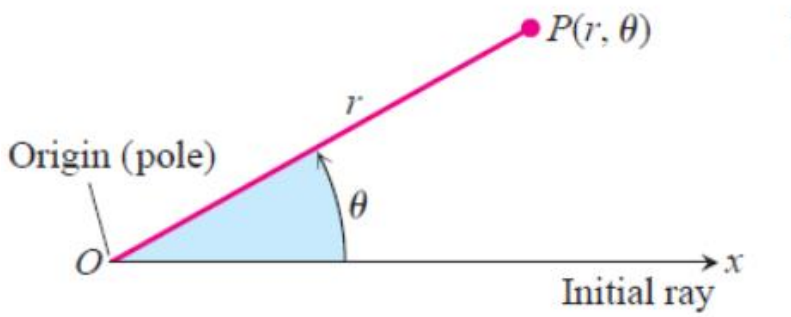
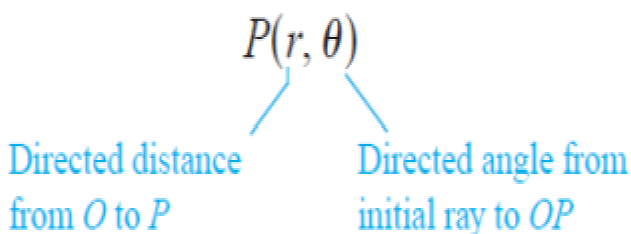
# Polar Coordinates

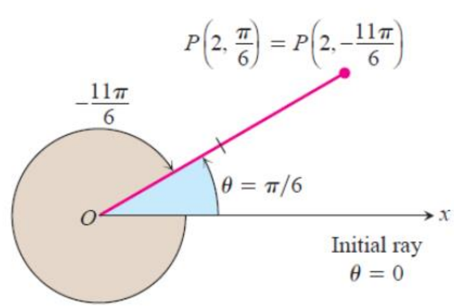
## Introduction

We first fix an origin O (called the pole) and an initial ray from O. Then each point P can be located by assigning to it a polar coordinate pair in which $(r, \theta)$ gives the directed distance from O to P and gives the directed angle from the initial ray to ray OP.

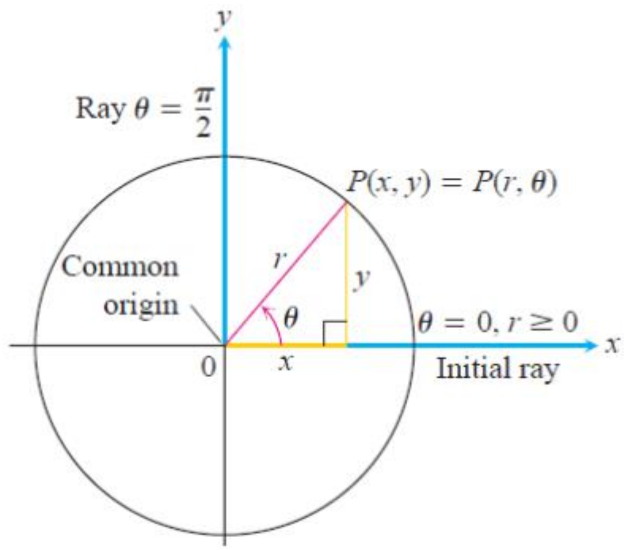



θ is positive when measured counterclockwise. In addition, θ is negative when measured clockwise.



## Relating Polar and Cartesian Coordinates




## Equations Relating Polar and Cartesian Coordinates

$$\boldsymbol{x = r\cos\theta, \qquad y = r\sin\theta, \qquad x^2 + y^2 = r^2}$$

(Converting Cartesian to Polar)

**Example :** Find a polar equation for the circle $\boldsymbol{x^2 + (y-3)^2 = 9}$

**Solution:**

$$x^2 + (y-3)^2 = 9$$

$$x^2 + y^2 - 6y + 9 = 9$$

$$r^2 - 6r\sin\theta = 0$$

$$r(r - 6\sin\theta) = 0$$

$$r = 0 \quad \text{or} \quad r = 6\sin\theta$$

**( Converting Polar to Cartesian )**

**Example: Replace the following polar equations by equivalent Cartesian equations, and identify their graphs**

$$(\boldsymbol{a})\ \ \boldsymbol{rcos\theta = -4}$$

$$(\boldsymbol{b})\ \ \boldsymbol{r^2 = 4rcos\theta}$$

$$(\boldsymbol{c})\ \ \boldsymbol{r = \frac{4}{2cos\theta - sin\theta}}$$

**Solution:** we use the substitutions $rcos\theta = x,\ \ rsin\theta = y,\ \ r^2 = x^2 + y^2$

$(\boldsymbol{a})\ \ \boldsymbol{rcos\theta = -4}$

The Cartesian equation: $\boldsymbol{rcos\theta = -4}$

$$x = -4$$

The graph: Vertical line through $x = -4$ on the $x - axis$

$(\boldsymbol{b})\ \ \boldsymbol{r^2 = 4r\cos\theta}$

The Cartesian equation: $r^2 = 4r\cos\theta$

$$x^2 + y^2 = 4x$$

$$x^2 - 4x + y^2 = 0$$

$$x^2 - 4x + 4 + y^2 = 4$$

$$(x-2)^2 + y^2 = 4$$

The graph: Equation of a circle whose center is $(2,0)$ and radius is 2.

$(\boldsymbol{c})\ \ \boldsymbol{r = \frac{4}{2cos\theta - sin\theta}}$

The Cartesian equation: $r(2\cos\theta - \sin\theta = 4$

$$2r\cos\theta - r\sin\theta = 4$$

$$2x - y = 4$$

$$y = 2x - 4$$

The graph: Equation of a straight line

## 2.3 Convert from Cartesian Coordinates to Polar Coordinates via Points

If $P$ is a point with Cartesian Coordinates $(x, y)$ the polar coordinates $(r, \theta)$ of $P$ is a given by:

$$r = \sqrt{x^2 + y^2} \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

**Example: Find the polar coordinates of the points with the following Cartesian Coordinates:**

$\boldsymbol{a})\ (\mathbf{2}, \mathbf{2}) \quad \boldsymbol{b})\ (-\mathbf{1}, \mathbf{1}) \quad$ **c)** $(\mathbf{1}, -\mathbf{1})$

**Solution:**

$\boldsymbol{a}) \quad (\boldsymbol{x}, \boldsymbol{y}) = (\mathbf{2}, \mathbf{2})$

$$r = \sqrt{x^2 + y^2} \qquad \theta = tan^{-1}\left(\frac{y}{x}\right)$$

$$r = \sqrt{2^2 + 2^2} = 2\sqrt{2} \qquad \theta = \tan^{-1}\left(\frac{2}{2}\right) = \frac{\pi}{4}$$

$\therefore x > 0, y > 0 \quad \Longrightarrow the\ first\ quadrant$

$$\therefore (r, \theta) = (2\sqrt{2}, \frac{\pi}{4})$$

$\boldsymbol{b}) \quad (\boldsymbol{x}, \boldsymbol{y}) = (-\mathbf{1}, \mathbf{1})$

$$r = \sqrt{x^2 + y^2}$$

$$r = \sqrt{-1^2 + 1^2} = \sqrt{2}$$

$$\tan\theta = \frac{y}{x}$$

$$\theta = \tan^{-1}\left(\frac{1}{-1}\right) = -\frac{\pi}{4}$$

$\therefore x < 0, y > 0 \quad \Longrightarrow the\ second\ quadrant$

$$\Longrightarrow \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

$$\therefore (r, \theta) = (\sqrt{2}, \frac{3\pi}{4})$$

**c)** $(\boldsymbol{x}, \boldsymbol{y}) = (\mathbf{1}, -\mathbf{1})$ **H.W**