Bending

1. Shear and Moment Diagrams

Members that are slender and support loadings that are applied perpendicular to their longitudinal axis are called ***beams***. In general, beams are long, straight bars having a constant cross-sectional area. Often they are classified as to how they are supported. For example, a *simply supported beam* is pinned at one end and roller supported at the other, Fig. 1, a *cantilevered beam* is fixed at one end and free at the other, and an *overhanging beam* has one or both of its ends freely extended over the supports. Beams are considered among the most important of all structural elements. They are used to support the floor of a building, the deck of a bridge, or the wing of an aircraft. Also, the axle of an automobile, the boom of a crane, even many of the bones of the body act as beams.

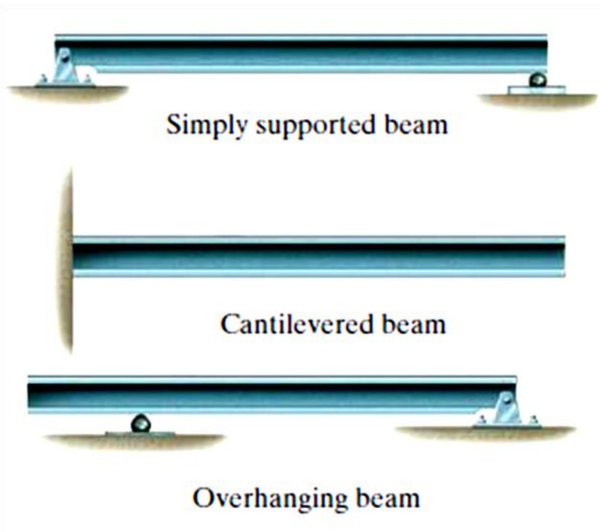


Fig. 6–1

Because of the applied loadings, beams develop an internal shear force and bending moment that, in general, vary from point to point along the axis of the beam. In order to properly design a beam it therefore becomes necessary to determine the *maximum* shear and moment in the beam. One way to do this is to express V and M as functions of their arbitrary position x along the beam's axis. These *shear and moment functions* can then be plotted and represented by graphs called ***shear and moment diagrams.*** The maximum values of V and M can then be obtained from these graphs. Also, since the shear and moment diagrams provide detailed information about the *variation* of the shear and moment along the beam's axis, they are often used by engineers to decide where to place reinforcement materials within the beam or how to proportion the size of the beam at various points along its length. In order to formulate V and M in terms of x we must choose the origin and the positive direction for x. Although the choice is

Arbitrary, most often the origin is located at the left end of the beam and the positive direction is to the right. In general, the internal shear and moment functions of *x* will be *discontinuous*, or their slope will be discontinuous, at points where a distributed loads changes or where concentrated forces or couple moments are applied. Because of this, the shear and moment functions must be determined for *each region* of the beam *between* any two discontinuities of loading. For example, coordinates $x1$, $x2$, and $x3$ will have to be used to describe the variation of *V* and *M* throughout the length of the beam in Fig. 2. These coordinates will be valid *only* within the regions from *A* to *B* for $x1$, from *B* to *C* for $x2$, and from *C* to *D* for $x3$.

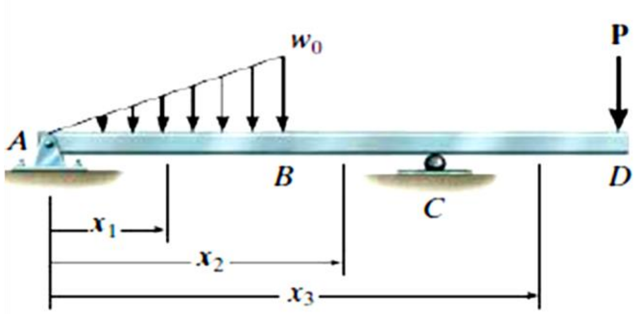
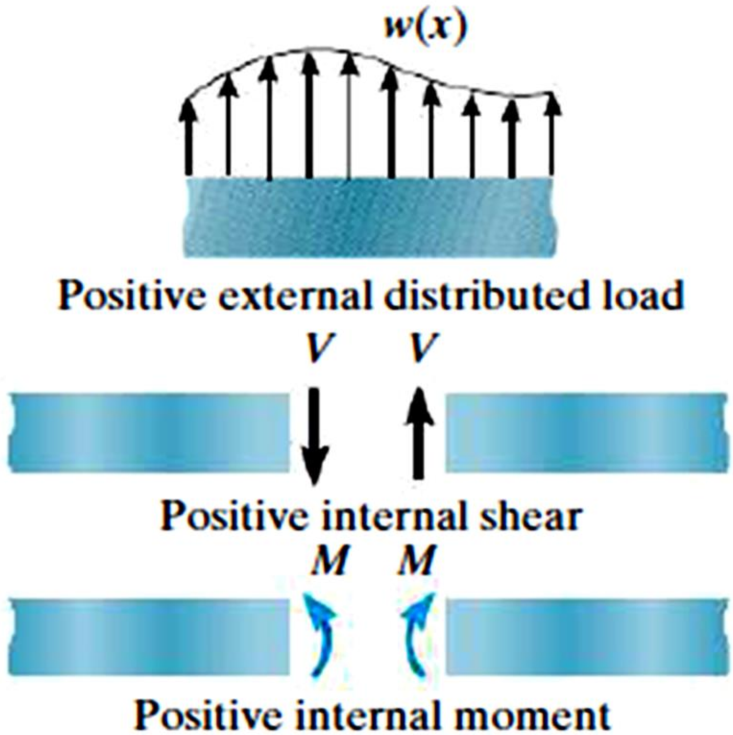


Fig. 6–2

2. Beam Sign Convention:

Before presenting a method for determining the shear and moment as functions of *x* and later *plotting* these functions (*shear and moment diagrams*), it is first necessary to establish a *sign convention* so as to define "positive" and "negative" values for *V* and *M* . Although the choice of a sign convention is arbitrary, here we will use the one often used in engineering practice and shown in Fig. 6–3. The *positive directions* are as follows: the *distributed load* acts *upward* on the beam; the internal *shear force* causes a *clockwise* rotation of the beam segment on which it acts; and the internal *moment* causes *compression* in the *top fibers* of the segment such that it bends the segment so that it "holds water". Loadings that are opposite to these are considered negative.

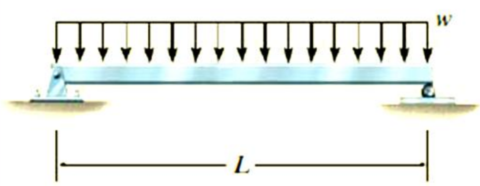


Beam sign convention

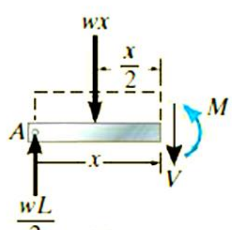
Fig. 6–3

EXAMPLE 6.1

Draw the shear and moment diagrams for the beam shown in Fig. 6–4*a*.



(a)



(b)

SOLUTION

Support Reactions. The support reactions are shown in Fig. 6–4*c*.

Shear and Moment Functions. A free-body diagram of the left segment of the beam is shown in Fig. 6–4*b*. The distributed loading on this segment, wx, is represented by its resultant force only *after* the segment is isolated as a free-body diagram. This force acts through the centroid of the area comprising the distributed loading, a distance of $x/2$ from the right end. Applying the two equations of equilibrium yields

$+\uparrow \Sigma F_y = 0;$

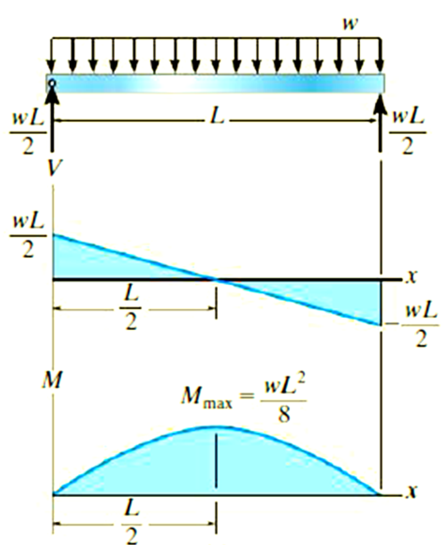
$$\frac{wL}{2} - wx - V = 0$$

$$V = w\left(\frac{L}{2} - x\right) \qquad (1)$$

$\circlearrowleft + \Sigma M = 0;$

$$-\left(\frac{wL}{2}\right)x + (wx)\left(\frac{x}{2}\right) + M = 0$$

$$M = \frac{w}{2}(Lx - x^2) \qquad (2)$$



(c)

Fig. 6–4

Shear and Moment Diagrams. The shear and moment diagrams shown in Fig. 6–4*c* are obtained by plotting Eqs. 1 and 2. The point of *zero shear* can be found from Eq. 1:

$$V = w\left(\frac{L}{2} - x\right) = 0$$

$$x = \frac{L}{2}$$

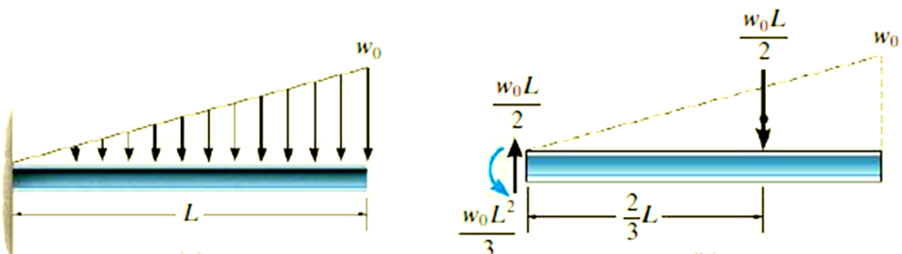
NOTE: From the moment diagram, this value of x represents the point on the beam where the *maximum moment* occurs, since by Eq. 6–2 (see Sec. 6.2) the *slope* $V = dM/dx = 0$. From Eq. 2, we have

$$M_{max} = \frac{w}{2}\left[L\left(\frac{L}{2}\right) - \left(\frac{L}{2}\right)^2\right]$$

$$= \frac{wL^2}{8}$$

EXAMPLE 6.2

Draw the shear and moment diagrams for the beam shown in Fig. 6–5*a*.

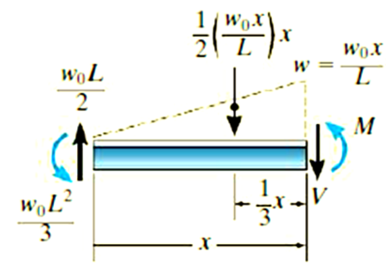


(a) (b)

SOLUTION

Support Reactions. The distributed load is replaced by its resultant force and the reactions have been determined as shown in Fig. 6–5*b*.

Shear and Moment Functions. A free-body diagram of a beam segment of length x is shown in Fig. 6–5*c*. Note that the intensity of the triangular load at the section is found by proportion, that is, $w/x = w_0/L$ or $w = w_0x/L$. With the load intensity known, the resultant of the distributed loading is determined from the area under the diagram. Thus,



(c)

$$+\uparrow \Sigma F_y = 0; \qquad \frac{w_0L}{2} - \frac{1}{2}\left(\frac{w_0x}{L}\right)x - V = 0$$

$$V = \frac{w_0}{2L}(L^2 - x^2) \qquad (1)$$

$$\circlearrowleft + \Sigma M = 0; \qquad \frac{w_0L^2}{3} - \frac{w_0L}{2}(x) + \frac{1}{2}\left(\frac{w_0x}{L}\right)x\left(\frac{1}{3}x\right) + M = 0$$

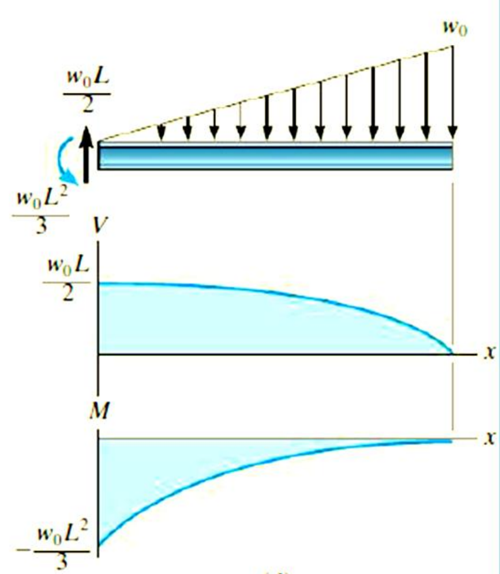
$$M = \frac{w_0}{6L}(-2L^3 + 3L^2x - x^3) \qquad (2)$$

These results can be checked by applying Eqs. 6–1 and 6–2 of Sec. 6.2, that is,

$$w = \frac{dV}{dx} = \frac{w_0}{2L}(0 - 2x) = -\frac{w_0x}{L} \qquad \text{OK}$$

$$V = \frac{dM}{dx} = \frac{w_0}{6L}(0 + 3L^2 - 3x^2) = \frac{w_0}{2L}(L^2 - x^2) \qquad \text{OK}$$

Shear and Moment Diagrams. The graphs of Eqs. 1 and 2 are shown in Fig. 6–5*d*.



(d)

Fig. 6–5

EXAMPLE 6.3

Draw the shear and moment diagrams for the beam shown in Fig. 6–6*a*.

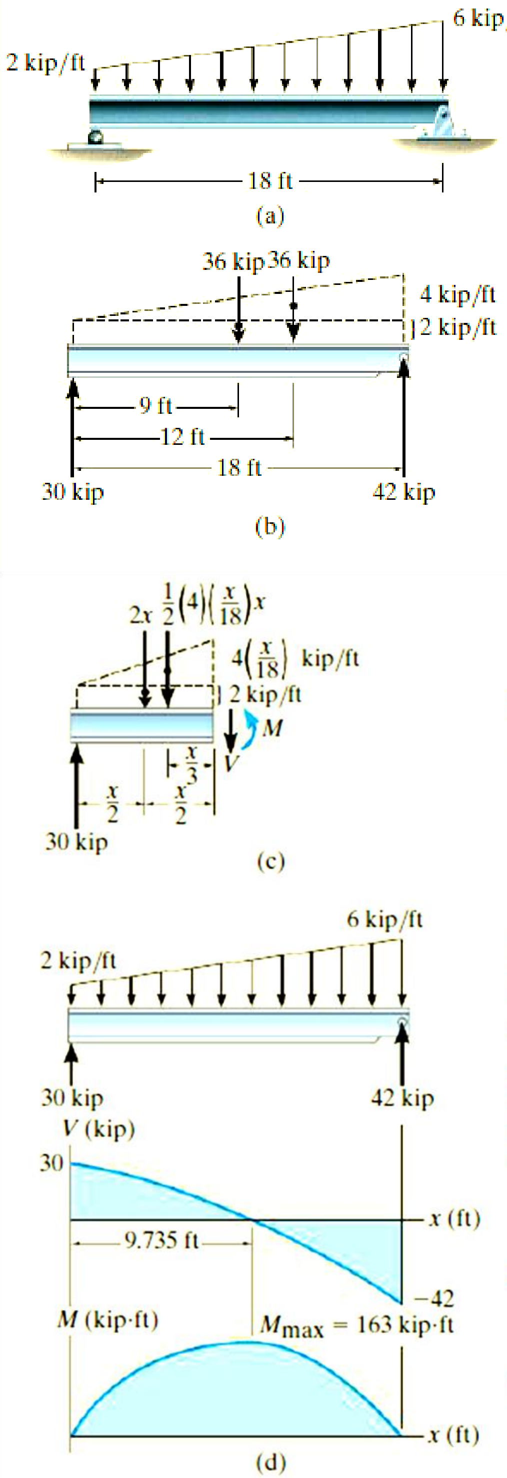


Fig. 6–6

SOLUTION

Support Reactions. The distributed load is divided into triangular and rectangular component loadings and these loadings are then replaced by their resultant forces. The reactions have been determined as shown on the beam's free-body diagram, Fig. 6–6*b*.

Shear and Moment Functions. A free-body diagram of the left segment is shown in Fig. 6–6*c*. As above, the trapezoidal loading is replaced by rectangular and triangular distributions. Note that the intensity of the triangular load at the section is found by proportion. The resultant force and the location of each distributed loading are also shown. Applying the equilibrium equations, we have

$$+\uparrow \Sigma F_y = 0; \qquad 30\text{ kip} - (2\text{ kip/ft})x - \frac{1}{2}(4\text{ kip/ft})\left(\frac{x}{18\text{ ft}}\right)x - V = 0$$

$$V = \left(30 - 2x - \frac{x^2}{9}\right)\text{kip} \qquad (1)$$

$$\circlearrowleft + \Sigma M = 0;$$

$$-30\text{ kip}(x) + (2\text{ kip/ft})x\left(\frac{x}{2}\right) + \frac{1}{2}(4\text{ kip/ft})\left(\frac{x}{18\text{ ft}}\right)x\left(\frac{x}{3}\right) + M = 0$$

$$M = \left(30x - x^2 - \frac{x^3}{27}\right)\text{kip}\cdot\text{ft} \qquad (2)$$

Equation 2 may be checked by noting that $dM/dx = V$, that is, Eq. 1. Also, $w = dV/dx = -2 - \frac{2}{9}x$. This equation checks, since when $x = 0$, $w = -2$ kip/ft, and when $x = 18$ ft, $w = -6$ kip/ft, Fig. 6–6*a*.

Shear and Moment Diagrams. Equations 1 and 2 are plotted in Fig. 6–6*d*. Since the point of maximum moment occurs when $dM/dx = V = 0$ (Eq. 6–2), then, from Eq. 1,

$$V = 0 = 30 - 2x - \frac{x^2}{9}$$

Choosing the positive root,

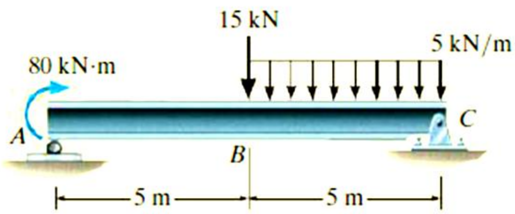
$$x = 9.735\text{ ft}$$

Thus, from Eq. 2,

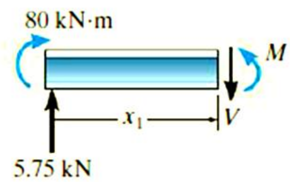
$$M_{max} = 30(9.735) - (9.735)^2 - \frac{(9.735)^3}{27}$$
$$= 163\text{ kip}\cdot\text{ft}$$

EXAMPLE 6.4

Draw the shear and moment diagrams for the beam shown in Fig. 6–7*a*.



(a)



(b)

SOLUTION

Support Reactions. The reactions at the supports have been determined and are shown on the free-body diagram of the beam, Fig. 6–7*d*.

Shear and Moment Functions. Since there is a discontinuity of distributed load and also a concentrated load at the beam's center, two regions of x must be considered in order to describe the shear and moment functions for the entire beam.

$0 \le x_1 < 5$ m, Fig. 6–7*b*:

$$+\uparrow \Sigma F_y = 0; \qquad 5.75 \text{ kN} - V = 0$$

$$V = 5.75 \text{ kN} \tag{1}$$

$$\circlearrowleft + \Sigma M = 0; \qquad -80 \text{ kN}\cdot\text{m} - 5.75 \text{ kN}\, x_1 + M = 0$$

$$M = (5.75x_1 + 80) \text{ kN}\cdot\text{m} \tag{2}$$

$5 \text{ m} < x_2 \le 10$ m, Fig. 6–7*c*:

$$+\uparrow \Sigma F_y = 0; \qquad 5.75 \text{ kN} - 15 \text{ kN} - 5 \text{ kN/m}(x_2 - 5 \text{ m}) - V = 0$$

$$V = (15.75 - 5x_2) \text{ kN} \tag{3}$$

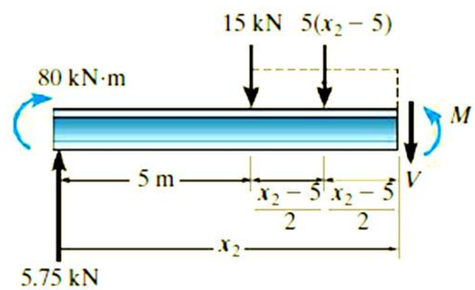
$$\circlearrowleft + \Sigma M = 0; \qquad -80 \text{ kN}\cdot\text{m} - 5.75 \text{ kN}\, x_2 + 15 \text{ kN}(x_2 - 5 \text{ m})$$

$$+ 5 \text{ kN/m}(x_2 - 5 \text{ m})\left(\frac{x_2 - 5 \text{ m}}{2}\right) + M = 0$$

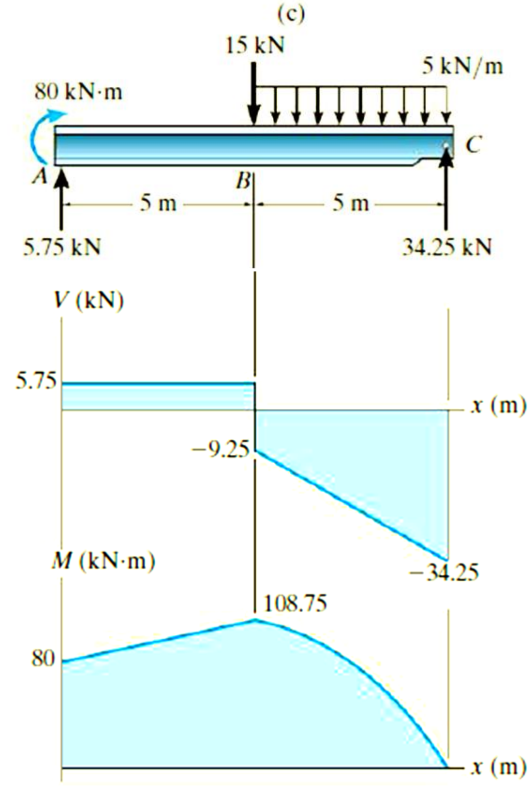
$$M = (-2.5x_2^2 + 15.75x_2 + 92.5) \text{ kN}\cdot\text{m} \tag{4}$$

These results can be checked in part by noting that $w = dV/dx$ and $V = dM/dx$. Also, when $x_1 = 0$, Eqs. 1 and 2 give $V = 5.75$ kN and $M = 80$ kN·m; when $x_2 = 10$ m, Eqs. 3 and 4 give $V = -34.25$ kN and $M = 0$. These values check with the support reactions shown on the free-body diagram, Fig. 6–7*d*.

Shear and Moment Diagrams. Equations 1 through 4 are plotted in Fig. 6–7*d*.



(c)



(d)

Fig. 6–7