Week-1-Fundamental of Laplace transform (L.T)

DAWAH-2025-2026

1-1 INTRODUCTION:

The Laplace transform is a mathematical tool that converts a function of a real variable (usually time, t) <u>into</u> a function of a complex variable (usually s). This transformation is achieved through an integral, effectively shifting the analysis from the time domain to the "s-domain" or "frequency domain." The fundamental idea is that operations like differentiation and integration in the time domain become simpler algebraic manipulations in the s-domain, making it useful for solving differential equations and analysing systems, particularly in engineering and physics [[Laplace Transform used for analog signals]]

LAPLACE TRANSFORM FORMULA

$$F(s) = \int_{0}^{+\infty} f(t) \cdot e^{-s \cdot t} \cdot dt$$

Where:

- **L{f(t)} represents the Laplace transform of f(t).**
- **f(t) is the original function in the time domain.**
- **s is the complex frequency variable.**
- **e^{-st} is the kernel of the transform.**
- **$\int_0^\infty$ indicates integration from 0 to infinity**

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S.no	$f(t)$	$\mathcal{L}\{f(t)\}$	S.no	$f(t)$	$\mathcal{L}\{f(t)\}$
1	1	$\frac{1}{s}$	11	$e^{at}\sinh bt$	$\frac{b}{(s-a)^2-b^2}$
2	e^{at}	$\frac{1}{s-a}$	12	$e^{at}\cosh bt$	$\frac{s-a}{(s-a)^2-b^2}$
3	t^n	$\frac{n!}{s^{n+1}}$	13	$t\cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
4	$\sin at$	$\frac{a}{s^2+a^2}$	14	$t\sin at$	$\frac{2as}{(s^2+a^2)^2}$
5	$\cos at$	$\frac{s}{s^2+a^2}$	15	$f'(t)$	$sF(s)-f(0)$
6	$\sinh at$	$\frac{a}{s^2-a^2}$	16	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
7	$\cosh at$	$\frac{s}{s^2-a^2}$	17	$\int_0^t f(u)du$	$\frac{1}{s}F(s)$
8	$e^{at}t^n$	$\frac{n!}{(s-a)^{n+1}}$	18	$t^n f(t)$ Where $n=1,2,3,..$	$(-1)^n\frac{d^n}{ds^n}\{F(s)\}$
9	$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$	19	$\frac{1}{t}\{f(t)\}$	$\int_s^\infty F(s)ds$
10	$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$	20	$e^{at}f(t)$	$F(s-a)$

1. $\mathrm{L}\,(1) = \frac{1}{s}$

Proof. $\mathrm{L}\,(1) = \int_0^\infty 1\,.\,e^{-st}\,dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = -\frac{1}{s}\left[\frac{1}{e^{st}}\right]_0^\infty = -\frac{1}{s}[0-1] = \frac{1}{s}$

Hence $\mathrm{L}\,(1) = \frac{1}{s}$

2. $\mathrm{L}\,(t^n) = \frac{n\,!}{s^{n+1}}$, where n and s are positive.

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Ex. : f(t)=t

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty t\,.\,e^{-st}\,dt$$

By using part method ($\int udv = uv - \int vdu$),

Let $u = t$ $\qquad dv = e^{-st}\,dt$

$du = dt$ $\qquad v = \frac{1}{-s}\,e^{-st}$

$$= t.\frac{1}{-s}[e^{-st}]_0^\infty - \int_0^\infty \frac{1}{-s}e^{-st}\,dt = \left[\frac{-t}{s}.e^{-st}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt$$

$$= \frac{-t}{s}[e^{-st}]_0^\infty - \left[\frac{1}{s^2}.e^{-st}\right]_0^\infty$$

So, $\mathcal{L}[t] = \frac{1}{s^2}$

Derivetive	Integral
(+) t	e^{-st}
(-) 1	$\frac{e^{-st}}{-s}$
(+) 0	$\frac{e^{-st}}{s^2}$

0 ≤ (variable t) < ∞

3. $\quad L(e^{at}) = \frac{1}{s-a}$ $\quad$ where $s > a$

Proof. $\quad L(e^{at}) = \int_0^\infty e^{-st}\,.\,e^{at}\,dt = \int_0^\infty e^{-st+at}\,.\,dt$

$$= \int_0^\infty e^{(-s+a)t}\,.\,dt = \int_0^\infty e^{-(s-a)t}\,.\,dt = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = -\frac{1}{s-a}\left[\frac{1}{e^{(s-a)t}}\right]_0^\infty$$

$$= \frac{-1}{(s-a)}(0-1) = \frac{1}{s-a}$$

Proved

4. $L(\cosh at) = \dfrac{s}{s^2 - a^2}$

Proof. $L(\cosh at) = L\left[\dfrac{e^{at} + e^{-at}}{2}\right]$ $\left(\because \cosh at = \dfrac{e^{at} + e^{-at}}{2}\right)$

$$= \frac{1}{2} L(e^{at}) + \frac{1}{2} L(e^{-at})$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] \qquad \left[L(e^{at}) = \frac{1}{s-a}\right]$$

$$= \frac{1}{2}\left[\frac{s+a+s-a}{s^2-a^2}\right] = \frac{s}{s^2-a^2}$$

Proved.

5. $L(\sinh at) = \dfrac{a}{s^2 - a^2}$

Proof. $L(\sinh at) = L\left[\dfrac{1}{2}(e^{at} - e^{-at})\right]$

$$= \frac{1}{2}[L(e^{at}) - L(e^{-at})] = \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{s+a-s+a}{s^2-a^2}\right]$$

$$= \frac{a}{s^2-a^2}$$

Proved.

6. $L(\sin at) = \dfrac{a}{s^2 + a^2}$

Proof. $L(\sin at) = L\left[\dfrac{e^{iat} - e^{-iat}}{2i}\right]$ $\left[\because \sin at = \dfrac{e^{iat} - e^{-iat}}{2i}\right]$

$$= \frac{1}{2i}[L(e^{iat} - e^{-iat})] = \frac{1}{2i}[L(e^{iat}) - L(e^{-iat})]$$

$$= \frac{1}{2i}\left[\frac{1}{s-ia} - \frac{1}{s+ia}\right] = \frac{1}{2i}\frac{s+ia-s+ia}{s^2+a^2}$$

$$= \frac{1}{2i}\frac{2ia}{s^2+a^2} = \frac{a}{s^2+a^2}$$

Proved

7. $\quad L(\cos at) = \dfrac{s}{s^2+a^2}$

Proof.

$$L(\cos at) = L\left(\frac{e^{iat}+e^{-iat}}{2}\right) \qquad \left[\because \cos at = \frac{e^{iat}+e^{-iat}}{2}\right]$$

$$= \frac{1}{2}[L(e^{iat}+e^{-iat})] = \frac{1}{2}[L(e^{iat}) + L(e^{-iat})]$$

$$= \frac{1}{2}\left[\frac{1}{s-ia}+\frac{1}{s+ia}\right] = \frac{1}{2}\,\frac{s+ia+s-ia}{s^2+a^2}$$

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$$= \frac{s}{s^2+a^2}$$

Proved