Torsion

Torsional Deformation of a Circular Shaft

Torque is a moment that tends to twist a member about its longitudinal axis. Its effect is of primary concern in the design of drive shafts used in vehicles and machinery. We can illustrate physically what happens when a torque is applied to a circular shaft by considering the shaft to be made of a highly deformable material such as rubber, Fig. 5–1*a*. When the torque is applied, the circles and longitudinal grid lines originally marked on the shaft tend to distort into the pattern shown in Fig. 5–1*b*. Note that twisting causes the circles to *remain circles*, and each longitudinal grid line deforms into a helix that intersects the circles at equal angles. Also, the cross sections at the ends of the shaft will remain *flat* — that is, they do not warp or bulge in or out — and radial lines *remain straight* during the deformation, Fig. 5–1*b*. From these observations we can assume that if the angle of twist is *small*, the *length of the shaft* and its *radius* will *remain unchanged*.

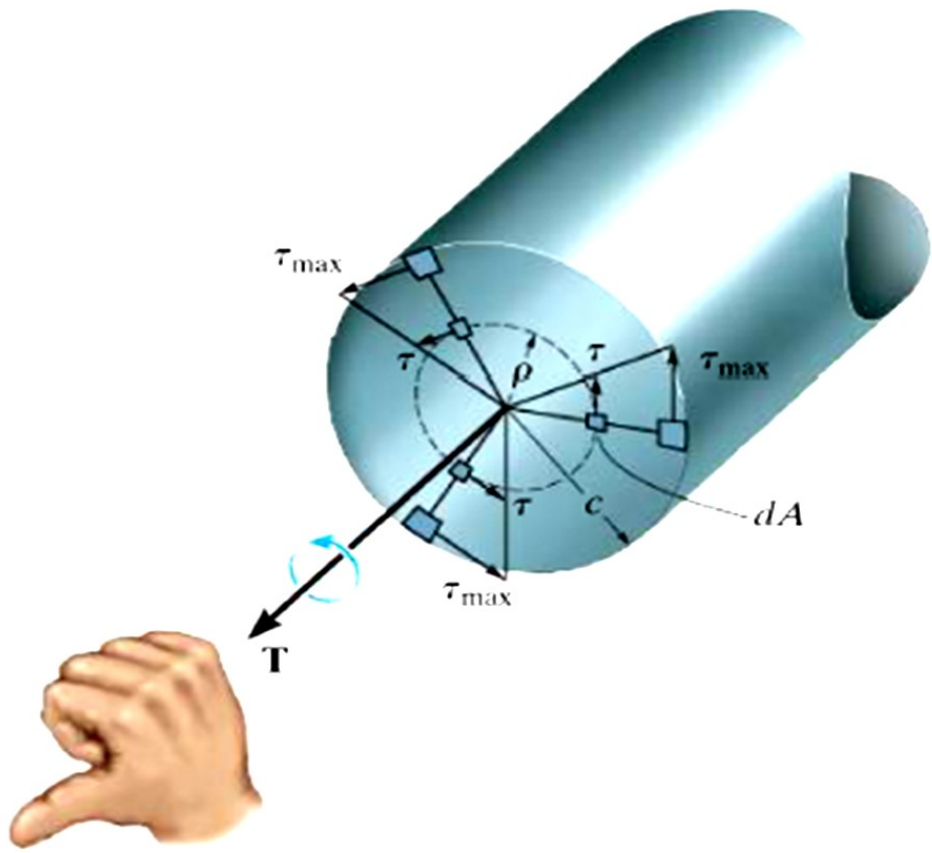
The Torsion Formula

When an external torque is applied to a shaft, it creates a corresponding internal torque within the shaft. In this section, we will develop an equation that relates this internal torque to the shear stress distribution on the cross section of a circular shaft or tube.

If the material is linear-elastic, then Hooke's law applies, $\tau = G\gamma$, and consequently a ***linear variation in shear strain***, as noted in the previous section, leads to a corresponding ***linear variation in shear stress*** along any radial line on the cross section. Hence, τ will vary from zero at the shaft's longitudinal axis to a maximum value, τ_{max}, at its outer surface. This variation is shown in Fig. 5–5 on the front faces of a selected number of elements, located at an intermediate radial position ρ and at the outer radius c. Due to the proportionality of triangles, we can write

$$\tau = \left(\frac{\rho}{c}\right)\tau_{max} \qquad (5\text{–}3)$$

This equation expresses the shear-stress distribution over the cross section in terms of the radial position ρ of the element. Using it, we can now apply the condition that requires the torque produced by the stress distribution over the entire cross section to be equivalent to the resultant internal torque T at the section, which holds the shaft in equilibrium, Fig. 5–5.



Shear stress varies linearly along each radial line of the cross section.

Fig. 5–5

Specifically, each element of area dA, located at ρ, is subjected to a force of $dF = \tau\, dA$. The torque produced by this force is $dT = \rho(\tau\, dA)$. We therefore have for the entire cross section

$$T = \int_A \rho(\tau\, dA) = \int_A \rho\left(\frac{\rho}{c}\right)\tau_{max}\, dA \tag{5–4}$$

Since τ_{max}/c is constant,

$$T = \frac{\tau_{max}}{c}\int_A \rho^2\, dA \tag{5–5}$$

The integral depends only on the geometry of the shaft. It represents the ***polar moment of inertia*** of the shaft's cross-sectional area about the shaft's longitudinal axis. We will symbolize its value as J, and therefore the above equation can be rearranged and written in a more compact form, namely,

$$\boxed{\tau_{max} = \frac{Tc}{J}} \tag{5–6}$$

Here

τ_{max} = the maximum shear stress in the shaft, which occurs at the outer surface

T = the resultant *internal torque* acting at the cross section. Its value is determined from the method of sections and the equation of moment equilibrium applied about the shaft's longitudinal axis

J = the polar moment of inertia of the cross-sectional area

c = the outer radius of the shaft

Combining Eqs. 5–3 and 5–6, the shear stress at the intermediate distance ρ can be determined from

$$\boxed{\tau = \frac{T\rho}{J}} \qquad (5\text{–}7)$$

Either of the above two equations is often referred to as the ***torsion formula***. Recall that it is used only if the shaft is circular and the material is homogeneous and behaves in a linear elastic manner, since the derivation is based on Hooke's law.

Solid Shaft. If the shaft has a solid circular cross section, the polar moment of inertia J can be determined using an area element in the form of a *differential ring* or annulus having a thickness $d\rho$ and circumference $2\pi\rho$, Fig. 5–6. For this ring, $dA = 2\pi\rho\, d\rho$, and so

$$J = \int_A \rho^2\, dA = \int_0^c \rho^2 (2\pi\rho\, d\rho) = 2\pi \int_0^c \rho^3\, d\rho = 2\pi\left(\frac{1}{4}\right)\rho^4\Big|_0^c$$

$$\boxed{J = \frac{\pi}{2} c^4} \qquad (5\text{–}8)$$

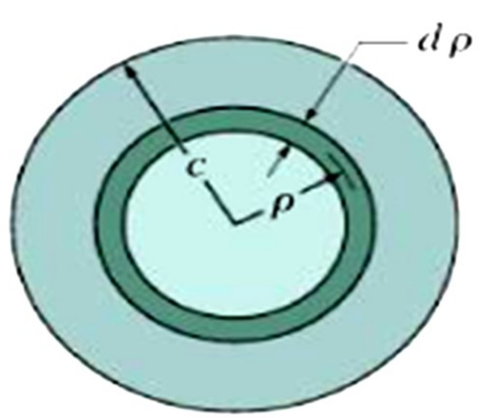
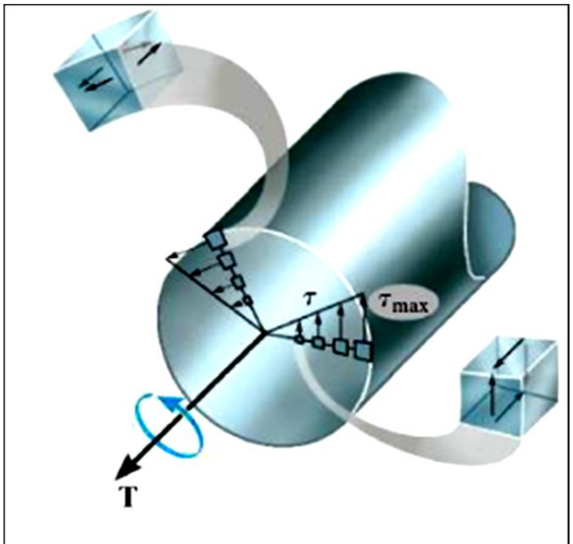


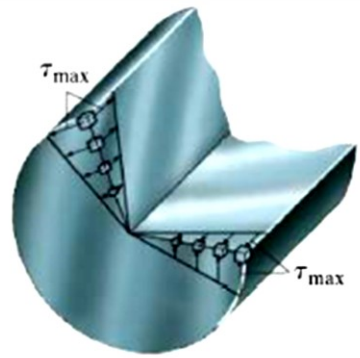
Fig. 5–6

Note that J is a *geometric property* of the circular area and is always positive. Common units used for its measurement are mm^4 or in^4.

The shear stress has been shown to vary linearly along each radial line of the cross section of the shaft. However, if an element of material on the cross section is isolated, then due to the complementary property of shear, equal shear stresses must also act on four of its adjacent faces as shown in Fig. 5–7*a*. Hence, ***not only does the internal torque T develop a linear distribution of shear stress along each radial line in the plane of the cross-sectional area, but also an associated shear-stress distribution is developed along an axial plane***, Fig. 5–7*b*. It is interesting to note that because of this axial distribution of shear stress, shafts made from wood tend to *split* along the axial plane when subjected to excessive torque, Fig. 5–8. This is because wood is an anisotropic material. Its shear resistance parallel to its grains or fibers, directed along the axis of the shaft, is much less than its resistance perpendicular to the fibers, directed in the plane of the cross section.

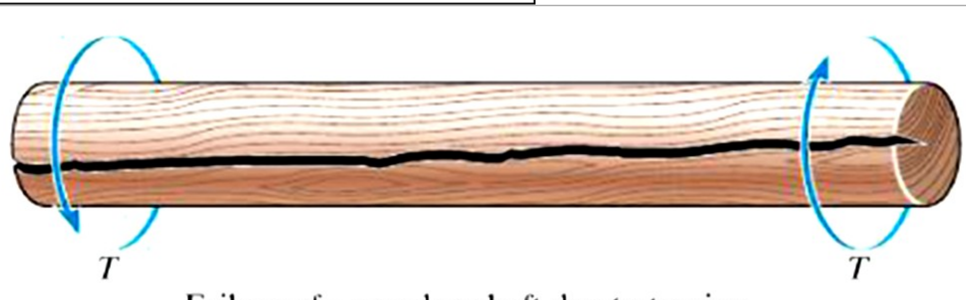


(a)



Shear stress varies linearly along each radial line of the cross section.
(b)

Fig. 5–7



Failure of a wooden shaft due to torsion.

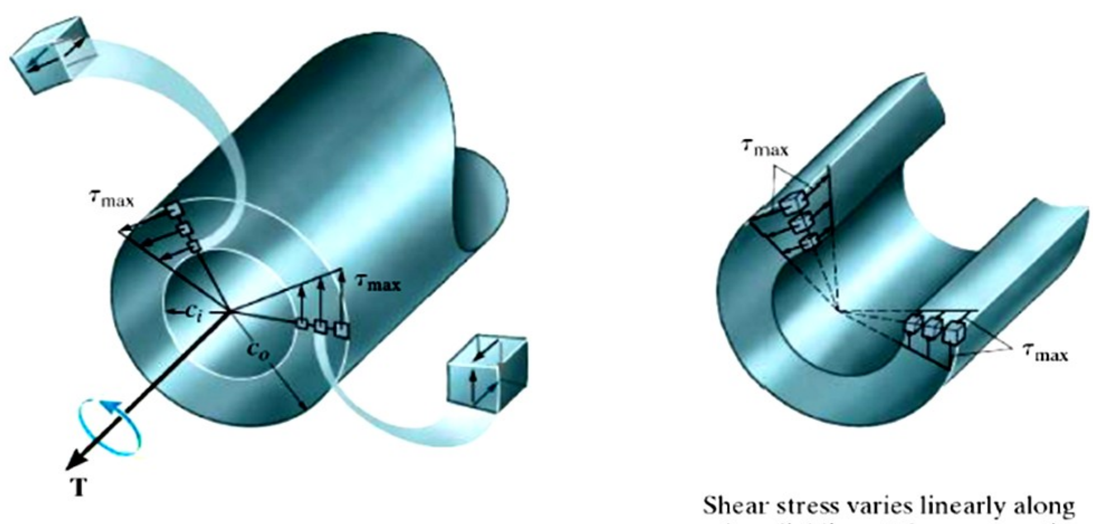
Fig. 5–8

Tubular Shaft. If a shaft has a tubular cross section, with inner radius c_i and outer radius c_o, then from Eq. 5–8 we can determine its polar moment of inertia by subtracting J for a shaft of radius c_i from that determined for a shaft of radius c_o. The result is

$$J = \frac{\pi}{2}(c_o^4 - c_i^4) \qquad (5\text{–}9)$$

Like the solid shaft, the shear stress distributed over the tube's cross-sectional area varies linearly along any radial line, Fig. 5–9*a*. Furthermore, the shear stress varies along an axial plane in this same manner, Fig. 5–9*b*.

Absolute Maximum Torsional Stress. If the absolute maximum torsional stress is to be determined, then it becomes important to find the location where the ratio Tc/J is a maximum. In this regard, it may be helpful to show the variation of the internal torque T at each section along the axis of the shaft by drawing a ***torque diagram***, which is a plot of the internal torque T versus its position x along the shaft's length. Once the internal torque throughout the shaft is determined, the maximum ratio of Tc/J can then be identified.



(a) (b)

Fig. 5–9

Example

The *solid* shaft of radius c is subjected to a torque $\mathbf{T}$, Fig. 5–10*a*. Determine the fraction of T that is resisted by the material contained within the outer core of the shaft, which has an inner radius of $c/2$ and outer radius c.

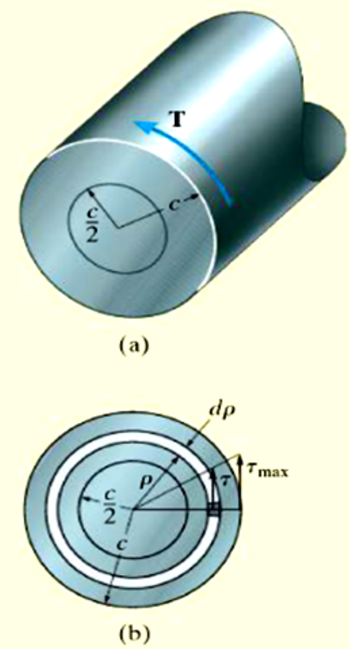


Fig. 5–10

SOLUTION

The stress in the shaft varies linearly, such that $\tau = (\rho/c)\tau_{max}$, Eq. 5–3. Therefore, the torque dT' on the ring (area) located within the outer core, Fig. 5–10*b*, is

$$dT' = \rho(\tau\, dA) = \rho(\rho/c)\tau_{max}(2\pi\rho\, d\rho)$$

For the entire outer core area the torque is

$$T' = \frac{2\pi\tau_{max}}{c}\int_{c/2}^{c}\rho^3 d\rho$$

$$= \frac{2\pi\tau_{max}}{c}\frac{1}{4}\rho^4\Big|_{c/2}^{c}$$

So that

$$T' = \frac{15\pi}{32}\tau_{max}c^3 \qquad (1)$$

This torque T' can be expressed in terms of the applied torque T by first using the torsion formula to determine the maximum stress in the shaft. We have

$$\tau_{max} = \frac{Tc}{J} = \frac{Tc}{(\pi/2)c^4}$$

or

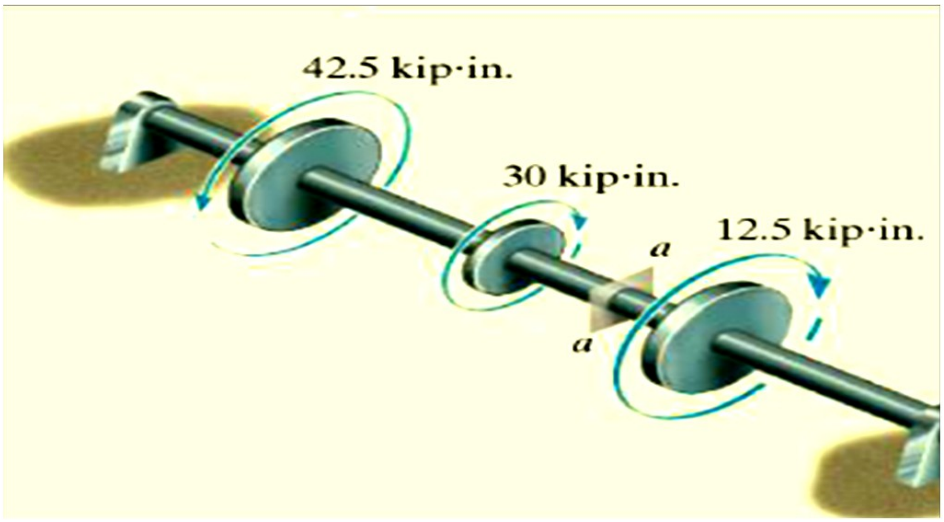
$$\tau_{max} = \frac{2T}{\pi c^3}$$

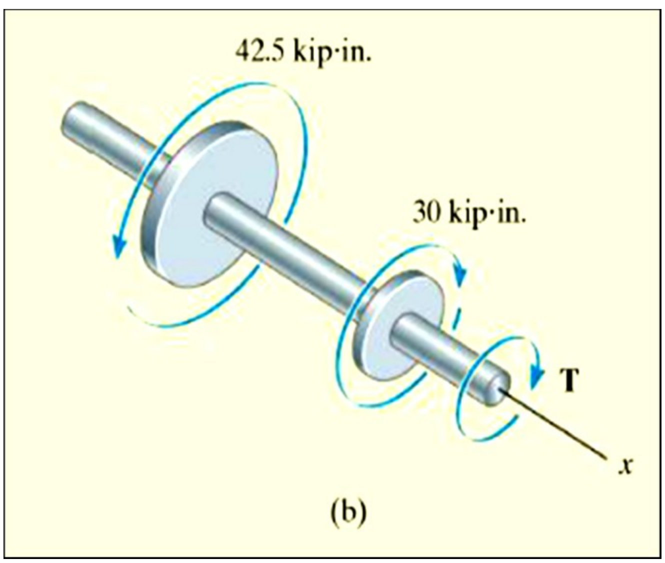
Substituting this into Eq. 1 yields

$$T' = \frac{15}{16}T$$ *Ans.*

Example

The shaft shown in Fig. 5–11*a* is supported by two bearings and is subjected to three torques. Determine the shear stress developed at points *A* and *B*, located at section *a–a* of the shaft, Fig. 5–11*c*.





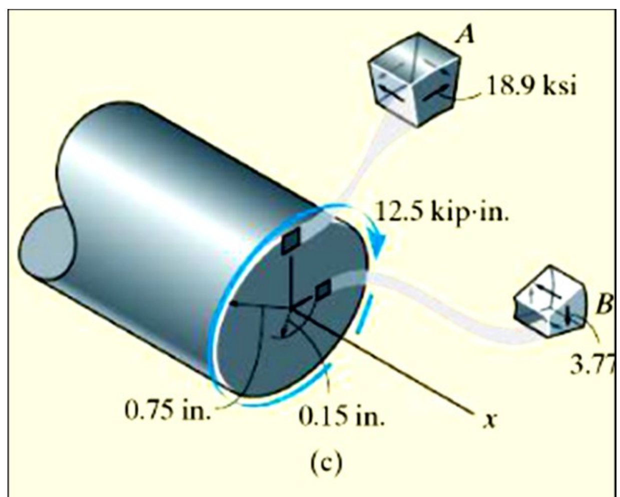


Fig. 5–11

SOLUTION

Internal Torque. Since the bearing reactions do not offer resistance to shaft rotation, the applied torques satisfy moment equilibrium about the shaft's axis.

The internal torque at section a–a will be determined from the free-body diagram of the left segment, Fig. 5–11b. We have

$\Sigma M_x = 0; \quad 42.5 \text{ kip} \cdot \text{in.} - 30 \text{ kip} \cdot \text{in.} - T = 0 \quad T = 12.5 \text{ kip} \cdot \text{in.}$

Section Property. The polar moment of inertia for the shaft is

$$J = \frac{\pi}{2}(0.75 \text{ in.})^4 = 0.497 \text{ in.}^4$$

Shear Stress. Since point A is at $\rho = c = 0.75$ in.,

$$\tau_A = \frac{Tc}{J} = \frac{(12.5 \text{ kip} \cdot \text{in.})(0.75 \text{ in.})}{(0.497 \text{ in.}^4)} = 18.9 \text{ ksi} \qquad \textit{Ans.}$$

Likewise for point B, at $\rho = 0.15$ in., we have

$$\tau_B = \frac{T\rho}{J} = \frac{(12.5 \text{ kip} \cdot \text{in.})(0.15 \text{ in.})}{(0.497 \text{ in.}^4)} = 3.77 \text{ ksi} \qquad \textit{Ans.}$$

NOTE: The directions of these stresses on each element at A and B, Fig. 5–11c, are established from the direction of the resultant internal torque $\mathbf{T}$, shown in Fig. 5–11b. Note carefully how the shear stress acts on the planes of each of these elements.