Graphical Method for Constructing Shear and Moment Diagrams

In cases where a beam is subjected to *several* different loadings, determining V and M as functions of x and then plotting these equations can become quite tedious. In this section a simpler method for constructing the shear and moment diagrams is discussed—a method based on two differential relations, one that exists between distributed load and shear, and the other between shear and moment.

Regions of Distributed Load

For purposes of generality, consider the beam shown in Fig. 6–8 *a*, which is subjected to an arbitrary loading. A free-body diagram for a very small segment Δx of the beam is shown in Fig. 6–8 *b* . Since this segment has been chosen at a position x where there is no concentrated force or couple moment, the results to be obtained will *not* apply at these points of concentrated loading.

Notice that all the loadings shown on the segment act in their positive directions according to the established sign convention, Fig. 6–3. Also, both the internal resultant shear and moment, acting on the right face of the segment, must be changed by a small amount in order to keep the segment in equilibrium. The distributed load, which is approximately constant over Δx, has been replaced by a resultant force $w(x)\, \Delta x$ that acts

At $\frac{1}{2}(\Delta x)$ from the right side, Applying the equations of equilibrium to the segment, we have

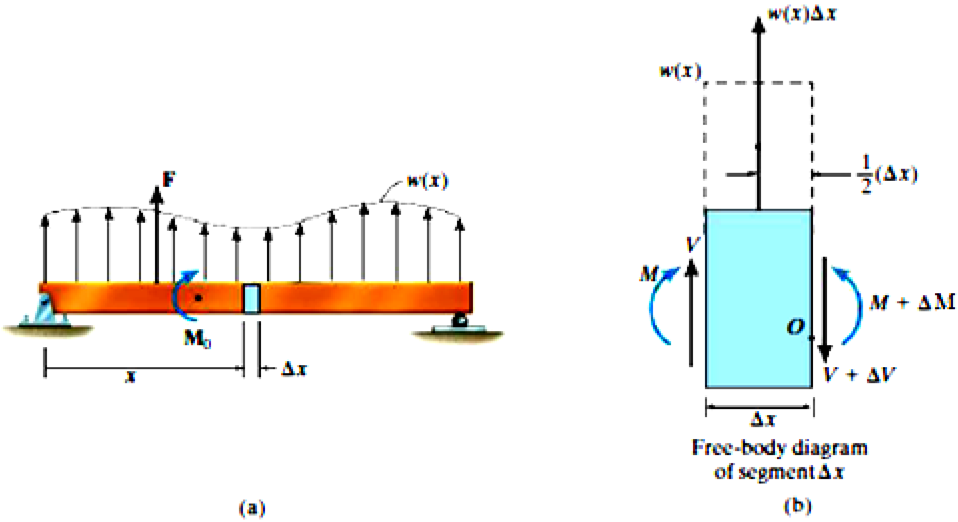


Fig. 6–8

$$+\uparrow \Sigma F_y = 0; \qquad V + w(x)\,\Delta x - (V + \Delta V) = 0$$
$$\Delta V = w(x)\,\Delta x$$

$$\circlearrowleft + \Sigma M_O = 0; \qquad -V\,\Delta x - M - w(x)\,\Delta x\left[\tfrac{1}{2}(\Delta x)\right] + (M + \Delta M) = 0$$
$$\Delta M = V\,\Delta x + w(x)\,\tfrac{1}{2}(\Delta x)^2$$

Dividing by Δx and taking the limit as $\Delta x \rightarrow 0$, the above two equations become

$$\frac{dV}{dx} = w(x) \tag{6–1}$$

slope of shear diagram at each point	=	distributed load intensity at each point

$$\frac{dM}{dx} = V(x) \tag{6–2}$$

slope of moment diagram at each point	=	shear at each point

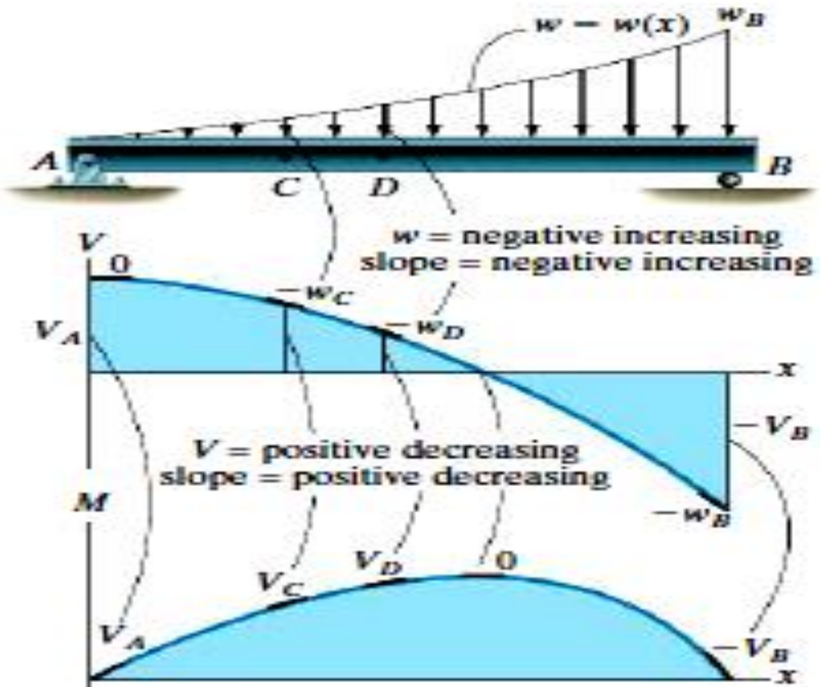
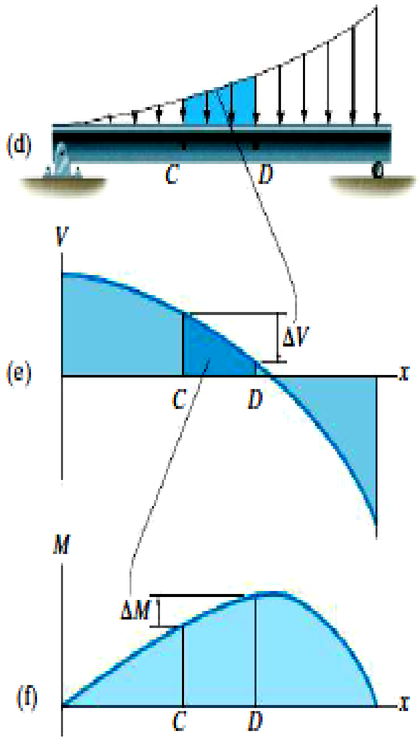


Fig. 6–9

These two equations provide a convenient means for quickly obtaining the shear and moment diagrams for a beam. Equation 6–1 states that at a point the *slope* of the shear diagram equals the intensity of the distributed loading. For example, consider the beam in Fig. 6–9*a*. The distributed loading is negative and increases from zero to w_B. Therefore, the shear diagram will be a curve that has a *negative slope,* increasing from zero to $-w_B$. Specific slopes $w_A = 0$, $-w_C$, $-w_D$, and $-w_B$ are shown in Fig. 6–9*b*.

In a similar manner, Eq. 6–2 states that at a point the *slope* of the moment diagram is equal to the shear. Notice that the shear diagram in Fig. 6–9*b* starts at $+V_A$, decreases to zero, and then becomes negative and decreases to $-V_B$. The moment diagram will then have an initial slope of $+V_A$ which decreases to zero, then the slope becomes negative and decreases to $-V_B$. Specific slopes V_A, V_C, V_D, 0, and $-V_B$ are shown in Fig. 6–9*c*.



Equations 6-1 and 6–2 may also be rewritten in the form $dV = w(x)dx$ and $dM = V dx$. Noting that $w(x)\,dx$ and $V\,dx$ represent differential areas under the distributed loading and shear diagram, respectively, we can integrate these areas between any two points C and D on the beam, Fig. 6–9*d*, and write

$$\Delta V = \int w(x)dx \qquad (6\text{–}3)$$

change in shear = area under distributed loading

$$\Delta M = \int V(x)dx \qquad (6\text{–}4)$$

change in moment = area under shear diagram

Equation 6–3 states that the *change in shear* between C and D is equal to the *area* under the distributed-loading curve between these two points, Fig. 6–9*d*. In this case the change is negative since the distributed load acts downward. Similarly, from Eq. 6–4, the change in moment between C and D, Fig. 6–9*f*, is equal to the area under the shear diagram within the region from C to D. Here the change is positive.

Since the above equations do not apply at points where a concentrated force or couple moment acts, we will now consider each of these cases.

Regions of Concentrated Force and Moment. A free-body diagram of a small segment of the beam in Fig. 6–8*a* taken from under the force is shown in Fig. 6–10*a*. Here it can be seen that force equilibrium requires

$$+\uparrow \Sigma F_y = 0; \qquad V + F - (V + \Delta V) = 0$$

$$\Delta V = F \tag{6–5}$$

Thus, when $\mathbf{F}$ acts *upward* on the beam, ΔV is *positive* so the shear will "jump" *upward*. Likewise, if $\mathbf{F}$ acts *downward*, the jump (ΔV) will be *downward*.

When the beam segment includes the couple moment M_0, Fig. 6–10*b*, then moment equilibrium requires the change in moment to be

$$\zeta + \Sigma M_O = 0; \qquad M + \Delta M - M_0 - V\,\Delta x - M = 0$$

Letting $\Delta x \rightarrow 0$, we get

$$\Delta M = M_0 \tag{6–6}$$

In this case, if $\mathbf{M}_0$ is applied *clockwise*, ΔM is *positive* so the moment diagram will "jump" *upward*. Likewise, when $\mathbf{M}_0$ acts *counterclockwise*, the jump (ΔM) will be *downward*.

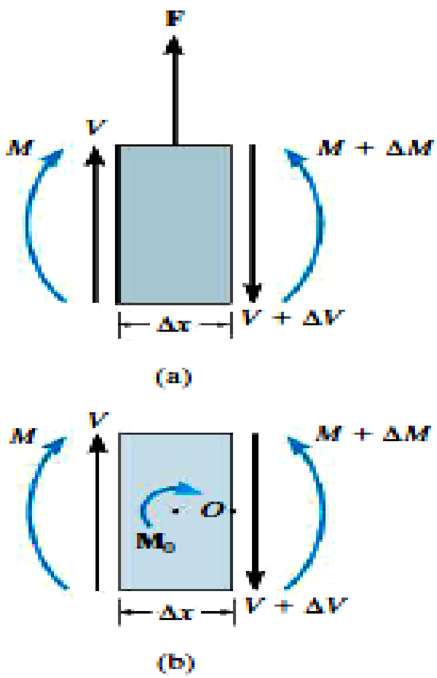


Fig. 6–10

Procedure for Analysis

The following procedure provides a method for constructing the shear and moment diagrams for a beam based on the relations among distributed load, shear, and moment.

Support Reactions.

- Determine the support reactions and resolve the forces acting on the beam into components that are perpendicular and parallel to the beam's axis.

Shear Diagram.

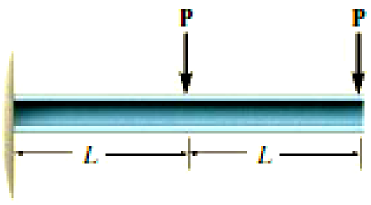
- Establish the V and x axes and plot the known values of the shear at the two *ends* of the beam.
- Notice how the values of the distributed load vary along the beam, and realize that each of these values indicates the way the shear diagram will slope ($dV/dx = w$). Here w is positive when it acts upward.
- If a numerical value of the shear is to be determined at a point, one can find this value either by using the method of sections and the equation of force equilibrium, or by using $\Delta V = \int w(x)\,dx$, which states that the *change in the shear* between any two points is equal to the *area under the load diagram* between the two points.

Moment Diagram.

- Establish the M and x axes and plot the known values of the moment at the *ends* of the beam.
- Notice how the values of the shear diagram vary along the beam, and realize that each of these values indicates the way the moment diagram will slope ($dM/dx = V$).
- At the point where the shear is zero, $dM/dx = 0$, and therefore this would be a point of maximum or minimum moment.
- If a numerical value of the moment is to be determined at the point, one can find this value either by using the method of sections and the equation of moment equilibrium, or by using $\Delta M = \int V(x)\,dx$, which states that the *change in moment* between any two points is equal to the *area under the shear diagram* between the two points.
- Since $w(x)$ must be *integrated* to obtain ΔV, and $V(x)$ is integrated to obtain $M(x)$, then if $w(x)$ is a curve of degree n, $V(x)$ will be a curve of degree $n + 1$ and $M(x)$ will be a curve of degree $n + 2$. For example, if $w(x)$ is uniform, $V(x)$ will be linear and $M(x)$ will be parabolic.

EXAMPLE 6.5

Draw the shear and moment diagrams for the beam shown in Fig. 6–11*a*.



(a)

SOLUTION

Support Reactions. The reaction at the fixed support is shown on the free-body diagram, Fig. 6–11*b*.

Shear Diagram. The shear at each end of the beam is plotted first, Fig. 6–11*c*. Since there is no distributed loading on the beam, the slope of the shear diagram is zero as indicated. Note how the force P at the center of the beam causes the shear diagram to jump downward an amount P, since this force acts downward.

Moment Diagram. The moments at the ends of the beam are plotted, Fig. 6–11*d*. Here the moment diagram consists of two sloping lines, one with a slope of $+2P$ and the other with a slope of $+P$.

The value of the moment in the center of the beam can be determined by the method of sections, or from the area under the shear diagram. If we choose the left half of the shear diagram,

$$M|_{x=L} = M|_{x=0} + \Delta M$$

$$M|_{x=L} = -3PL + (2P)(L) = -PL$$

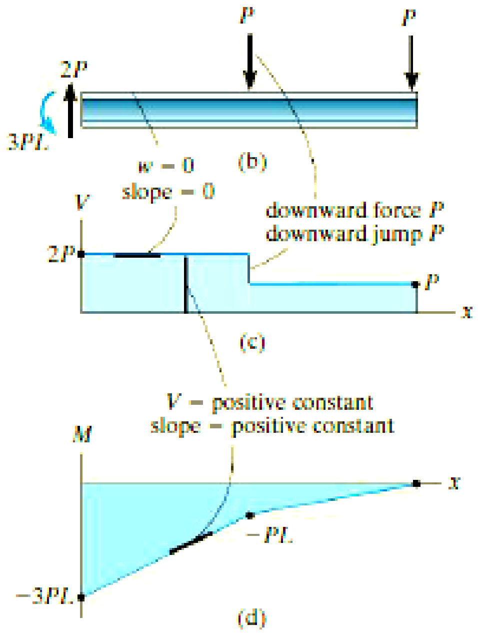
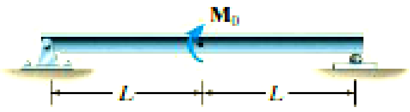


Fig. 6–11

EXAMPLE 6.6

Draw the shear and moment diagrams for the beam shown in Fig. 6–12*a*.



(a)

SOLUTION

Support Reactions. The reactions are shown on the free-body diagram in Fig. 6–12*b*.

Shear Diagram. The shear at each end is plotted first, Fig. 6–12*c*. Since there is no distributed load on the beam, the shear diagram has zero slope and is therefore a horizontal line.

Moment Diagram. The moment is zero at each end, Fig. 6–12*d*. The moment diagram has a constant negative slope of $-M_0/2L$ since this is the shear in the beam at each point. Note that the couple moment M_0 causes a jump in the moment diagram at the beam's center, but it does not affect the shear diagram at this point.

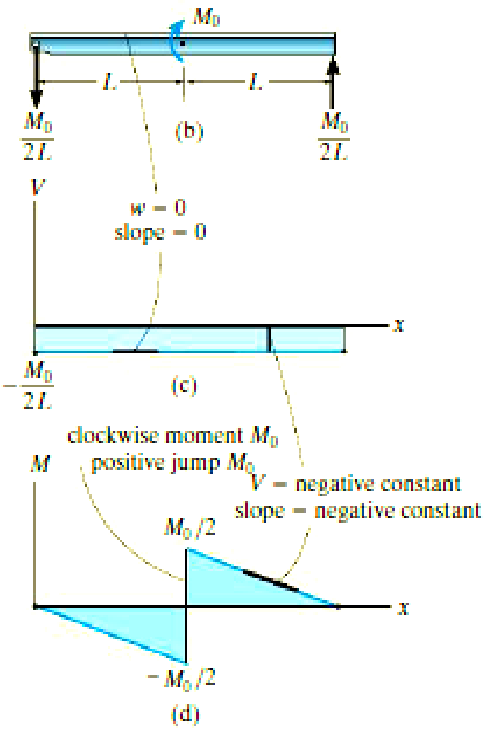


Fig. 6–12

EXAMPLE 6.7

Draw the shear and moment diagrams for each of the beams shown in Figs. 6–13*a* and 6–14*a*.

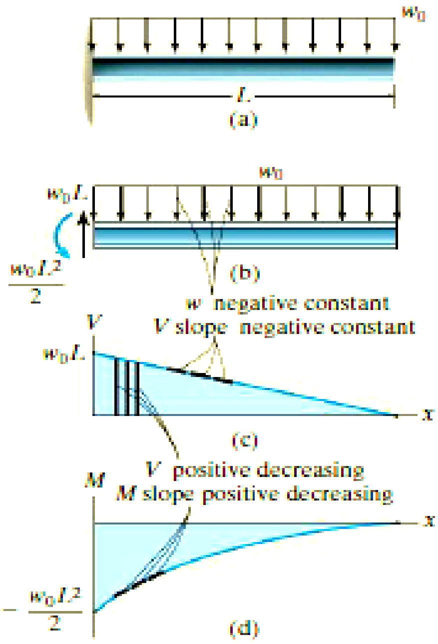


Fig. 6–13

SOLUTION

Support Reactions. The reactions at the fixed support are shown on each free-body diagram, Figs. 6–13*b* and 6–14*b*.

Shear Diagram. The shear at each end point is plotted first, Figs. 6–13*c* and 6–14*c*. The distributed loading on each beam indicates the slope of the shear diagram and thus produces the shapes shown.

Moment Diagram. The moment at each end point is plotted first, Figs. 6–13*d* and 6–14*d*. Various values of the shear at each point on the beam indicate the slope of the moment diagram at the point. Notice how this variation produces the curves shown.

NOTE: Observe how the degree of the curves from w to V to M increases by one due to the integration of $dV = w\,dx$ and $dM = V dx$. For example, in Fig. 6–14, the linear distributed load produces a parabolic shear diagram and cubic moment diagram.

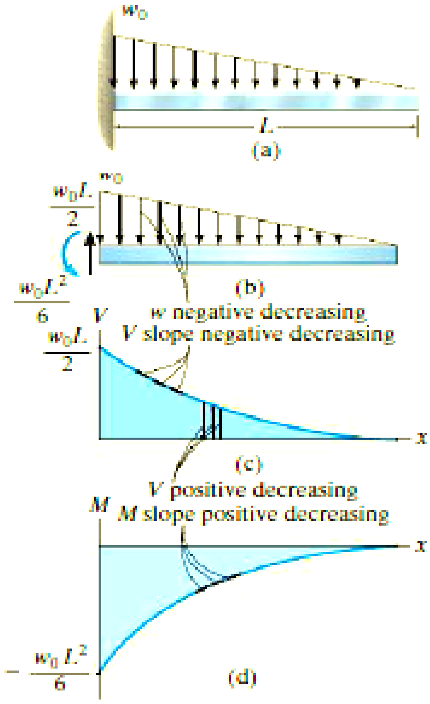
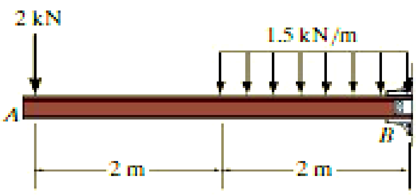


Fig. 6–14

EXAMPLE 6.8

Draw the shear and moment diagrams for the cantilever beam in Fig. 6–15*a*.



(a)

SOLUTION

Support Reactions. The support reactions at the fixed support B are shown in Fig. 6–15*b*.

Shear Diagram. The shear at end A is –2 kN. This value is plotted at $x = 0$, Fig. 6–15*c*. Notice how the shear diagram is constructed by following the slopes defined by the loading w. The shear at $x = 4$ m is –5 kN, the reaction on the beam. This value can be verified by finding the area under the distributed loading, Eq. 6–3.

$$V|_{x=4\text{ m}} = V|_{x=2\text{ m}} + \Delta V = -2\text{ kN} - (1.5\text{ kN/m})(2\text{ m}) = -5\text{ kN}$$

Moment Diagram. The moment of zero at $x = 0$ is plotted in Fig. 6–15*d*. Notice how the moment diagram is constructed based on knowing its slope, which is equal to the shear at each point. The change of moment from $x = 0$ to $x = 2$ m is determined from the area under the shear diagram. Hence, the moment at $x = 2$ m is

$$M|_{x=2\text{ m}} = M|_{x=0} + \Delta M = 0 + [-2\text{ kN}(2\text{ m})] = -4\text{ kN}\cdot\text{m}$$

This same value can be determined from the method of sections, Fig. 6–15*e*.

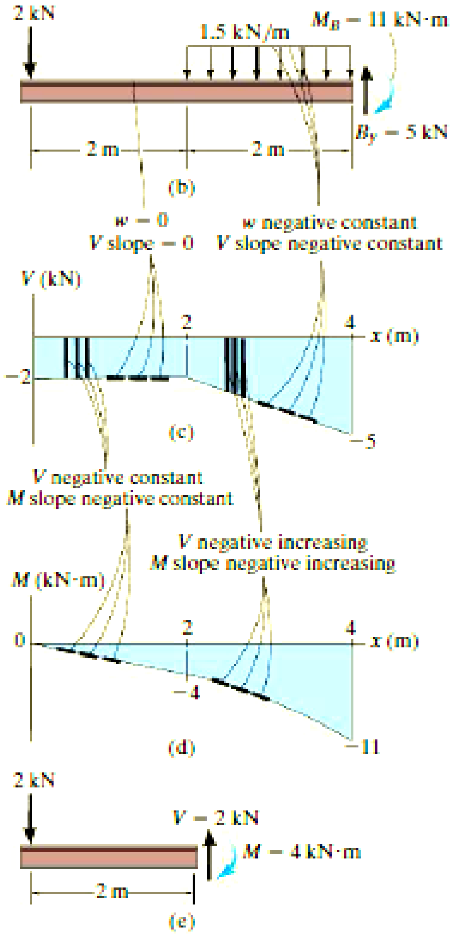


Fig. 6–15