Determinants and their properties

The determinant of a Matrix: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix, then $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is called determinant of a Matrix A, and denoted by Δ or $det(A)$ where

$$\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example: Find the determinant of the following matrices

$$A = \begin{bmatrix} 4 & 1 \\ -2 & 3 \end{bmatrix}, B = \begin{bmatrix} \sqrt{2} & -3 \\ 2 & 3\sqrt{2} \end{bmatrix}$$

Solution.

$$det(A) = \begin{vmatrix} 4 & 1 \\ -2 & 3 \end{vmatrix} = 4 \times 3 - (1 \times -2) = 12 + 2 = 14$$

$$det(B) = \begin{vmatrix} \sqrt{2} & -3 \\ 2 & 3\sqrt{2} \end{vmatrix} = \sqrt{2} \times 3\sqrt{2} - (-3 \times 2) = 6 + 6 = 12$$

Example: Find the value of h if

$$\begin{vmatrix} 3h & -2 \\ 3 & h \end{vmatrix} = 9$$

Solution.

$$3h \times h - (-2 \times 3) = 9$$

$$3h^2 + 6 = 9$$

$$3h^2 = 3$$

$$h^2 = 1$$

$$h = \pm 1$$

Simultaneous Equations

Cramer's rule is a method for solving linear simultaneous equations. It makes use of determinants and so a knowledge of these is necessary before proceeding.

Cramer's Rule (two equations)

If we are given a pair of simultaneous equations

$$ax_1 + by_1 = c_1$$

$$ax_2 + by_2 = c_2$$

Determinants are used to solve two first-degree equations with two variables, where

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

,

$$\Delta x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$$

And

$$\Delta y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

So,

$$x = \frac{\Delta x}{\Delta} \text{ and } y = \frac{\Delta y}{\Delta}$$

Example: Using Cramer's rule to solve the following equations

$$5x - 2y = 11$$

$$2x + 3y = 12$$

Solution.

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} 5 & -2 \\ 2 & 3 \end{vmatrix} = 5 \times 3 - (-2) \times 2 = 15 + 4 = 19$$

$$\Delta x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = \begin{vmatrix} 11 & -2 \\ 12 & 3 \end{vmatrix} = 11 \times 3 - (-2) \times 12 = 33 + 24 = 57$$

$$\Delta y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = \begin{vmatrix} 5 & 11 \\ 2 & 12 \end{vmatrix} = 5 \times 12 - 11 \times 2 = 60 - 22 = 38$$

$$\therefore\ x = \frac{\Delta x}{\Delta} = \frac{57}{19} = 3$$

$$y = \frac{\Delta y}{\Delta} = \frac{38}{19} = 2$$