2. TRANSCENDENTAL FUNCTIONS

A ***function*** that is ***not algebraic*** (cannot expressed in terms of algebra) is called ***transcendental function***. The transcendental functions are:

1. Trigonometric and inverse trigonometric functions

2. Hyperbolic and inverse hyperbolic functions

3. Exponential functions and logarithmic functions

Inverse Functions

A function that undoes, or inverts, the effect of a function f is called the ***inverse*** of f.

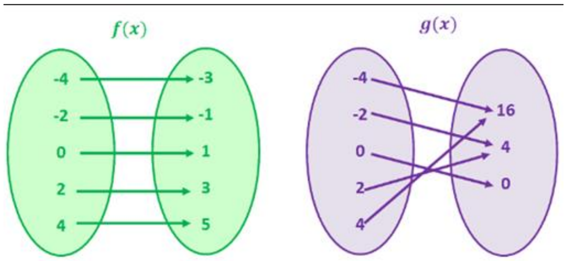
One-to-One Functions

A ***function*** is a rule that assigns a value from its range to each element in its domain. (i.e for each value of x, there is only one value of y)

For example: $y = x^3$ one-to-one function

$y = x + 1$ one-to-one function

$y = x^2$ not one-to-one function



Some functions assign the same range value to more than one element in the domain.

The symbol for the inverse function is f^{-1}

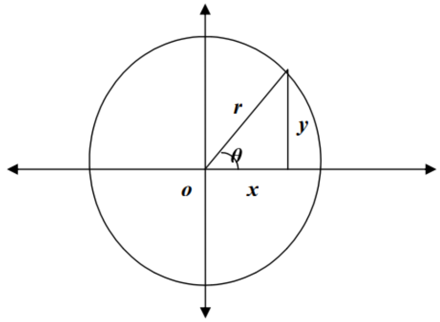
$$x \to f \to f(x) \to f^{-1} \to x$$

$$f^{-1}f(x) = x$$

$$f\,f^{-1}(x) = x$$

• Only one-to-one function have inverse.

2.1 Trigonometric Functions : When an angle of measure is placed in standard position at the center of a circle of radius r, the trigonometric functions of are defined by the equations:



$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$

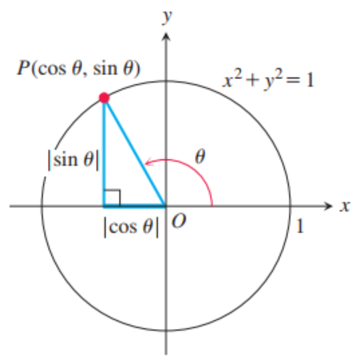
$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

Notice also that whenever the quotients are defined,

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

Trigonometric Identities: The coordinates of any point $P(x, y)$ in the plane can be expressed in terms of the point's distance r from the origin and the angle θ that ray OP makes with the positive x-axis (Figure 2). Since $x/r = \cos\theta$ and $y/r = \sin\theta$, we have $x = r\cos\theta$, $y = r\sin\theta$



The following are some properties of these functions:

1) $Sin^2\theta + Cos^2\theta = 1$

2) $1 + tan^2\theta = sec^2\theta \quad and \quad 1 + Cot^2\theta = csc^2\theta$

3) $Sin(\theta \mp \beta) = Sin\theta.Cos\beta \mp Cos\theta.Sin\beta$

4) $Cos(\theta \mp \beta) = Cos\theta.Cos\beta \pm Sin\theta.Sin\beta$

5) $tan(\theta \mp \beta) = \dfrac{tan\theta \mp tan\beta}{1 \pm tan\theta.tan\beta}$

6) $Sin2\theta = 2Sin\theta.Cos\theta \quad and \quad Cos2\theta = Cos^2\theta - Sin^2\theta$

7) $Cos^2\theta = \dfrac{1 + Cos2\theta}{2} \quad and \quad Sin^2\theta = \dfrac{1 - Cos2\theta}{2}$

8) $Sin(\theta \mp \dfrac{\pi}{2}) = \mp Cos\theta \quad and \quad Cos(\theta \mp \dfrac{\pi}{2}) = \pm Sin\theta$

9) $Sin(-\theta) = -Sin\theta \quad and \quad Cos(-\theta) = Cos\theta \quad and \quad tan(-\theta) = -tan\theta$

10) $Sin\theta.Sin\beta = \dfrac{1}{2}[Cos(\theta - \beta) - Cos(\theta + \beta)]$

$Cos\theta.Cos\beta = \dfrac{1}{2}[Cos(\theta - \beta) + Cos(\theta + \beta)]$

$Sin\theta.Cos\beta = \dfrac{1}{2}[Sin(\theta - \beta) + Sin(\theta + \beta)]$

11) $Sin\theta + Sin\beta = 2Sin\dfrac{\theta + \beta}{2}.Cos\dfrac{\theta - \beta}{2}$

$Sin\theta - Sin\beta = 2Cos\dfrac{\theta + \beta}{2}.Sin\dfrac{\theta - \beta}{2}$

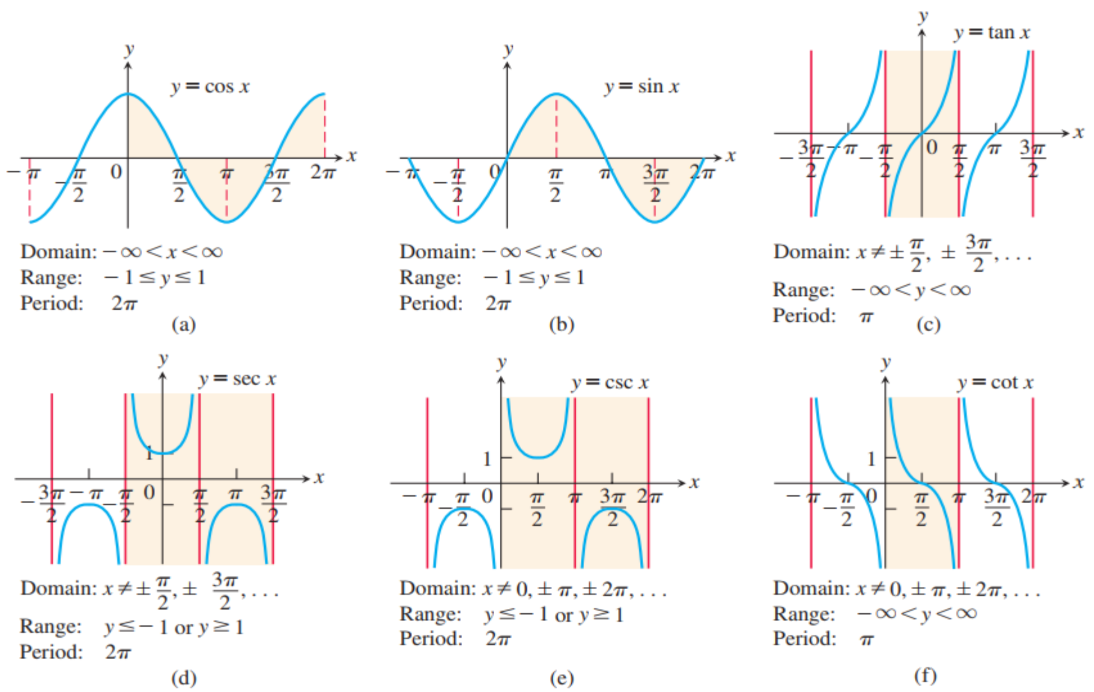
12) $Cos\theta + Cos\beta = 2Cos\dfrac{\theta + \beta}{2}.Cos\dfrac{\theta - \beta}{2}$

$Cos\theta - Cos\beta = -2Sin\dfrac{\theta + \beta}{2}.Sin\dfrac{\theta - \beta}{2}$

TABLE : **Values of $sin\theta$, $cos\theta$, and $tan\theta$ for selected values of θ**

Degrees	−180	−135	−90	−45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
sin θ	0	$\frac{-\sqrt{2}}{2}$	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	−1	0
cos θ	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{3}}{2}$	−1	0	1
tan θ	0	1		−1	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$		$-\sqrt{3}$	−1	$\frac{-\sqrt{3}}{3}$	0		0

Graphs of the trigonometric functions are :



Example : Solve the following equations, for values of θ from 0 to 360 inclusive.

$$a)\ tan\theta = 2sin\theta \qquad , \qquad b)\ \ 1 + cos\theta = 2sin^2\theta$$

Solution:

$$a)\ tan\theta = 2sin\theta \ \Rightarrow\ \frac{sin\theta}{cos\theta} = 2sin\theta$$

$$sin\theta = 2sin\theta cos\theta$$

$$\Rightarrow\ sin\theta(1 - 2cos\theta) = 0$$

either $sin\theta = 0 \ \Rightarrow\ \theta = 0^\circ, 180^\circ, 360^\circ$

or $cos\theta = \frac{1}{2} \quad \Rightarrow \theta = 60^\circ, 300^\circ$

Therefore the required values of are $0^\circ, 180^\circ, 360^\circ, 60^\circ, 300^\circ$.

$$b)\ \ 1 + cos\theta = 2sin^2\theta \ \Rightarrow\ 1 + cos\theta = 2(1 - cos^2\theta)$$

$$1 + cos\theta = 2(1 + cos\theta)(1 - cos\theta)$$

$$2(1 + cos\theta)(1 - cos\theta) - (1 + cos\theta) = 0$$

$$(1 + cos\theta)[2(1 - cos\theta) - 1] = 0$$

$$\Rightarrow\ (cos\theta + 1)(1 - 2cos\theta) = 0$$

$$\text{either } \cos\theta = -1 \quad \Rightarrow \theta = 180^\circ$$

$$\text{or } 1 - 2\cos\theta = 0 \Rightarrow \cos\theta = \frac{1}{2} \quad \Rightarrow \theta = 60^\circ,\ 300^\circ$$

There the roots of the equation between 0° and 360° are 60°, 180° and 300°.

Example : **If $\tan\theta = \frac{7}{24}$, find without using tables the values of $Sec\theta$ and $Sin\theta$.**

Solution.

$$\tan\theta = \frac{y}{x} = \frac{7}{24} \Rightarrow r = \sqrt{24^2 + 7^2} \Rightarrow r = \sqrt{625} \Rightarrow r = 25$$

$$\sec\theta = \frac{r}{x} = \frac{25}{24} \text{ and } \sin\theta = \frac{y}{r} = \frac{7}{25}$$

Example : **Prove the following identities:**

$$a)\ \csc\theta + \tan\theta\sec\theta = \csc\theta\sec^2\theta$$

$$b)\ \cos^4\theta - \sin^4\theta = \cos^2\theta - \sin^2\theta$$

$$c)\ \frac{\sec\theta - \csc\theta}{\tan\theta - \cot\theta} = \frac{\tan\theta + \cot\theta}{\sec\theta + \csc\theta}$$

Solution.

$$a)\ L.H.S \quad \csc\theta + \tan\theta\sec\theta = \frac{1}{\sin\theta} + \frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\cos\theta}$$

$$\frac{1}{\sin\theta} + \frac{\sin\theta}{\cos^2\theta} = \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos^2\theta} = \frac{1}{\sin\theta}\cdot\frac{1}{\cos^2\theta} = \csc\theta.\sec^2\theta \quad R.H.S$$

$$b)\ L.H.S \quad \cos^4\theta - \sin^4\theta = (\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta) = \cos^2\theta - \sin^2\theta \quad R.H.S$$

$$c)\ L.H.S \quad \frac{\sec\theta - \csc\theta}{\tan\theta - \cot\theta} = \frac{\frac{1}{\cos\theta} - \frac{1}{\sin\theta}}{\frac{\sin\theta}{\cos\theta} - \frac{\cos\theta}{\sin\theta}} = \frac{\frac{\sin\theta - \cos\theta}{\cos\theta\sin\theta}}{\frac{\sin^2\theta - \cos^2\theta}{\cos\theta\sin\theta}} = \frac{\frac{\sin\theta - \cos\theta}{\cos\theta\sin\theta}}{\frac{(\sin\theta + \cos\theta)(\sin\theta - \cos\theta)}{\cos\theta\sin\theta}}$$

$$= \frac{\frac{1}{\cos\theta\sin\theta}}{\frac{(\sin\theta + \cos\theta)}{\cos\theta\sin\theta}} = \frac{\frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}}{\frac{\sin\theta}{\cos\theta\sin\theta} + \frac{\cos\theta}{\cos\theta\sin\theta}} = \frac{\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}}{\frac{1}{\cos\theta} + \frac{1}{\sin\theta}}$$

$$= \frac{\tan\theta + \cot\theta}{\sec\theta + \csc\theta} \quad R.H.S$$

Example : Simplify $\frac{1}{\sqrt{x^2-a^2}}$ when $x = acsc\theta$.

Solution.

$$\frac{1}{\sqrt{x^2-a^2}} = \frac{1}{\sqrt{a^2csc^2\theta - a^2}} = \frac{1}{a\sqrt{csc^2\theta - 1}} = \frac{1}{a\sqrt{cot^2\theta}} = \frac{1}{a}tan\theta$$

Example : Prove that the following identities

a) $Sin(A+B) + Sin(A-B) = 2.SinA.CosB$

b) $tan\,A + tan\,B = \frac{Sin(A+B)}{CosA.CosB}$

c) $Sec(A+B) = \frac{SecA.SecB.CscA.CscB}{CscA.CscB - SecA.SecB}$

d) $\frac{Sin\,2\theta + Cos\,2\theta + 1}{Sin\,2\theta - Cos\,2\theta + 1} = Cot\,\theta$

Solution.

a) L.H.S $Sin(A+B) + Sin(A-B) = SinA.CosB + CosA.SinB + SinA.CosB - CosA.SinB$

$$= 2SinA.CosB \text{ R.H.S}$$

b) R.H.S $\frac{Sin(A+B)}{CosA.CosB} = \frac{SinACosB+CosASinB}{CosA.CosB} = \frac{SinACosB}{CosA.CosB} + \frac{CosASinB}{CosA.CosB} = tanA + tanB$ L.H.S

c) R.H.S $\frac{SecA.SecB.CscA.CscB}{CscA.CscB-SecA.SecB} = \frac{\frac{1}{CosA}\cdot\frac{1}{CosB}\cdot\frac{1}{SinA}\cdot\frac{1}{SinB}}{\frac{1}{SinA}\cdot\frac{1}{SinB}-\frac{1}{CosA}\cdot\frac{1}{CosB}} = \frac{\frac{1}{CosA.CosB.SinA.SinB}}{\frac{CosA.CosB-SinA.SinB}{CosA.CosB.SinA.SinB}} = \frac{1}{CosA.CosB-SinA.SinB}$

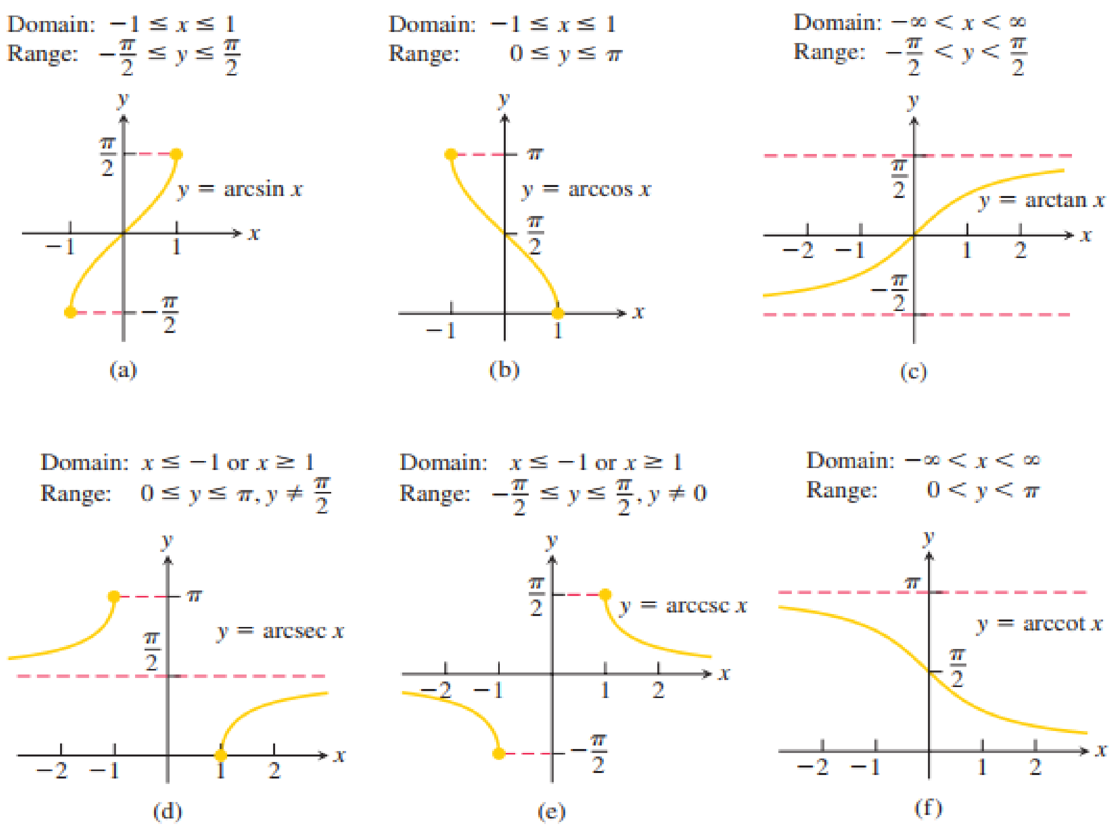
$$= \frac{1}{Cos(A+B)} = Sec(A+B) \text{ L.H.S}$$

d) L.H.S $\frac{Sin2\theta+Cos2\theta+1}{Sin2\theta-Cos2\theta+1} = \frac{2Sin\theta Cos\theta+Cos^2\theta-Sin^2\theta+1}{2Sin\theta Cos\theta-(Cos^2\theta-Sin^2\theta)+1} = \frac{2Sin\theta Cos\theta+Cos^2\theta-\cancel{Sin^2\theta}+Cos^2\theta+\cancel{Sin^2\theta}}{2Sin\theta Cos\theta-\cancel{Cos^2\theta}+Sin^2\theta+\cancel{Cos^2\theta}+Sin^2\theta}$

$$= \frac{2Sin\theta Cos\theta+2Cos^2\theta}{2Sin\theta Cos\theta+2Sin^2\theta} = \frac{2Cos\theta\cancel{(Sin\theta+Cos\theta)}}{2Sin\theta\cancel{(Cos\theta+Sin\theta)}} = Cot\theta \quad \text{R.H.S}$$

2.3 The inverse trigonometric functions : The inverse trigonometric functions arise in problems that require finding angles from side measurements in triangles:

$$y = Sinx \quad \Leftrightarrow \quad x = Sin^{-1} y$$



The following are some properties of the inverse trigonometric functions :

1. $Sin^{-1}(-x) = -Sin^{-1} x$
2. $Cos^{-1}(-x) = \pi - Cos^{-1} x$
3. $Sin^{-1} x + Cos^{-1} x = \frac{\pi}{2}$
4. $tan^{-1}(-x) = -tan^{-1} x$
5. $Cot^{-1} x = \frac{\pi}{2} - tan^{-1} x$
6. $Sec^{-1} x = Cos^{-1} \frac{1}{x}$
7. $Csc^{-1} x = Sin^{-1} \frac{1}{x}$
8. $Sec^{-1}(-x) = \pi - Sec^{-1} x$

and noted that $(Sinx)^{-1} = \frac{1}{Sinx} = Cscx \ne Sin^{-1} x$

Example : **Given that $\alpha = sin^{-1}\frac{\sqrt{3}}{2}$, find:**

$$csc\alpha,\ cos\alpha,\ sec\alpha,\ tan\alpha, \text{ and, } cot\alpha$$

Solution.

$$\alpha = sin^{-1}\frac{\sqrt{3}}{2} \Rightarrow sin\alpha = \frac{\sqrt{3}}{2} = \frac{y}{r}$$

$$x^2 + y^2 = r^2 \Rightarrow x = \sqrt{r^2 - y^2} \Rightarrow x = \sqrt{2^2 - (\sqrt{3})^2} = 1$$

$$csc\alpha = \frac{r}{y} = \frac{2}{\sqrt{3}},\ cos\alpha = \frac{x}{r} = \frac{1}{2},\ sec\alpha = \frac{r}{x} = 2,\ tan\alpha = \frac{y}{x} = \sqrt{3},\ cot\alpha = \frac{x}{y} = \frac{1}{\sqrt{3}}$$

Example : **Evaluate the following expressions :**

$$a)\ sec\left(cos^{-1}\frac{1}{2}\right),\ \ b)\ \ sin^{-1}(1) - sin^{-1}(-1),\ \ c)\ cos^{-1}\left(-sin\frac{\pi}{6}\right)$$

Solution.

$$a)\ sec\left(cos^{-1}\frac{1}{2}\right) = sec\left(\frac{\pi}{3}\right) = \frac{1}{cos\left(\frac{\pi}{3}\right)} = \frac{1}{\frac{1}{2}} = 2$$

$$b)\ \ sin^{-1}(1) - sin^{-1}(-1) = sin^{-1}(1) + sin^{-1}(1) = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

$$c)\ cos^{-1}\left(-sin\frac{\pi}{6}\right) = cos^{-1}\left(-\frac{1}{2}\right) = \pi - cos^{-1}\left(\frac{1}{2}\right) = \pi - \frac{\pi}{3} = \frac{2}{3}\pi$$

Example : **Prove that**

$$a)\ \ Sec^{-1}x = Cos^{-1}\frac{1}{x} \qquad b)\ \ Sin^{-1}(-x) = -Sin^{-1}x$$

Solution.

a) Let $y = Sec^{-1}x \Rightarrow x = Secy \Rightarrow x = \frac{1}{Cosy} \Rightarrow Cosy = \frac{1}{x} \Rightarrow y = Cos^{-1}\frac{1}{x} \Rightarrow Sec^{-1}x = Cos^{-1}\frac{1}{x}$

b) Let $y = -Sin^{-1}x \Rightarrow -y = Sin^{-1}x \Rightarrow x = Sin(-y) \Rightarrow x = -Siny$

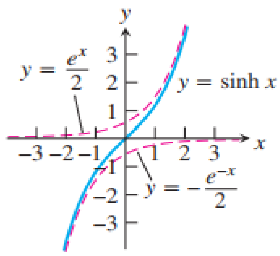
$$\Rightarrow x = -Siny \Rightarrow -x = Siny \Rightarrow y = Sin^{-1}(-x) \Rightarrow -Sin^{-1}x = Sin^{-1}(-x)$$

Hyperbolic and Inverse Hyperbolic Functions

Hyperbolic functions: **Hyperbolic functions are used to describe the motions of waves in elastic solids; the shapes of electric power lines; temperature distributions in metal fins that cool pipes …etc. The hyperbolic sine (Sinh) and hyperbolic cosine (Cosh) are defined by the following equations:**

1. $Sinhu = \dfrac{e^u - e^{-u}}{2}$ and $Coshu = \dfrac{e^u + e^{-u}}{2}$
2. $tanh\,u = \dfrac{Sinhu}{Coshu} = \dfrac{e^u - e^{-u}}{e^u + e^{-u}}$ and $Cothu = \dfrac{Coshu}{Sinhu} = \dfrac{e^u + e^{-u}}{e^u - e^{-u}}$
3. $Sechu = \dfrac{1}{Coshu} = \dfrac{2}{e^u + e^{-u}}$ and $Cschu = \dfrac{1}{Sinhu} = \dfrac{2}{e^u - e^{-u}}$
4. $Cosh^2u - Sinh^2u = 1$
5. $tanh^2\,u + Sech^2u = 1$ and $Coth^2u - Csch^2u = 1$
6. $Coshu + Sinhu = e^u$ and $Coshu - Sinhu = e^{-u}$
7. $Cosh(-u) = Coshu$ and $Sinh(-u) = -Sinhu$
8. $Cosh0 = 1$ and $Sinh0 = 0$
9. $Sinh(x+y) = Sinhx.Coshy + Coshx.Sinhy$
10. $Cosh(x+y) = Coshx.Coshy + Sinhx.Sinhy$
11. $Sinh2x = 2.Sinhx.Coshx$
12. $Cosh2x = Cosh^2x + Sinh^2x$
13. $Cosh^2x = \dfrac{Cosh2x + 1}{2}$ and $Sinh^2x = \dfrac{Cosh2x - 1}{2}$

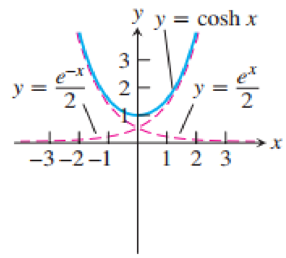
The graph of the six basic hyperbolic functions



(a)

Hyperbolic sine:

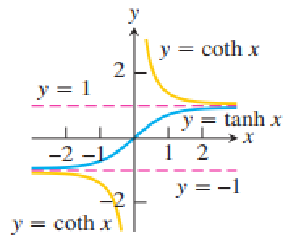
$$\sinh x = \frac{e^x - e^{-x}}{2}$$



(b)

Hyperbolic cosine:

$$\cosh x = \frac{e^x + e^{-x}}{2}$$



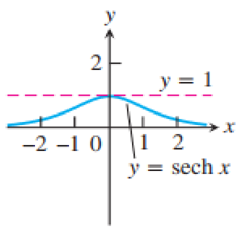
(c)

Hyperbolic tangent:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Hyperbolic cotangent:

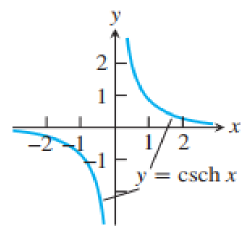
$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$



(d)

Hyperbolic secant:

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$



(e)

Hyperbolic cosecant:

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

Example : Let $Tanh\ u = \frac{-7}{25}$, determine the values of the remaining five hyperbolic functions.

Solution:

$$Cothu = \frac{1}{Tanhu} = -\frac{25}{7}$$

$$Tanh^2u + Sech^2u = 1 \Rightarrow \left(\frac{-7}{25}\right)^2 + Sech^2u = 1 \Rightarrow \frac{49}{625} + Sech^2u = 1 \Rightarrow Sech^2u = \frac{576}{625}$$

$$Sechu = \frac{24}{25}$$

$$Coshu = \frac{1}{Sechu} = \frac{25}{24}$$

$$Tanhu = \frac{Sinhu}{Coshu} \Rightarrow -\frac{7}{25} = \frac{Sinhu}{\frac{24}{25}} \Rightarrow Sinhu = -\frac{7}{24}$$

$$Cschu = \frac{1}{Sinhu} = -\frac{24}{7}$$

Example : **Rewrite the following expressions in terms of exponentials. Write the final result as simply as you can:**

a) 2Cosh(ln x) *b) tanh(ln x)*

c) Cosh5x + Sinh5x *d) (Sinhx + Coshx)⁴*

Solution:

a) $2Cosh(x) = 2\frac{e^{lnx}+e^{-lnx}}{2} = e^{lnx} + e^{lnx^{-1}} = x + x^{-1} = x + \frac{1}{x} = \frac{x^2+1}{x}$

b) $tanh(lnx) = \frac{e^{lnx}-e^{-lnx}}{e^{lnx}+e^{-lnx}} = \frac{e^{lnx}-e^{lnx^{-1}}}{e^{lnx}+e^{lnx^{-1}}} = \frac{x-x^{-1}}{x+x^{-1}} = \frac{x-x^{-1}}{x+x^{-1}} = \frac{x-\frac{1}{x}}{x+\frac{1}{x}} = \frac{\frac{x^2-1}{x}}{\frac{x^2+1}{x}}$

$= \frac{x^2-1}{x^2+1}$

c) $Cosh5x + Sinh5x = \frac{e^{5x}+e^{-5x}}{2} + \frac{e^{5x}-e^{-5x}}{2} = \frac{e^{5x}}{2} + \frac{e^{5x}}{2} = e^{5x}$

d) $(Sinhx + Coshx)^4 = \left(\frac{e^x-e^{-x}}{2} + \frac{e^x+e^{-x}}{2}\right)^4 = \left(\frac{e^x}{2} + \frac{e^x}{2}\right)^4 = (e^x)^4 = e^{4x}$

Example : **Verify the following identity:**

a) ***Sinh(u+v)=Sinh u. Cosh v + Cosh u.Sinh v***
b) **then verify** ***Sinh(u-v)=Sinh u. Cosh v - Cosh u.Sinh v***

Solution:

a) **R.H.S** $Sinhu.Coshv + Coshu.Sinhv = \frac{e^u-e^{-u}}{2}.\frac{e^v+e^{-v}}{2} + \frac{e^u+e^{-u}}{2}.\frac{e^v-e^{-v}}{2}$

$= \frac{e^{u+v} + e^{u-v} - e^{-u+v} - e^{-u-v}}{4} + \frac{e^{u+v} - e^{u-v} + e^{-u+v} - e^{-u-v}}{4}$

$= \frac{e^{u+v} - e^{-u-v}}{4} + \frac{e^{u+v} - e^{-u-v}}{4} = \frac{2e^{u+v} - 2e^{-u-v}}{4} = \frac{2\left(e^{u+v} - e^{-(u+v)}\right)}{4}$

$= \frac{e^{u+v}-e^{-(u+v)}}{2} = Sinh(u + v)$ **L.H.S**

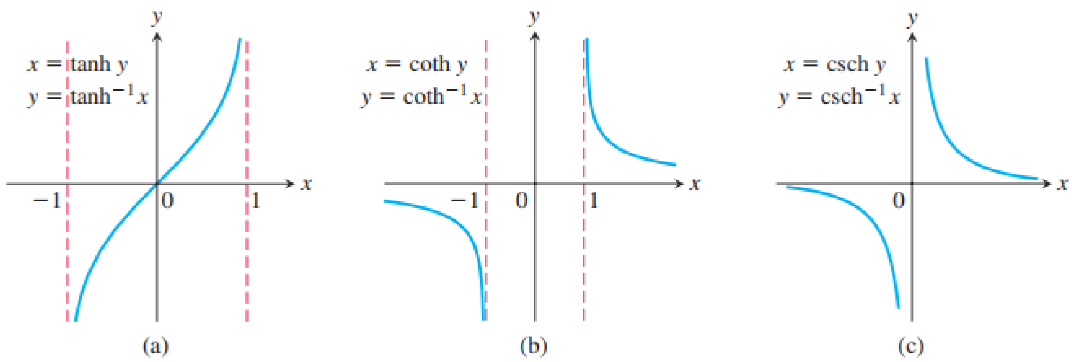
b) **L.H.S** $Sinh(u-v) = Sinh(u+(-v)) = Sinhu.Cosh(-v) + Coshu.Sinh(-v)$

$$= Sinhu.Coshv - Coshu.Sinhv \quad \textbf{R.H.S}$$

Inverse Hyperbolic Functions

All hyperbolic functions have inverses that are useful in integration and interesting as differentiable functions in their own right.

The graphs of the inverse hyperbolic tangent, cotangent, and cosecant of x.



Some useful identities:

1. $Sinh^{-1}x = ln(x + \sqrt{x^2+1})$
2. $Cosh^{-1}x = ln(x + \sqrt{x^2-1})$
3. $tanh^{-1}x = \frac{1}{2}.ln\left(\frac{1+x}{1-x}\right)$
4. $Coth^{-1}x = \frac{1}{2}.ln\left(\frac{x+1}{x-1}\right) = tanh^{-1}\frac{1}{x}$
5. $Sech^{-1}x = ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) = Cosh^{-1}\frac{1}{x}$
6. $Csch^{-1}x = ln\left(\frac{1}{x} + \frac{\sqrt{x^2+1}}{|x|}\right) = Sinh^{-1}\frac{1}{x}$

Example : **Derive the formula:**

$$Sinh^{-1}x = ln(x + \sqrt{x^2 + 1})$$

Solution:

Let $y = Sinh^{-1}x \implies x = Sinhy \implies x = \frac{e^y - e^{-y}}{2} \implies e^y - e^{-y} = 2x$

$$\implies e^y - 2x - e^{-y} = 0 \implies e^{2y} - 2xe^y - 1 = 0$$

$$a = 1, b = -2x, c = -1$$

$$e^y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \implies e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} \implies e^y = \frac{2x \pm \sqrt{4(x^2 + 1)}}{2}$$

$$e^y = x \pm \sqrt{x^2 + 1}$$

either $y = ln(x - \sqrt{x^2 + 1})$ neglected since $x - \sqrt{x^2 + 1} < 0$

or $y = ln\left(x + \sqrt{x^2 + 1}\right) \implies Sinhx = ln(x + \sqrt{x^2 + 1})$

Exponential and Logarithm Functions:

Exponential functions: If a is a positive number and x is any number, we define the exponential function as:

$$y = a^x \text{ with domain: } -\infty < x < \infty$$

$$\text{Range: } y > 0$$

The properties of the exponential functions are:

1. If $a > 0 \Leftrightarrow a^x > 0.$ — 1- اذا كان الاساس موجب فالدالة الاسية موجبة
2. $a^x . a^y = a^{x+y}$ — 2- عند الضرب تجمع الاسس للاساسات المتشابهه
3. $\frac{a^x}{a^y} = a^{x-y}$ — 3- عند القسمة تطرح الاسس للاساسات المتشابهه
4. $(a^x)^y = a^{x.y}$ — 4- عند الرفع تضرب الاسس
5. $(a.b)^x = a^x . b^x$ — 5- الاس يتوزع على الضرب
6. $a^{\frac{x}{y}} = \sqrt[y]{a^x} = \left(\sqrt[y]{a}\right)^x$ — 6- تحويل الاس الكسري الى جذر
7. $a^{-x} = \frac{1}{a^x}$ and $a^x = \frac{1}{a^{-x}}$ — 7- البسط ينزل للمقام بعكس اشارة الاس والمقام يصعد للبسط بعكس اشارة الاس
8. $a^x = a^y \Leftrightarrow x = y$ — 8- اذا تساوت الاساسات فالاسس متساوية والعكس
9. $a^0 = 1, a^\infty = \infty, a^{-\infty} = 0,$ where $a > 0.$

   $a^\infty = 0, a^{-\infty} = \infty,$ where $a < 0.$

The graph of the exponential function $y = a^x$ is:

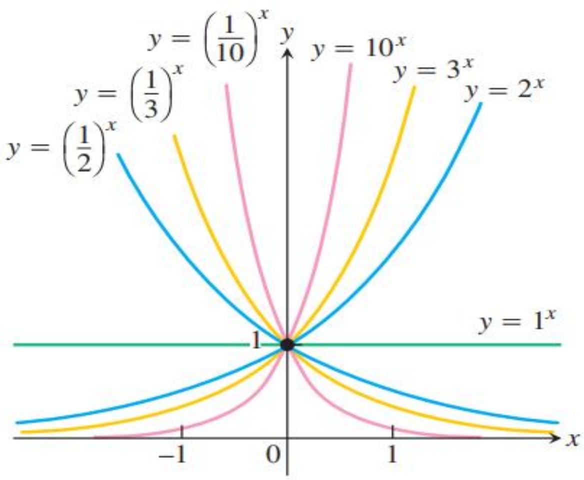


FIGURE 6 Exponential functions decrease if $0 < a < 1$ and increase if $a > 1$. As $x \to \infty$, we have $\boldsymbol{a^x} \to \mathbf{0}$ if $0 < a < 1$ and $\boldsymbol{a^x} \to \infty$ if $a > 1$. As $x \to -\infty$, we have $\boldsymbol{a^x} \to \infty$ if $0 < a < 1$ and $\boldsymbol{a^x} \to \mathbf{0}$ if $a > 1$.

Logarithm function: If a is any positive number other than 1, then the logarithm of x to the base a denoted by :

$$y = \log_a x \text{ where } x > 0$$

At $a = e = 2.7182828 \ldots$, we get the natural logarithm and denoted by:

$$y = \ln x$$

Let $x, y > 0$ then the properties of logarithm functions are:

1. $y = a^x \leftrightarrow x = \log_a y$ and $y = e^x \leftrightarrow x = \ln y$.
2. $\log_e x = \ln x$.
3. $\log_a x = \ln x / \ln a$.

4. $\ln (x.y) = \ln x + \ln y$.
5. $\ln (x / y) = \ln x - \ln y$.
6. $\ln x^n = n. \ln x$.
7. $\ln e = \log_a a = 1$ and $\ln 1 = \log_a 1 = 0$.
8. $a^x = e^{x. \ln a}$.
9. $e^{\ln x} = x$.

The graph of the function $y = ln\, x$ is:

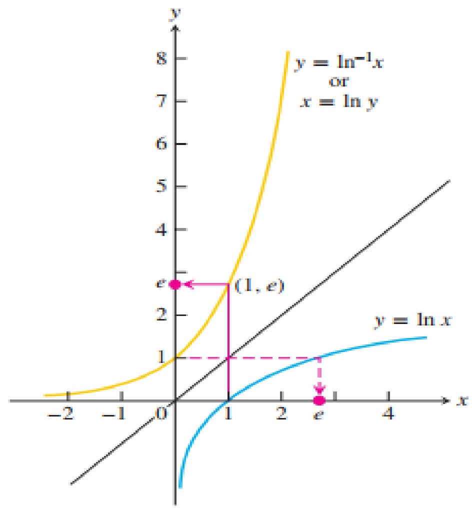


FIGURE The graphs of $y = ln\, x$ and $y = ln^{-1}\, x = exp\, x$.
The number e is $ln^{-1}\, 1 = exp\,(1)$.

Application of exponential and logarithm functions :

We take Newton's law of cooling:

$$\boldsymbol{T - T_0 = (T_0 - T_s)\, e^{tk}}$$

where T is the temperature of the object at time t.

T_s is the surrounding temperature.

T_0 is the initial temperature of the object.

k is a constant.

Example: The temperature of an ingot of metal is 80 C° and the room temperature is 20 C°. After twenty minutes, it was 70 C°.

a) What is the temperature will the metal be after 30 minutes ?

b) What is the temperature will the metal be after two hours ?

c) When will the metal be 30 C° ?

Solution:

$$T - T_0 = (T_0 - T_s)e^{tk} \Longrightarrow 50 = 60e^{20k} \Longrightarrow k = \frac{ln5 - ln6}{20} = -0.0091$$

$a)$ $T - 20 = 60e^{30(-0.0091)} = 60 \times 0.761 = 45.6C^{\circ} \Longrightarrow T = 65.6C^{\circ}$

$b)$ $T - T_s = 60e^{120(-0.0091)} = 60 \times 0.335 = 20.1C^{\circ} \Longrightarrow T = 65.6C^{\circ}$

$c)$ $10 = 60e^{-0.0091t} = -0.0091t = -ln6 \Longrightarrow t = 3.3\ hrs.$