2.5 - \frac{4.875}{32.25} = 2.5 - 0.15 = 2.35$$

$$f(2.35) = 38.93 - 22.09 - 9.4 - 7 = 0.44$$
$$f'(2.35) = 49.7 - 18.8 - 4 = 26.9$$

$$a_2 = 2.35 - \frac{f(2.35)}{f'(2.35)} = 2.35 - \frac{0.44}{26.9} = 2.35 - 0.016 = 2.334$$

$$f(2.334) = 38.14 - 21.79 - 9.34 - 7 = 0.01 \text{ which is nearly zero.}$$

the required root is 2.334 AUC...DAWAH **Ans.**

Solution of Algebraic and Transcendental Equations

Introduction A polynomial equation of the form

$$f(x) = p_n(x) = a_0 x^{n-1} + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n = 0$$

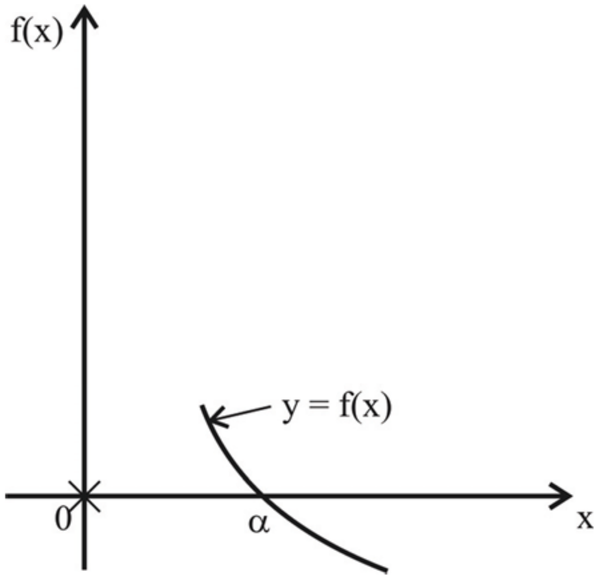
is called an Algebraic equation. For example,

$$x^4 - 4x^2 + 5 = 0, \; 4x^2 - 5x + 7 = 0; \; 2x^3 - 5x^2 + 7x + 5 = 0$$

are algebraic equations. An equation which contains polynomials, trigonometric functions, logarithmic functions, exponential functions etc., is called a Transcendental equation. For example,

$$\tan x - e^{x} = 0; \;\; \sin x - xe^{2x} = 0; \qquad x\,e^{x} = \cos x$$

are transcendental equations. Finding the roots or zeros of an equation of the form f(x) = 0 is an important problem in science and engineering. We assume that f (x) is continuous in the required interval. A root of an equation f (x) = 0 is the value of x, say x = a for which f (a) = 0. Geometrically, a root of an equation f (x) = 0 is the value of x at which the graph of the equation y = f (x) intersects the x – axis (see Fig.)



Geometrical Interpretation of a root of $f(x) = 0$

A number a is a simple root of f (x) = 0

$f(x) = 0$; if $f(\alpha) = 0$ and $f'(\alpha) \neq 0$

Then, we can write

f (x) as,

$$f(x) = (x - \alpha)\, g(x),\ g(\alpha) \neq 0$$

A number a is a multiple root of multiplicity m of f (x) = 0,

$$\text{if } f(\alpha) = f^{1}(\alpha) = \ldots = f^{(m-1)}(\alpha) = 0$$

$$f^{m}(\alpha) = 0.$$

Then, f (x) can be writhen as,

$$f(x) = (x - \alpha)^{m}\, g(x),\ g(\alpha) \neq 0$$

A polynomial equation of degree n will have exactly n roots, real or complex, simple or multiple. A transcendental equation may have one root or no root or infinite number of roots depending on the form of f (x). The methods of finding the roots of

f (x) = 0 are classified as,

1. Direct Methods
2. Numerical Methods.

Direct methods give the exact values of all the roots in a finite number of steps. Numerical methods are based on the idea of successive approximations. In these methods, we start with one or two initial approximations to the root and obtain a sequence of approximations $x_0, x_1, \ldots x_k$ which in the limit as $k \to \infty$ converge to the exact root $x = a$.

There are no direct methods for solving higher degree algebraic equations or transcendental equations. Such equations can be solved by Numerical methods. In these methods, we first find an interval in which the root lies. If a and b are two numbers such that $f(a)$ and $f(b)$ have opposite signs, then a root of $f(x) = 0$ lies in between *a* and *b*. We take *a* or *b* or any valve in between *a* or *b* as first approximation *x*1. This is further improved by numerical methods. Here we discuss few important Numerical methods to find a root of $f(x) = 0$.