Statically Indeterminate Axially Loaded Member

Consider the bar shown in Fig. 4 -*a* ,which is fixed supported at both of its ends. From the free-body diagram, Fig. 4- *b*, equilibrium requires

$\boxed{+\uparrow \Sigma F = 0;}$

$$F_B + F_A - P = 0$$

This type of problem is called statically indeterminate, since the equilibrium equation(s) are not sufficient to determine the two reactions on the bar.

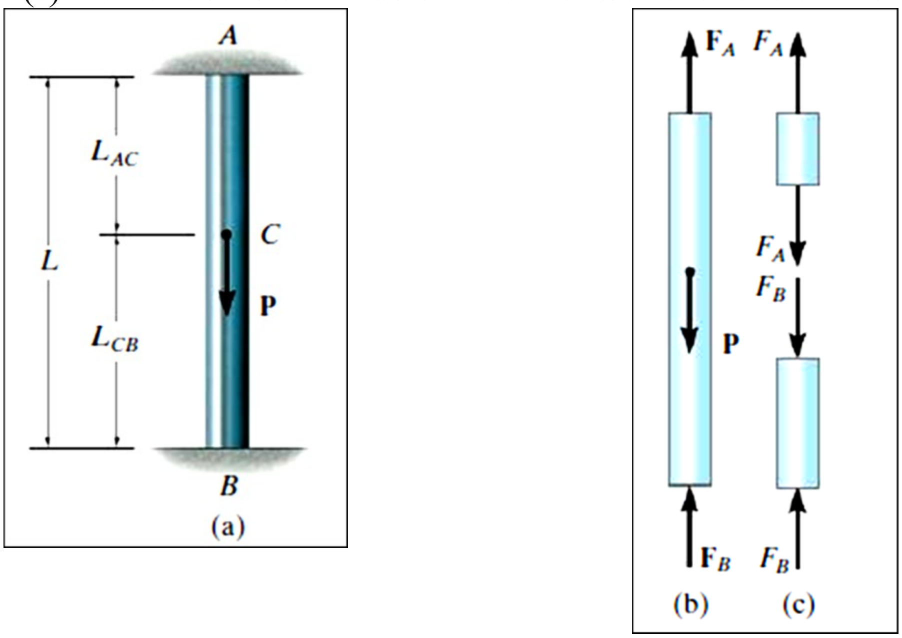


Fig. 4

This type of problem is called statically indeterminate, since the equilibrium equation(s) are not sufficient to determine the two reactions on the bar. In order to establish an additional equation needed for solution, it is necessary to consider how points on the bar displace. Specifically, an equation that specifies the conditions for displacement is referred to as a compatibility or kinematic condition. In this case, a suitable compatibility condition would require the displacement of one end of the bar with respect to the other end to be equal to zero, since the end supports are fixed. Hence, the compatibility condition becomes

$$\boxed{\delta_{A/B} = 0}$$

This equation can be expressed in terms of the applied loads by using a load – displacement relationship, which depends on the material behavior. For example, if linear-elastic behavior occurs, $\delta = P\ L\ /A\ E$ can be used. Realizing that the internal force in segment AC is $+F_A$, and in segment CB the internal force is $-F_B$, Fig.- c , the above equation can be written as

$$\boxed{\frac{F_A L_{AC}}{AE} - \frac{F_B L_{CB}}{AE} = 0}$$

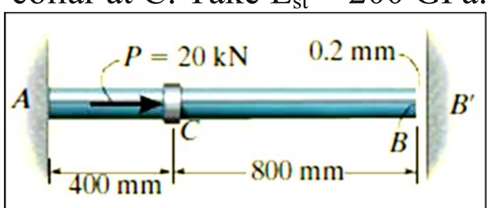
Since AE is constant, then $F_A = F_B\,(L_{CB}/L_{AC})$, so that using the equilibrium equation, the equations for the reactions become

$$\boxed{F_A = P\left(\frac{L_{CB}}{L}\right) \quad \text{and} \quad F_B = P\left(\frac{L_{AC}}{L}\right)}$$

Since both of these results are positive, the direction of the reactions is shown correctly on the free-body diagram.

Example 4

The steel rod shown in Fig-a has a diameter of 10 mm. It is fixed to the wall at A, and before it is loaded, there is a gap of 0.2 mm between the walls at B' and the rod. Determine the reactions at A and B' if the rod is subjected to an axial force of P = 20 KN as shown.
Neglect the size of the collar at C. Take E_{st} = 200 GPa.

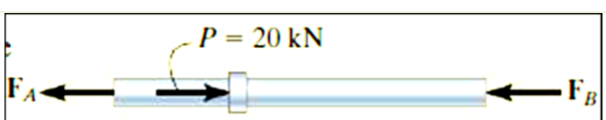


(a)

SOLUTION
Equilibrium.

As shown on the free-body diagram, Fig. b, we will assume that force P is large enough to cause the rod's end B to contact the wall at B'. The problem is statically indeterminate since there are two unknowns and only one equation of equilibrium.

$$\xrightarrow{+} \Sigma F_x = 0; \qquad -F_A - F_B + 20(10^3)\text{ N} = 0 \qquad (1)$$



(b)

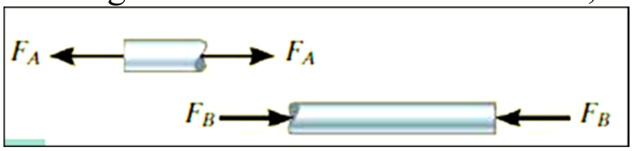
Compatibility.

The force P causes point B to move to B', with no further displacement. Therefore the compatibility condition for the rod is

$$\boxed{\delta_{B/A} = 0.0002 \text{ m}}$$

Load-Displacement.

This displacement can be expressed in terms of the unknown reactions using the load–displacement relationship, Eq. 2 , applied to segments AC and CB , Fig. c . Working in units of Newton and meters, we have



(c)

$$\delta_{B/A} = 0.0002 \text{ m} = \frac{F_A L_{AC}}{AE} - \frac{F_B L_{CB}}{AE}$$

$$0.0002 \text{ m} = \frac{F_A(0.4 \text{ m})}{\pi(0.005 \text{ m})^2\,[200(10^9)\text{ N/m}^2]} - \frac{F_B(0.8 \text{ m})}{\pi(0.005 \text{ m})^2\,[200(10^9)\text{ N/m}^2]}$$

or

$$F_A(0.4 \text{ m}) - F_B(0.8 \text{ m}) = 3141.59 \text{ N}\cdot\text{m} \qquad (2)$$

Solving Eqs. 1 and 2 yields

$$\boxed{F_A = 16.0 \text{ kN} \quad F_B = 4.05 \text{ kN} \qquad \textit{Ans.}}$$

Since the answer for F_B is positive, indeed end B contacts the wall at B' as originally assumed.

NOTE: If F_B were a negative quantity, the problem would be statically determinate, so that $F_B = 0$ and $F_A = 20$ kN.

Thermal Stress

A change in temperature can cause a body to change its dimensions. Generally, if the temperature increases, the body will expand, whereas if the temperature decreases, it will contract. Ordinarily this expansion or contraction is linearly related to the temperature increase or decrease that occurs. If this is the case, and the material is homogeneous and isotropic, it has been found from experiment that the displacement of a member having a length L can be calculated using the formula

$$\boxed{\delta_T = \alpha\,\Delta T L} \qquad (4)$$

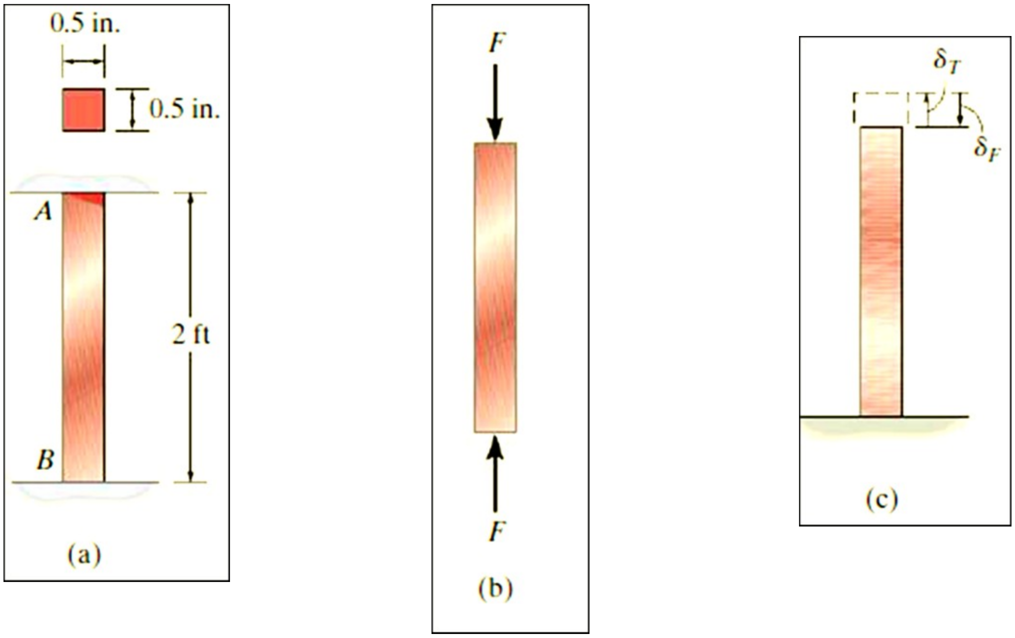
where
α = a property of the material, referred to as the linear coefficient of thermal expansion . The units measure strain per degree of temperature. They are $1/^{o}F$ (Fahrenheit) in the FPS system, and $1/^{o}C$ (Celsius) or 1/K (Kelvin) in the SI system. Typical values are given on the inside back cover
ΔT = the algebraic change in temperature of the member
L = the original length of the member
δ_T = the algebraic change in the length of the member

The change in length of a statically determinate member can easily be calculated using Eq. 4, since the member is free to expand or contract when it undergoes a temperature change. However, in a statically indeterminate member, these thermal displacements will be constrained by the supports, thereby producing thermal stresses that must be considered in design. Determining these thermal stresses is possible using the methods outlined in the previous sections. The following examples illustrate some applications.

Most traffic bridges are designed with expansion joints to accommodate the thermal movement of the deck and thus avoid any thermal stress.

Example

The A-36 steel bar shown in Fig- a is constrained to just fit between two fixed supports when $T_1 = 60^{o}F$. If the temperature is raised to $T_2 = 120^{o}F$, determine the average normal thermal stress developed in the bar.



SOLUTION

Equilibrium:

The free-body diagram of the bar is shown in Fig- b. Since there is no external load, the force at A is equal but opposite to the force at B; that is,

$$+\uparrow \Sigma F_y = 0; \qquad F_A = F_B = F$$

The problem is statically indeterminate since this force cannot be determined from equilibrium.

Compatibility:

Since $\delta_{A/B} = 0$, the thermal displacement δ_T at A that occurs, Fig-c , is counteracted by the force F that is required to push the bar δ F back to its original position. The compatibility condition at A becomes

$$(+\uparrow) \qquad \delta_{A/B} = 0 = \delta_T - \delta_F$$

Load-Displacement:

Applying the thermal and load–displacement relationships, we have

$$\boxed{0 = \alpha \Delta T L - \frac{FL}{AE}}$$

Thus, from the data on the inside back cover,

$$
\begin{aligned}
F &= \alpha \Delta T A E \\
&= [6.60(10^{-6})/°\text{F}](120°\text{F} - 60°\text{F})(0.5 \text{ in.})^2 \, [29(10^3) \text{ kip/in}^2] \\
&= 2.871 \text{ kip}
\end{aligned}
$$

Since F also represents the internal axial force within the bar, the average normal compressive stress is thus

$$\sigma = \frac{F}{A} = \frac{2.871 \text{ kip}}{(0.5 \text{ in.})^2} = 11.5 \text{ ksi} \qquad \textit{Ans.}$$