University of Al Maarif

SIGNAL PROCESSING

Department of Medical Instruments
Techniques Engineering

Class: 3rd

Lecture 4

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2024-2025

Discrete-Time Fourier Transform (DTFT)

In this chapter we present the Fourier analysis in the context of discrete-time signals (sequences) and systems. The Fourier analysis plays the same fundamental role in discrete time as in continuous time. As we will see, there are many similarities between the techniques of discrete-time Fourier analysis and their continuous-time counterparts, but there are also some important differences.

Consider a discrete-time signal x[n]. Its discrete-time Fourier transform (DTFT) is defined as

$$DTFT\{x[n]\} = X(\Omega) = \sum_{n=-\infty}^{\infty} x[n].e^{-j\Omega n} \qquad (1)$$

where **$\Omega = 2\pi/N_o$**, is discrete time frequency in 'rad/numbers'. The transform X(**Ω**) is frequency domain description of x[n]. Thus, x[n] and X(**Ω**) makes Fourier transform pair as

$$x[n] \overset{DTFT}{\longleftrightarrow} X(\Omega) \qquad (2)$$

and the inverse DTFT which maps the frequency domain description X(**Ω**) back into time is given as

$$IDTFT\{X(\Omega)\} = x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega).e^{j\Omega n}\, d\Omega \qquad (3)$$

Discrete-time Fourier Transform of Unit impulse:

$$\boldsymbol{x[n] = \delta[n]}$$

By definition

$$X(\Omega) = \sum_{n=-\infty}^{\infty} \delta[n].e^{-j\Omega n} = e^{-j\Omega n}|_{n=0} = 1$$

Therefore, $$\delta[n] \overset{DTFT}{\longleftrightarrow} 1 \qquad (4)$$

Z - Transform

The discrete-time counterpart of Laplace transform is z-transform. The frequency domain analysis of discrete-time system allows us to represent any arbitrary signal x[n] as a sum of exponentials of the form z^n. There are two varieties of z-transform: bilateral and unilateral. The bilateral one, also known as two-sided z-transform can handle all causal and non-causal signals. It provides insights about system's characteristics such as stability, causality and frequency response. The unilateral one, also known as one-sided z-transform can handle only causal signals.

Consider a discrete-time signal x[n]. Its z-transform is defined as

$$ZT\{x[n]\} = X(z) = \sum_{n=-\infty}^{\infty} x[n].Z^{-n} \qquad (5)$$

Where 'z' is complex variable which is given as

$$Z = re^{j\Omega}$$

r = distance from origin in z-plane (magnitude of z)

Ω = an angle from positive real axis in z-plane (angle of z)

The signal x[n] and X[z] makes z-transform pair as

$$x[n] \overset{ZT}{\longleftrightarrow} X(z) \qquad (6)$$

and the inverse z-transform which gives x[n] back from X(z) can be obtained as

$$IZT\{X(z)\} = ZT^{-1}\{X[z]\} = x[n] = \frac{1}{2\pi j}\oint X(z).Z^{n-1}dz \qquad (7)$$

The symbol $\oint$ indicates an integration in counter clockwise direction around a closed path in the complex plane.

1. Relationship between z-transform and DTFT

For a signal x[n], its z-transform X(z) is given by

$$ZT\{x[n]\} = X(z) = \sum_{n=-\infty}^{\infty} x[n].Z^{-n}$$

Put $Z = re^{j\Omega}$, we get

$$X(z) = \sum_{n=-\infty}^{\infty} x[n].(re^{j\Omega})^{-n}$$

$$X(z) = \sum_{n=-\infty}^{\infty} (x[n].r^{-n}).e^{-j\Omega n}$$

$$X(z) = DTFT\{(x[n].r^{-n})\} \qquad (8)$$

2. Region of Convergence for z-transform

The z-transform is guaranteed to converge if $x[n].r^{-n}$ is absolutely summable.

Hence $$\sum |x[n].r^{-n}| < \infty$$

This condition shows the X(z) will be finite

$$|X(z)| < \infty$$

Thus, ROC consist of those values of:
$|Z| = |re^{j\Omega}| = r$ for which z-transform converges.

Properties of ROC

(i) The ROC of X(z) consists of a ring in the z-plane centered about the origin.

(ii) The ROC does not contain any poles.

(iii) If x[n] is of finite duration, then the ROC is entire z-plane, except possibly $z = 0$ and/or $z = \infty$.

(iv) If x[n] is a right-sided sequence, and if the circle $|z| = r_0$ is in the ROC then all finite values of z for which $|z| > r_0$ will also be in the ROC.

(v) If x[n] is a left-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then all values of z for which $0 < |z| < r_0$ will also be in the ROC.

(vi) If x[n] is two sided, and if the circle $|z| = r_0$ is in the ROC, then the ROC will consist of a ring in the z-plane that includes the circle $|z|= r_0$

(vii) If the z-transform X(z) of x[n] is rational, then its ROC is bounded by poles or extends to infinity.

(viii) If the z-transform X(z) of x[n] is rational and if x[n] is right sided, then the ROC is the region in the z-plane outside the outer most pole i.e., outside the circle of radius equal to the largest magnitude of the poles of X(z). Furthermore, if x[n] is causal (i.e., if it is right sided and equal to 0 for $n < 0$), then the ROC also includes $z = \infty$,

(ix) If the z-transform X(z) of x[n] is rational, and if x[n] is left sided, then the ROC is the region in the z-plane inside the innermost non-zero pole i.e. magnitude of the poles of X(z) other than any at $z = 0$ and extending inward to and possibly including $z = 0$. In particular, if x[n] is anti-causal (i.e., if it is left sided and equal to 0 for $n > 0$), then the ROC also includes $z = 0$.

3. The z-plane and poles and zeros

The graphical representation of complex number $z = re^{j\Omega}$ in terms of complex plane is called as z-plane.

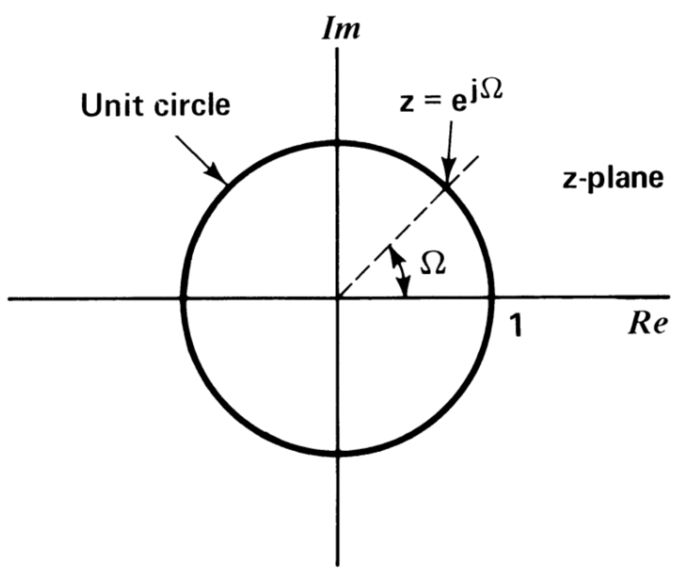

Figure 1 z-plane

As we have seen z-transform reduces to DTFT on the contour in z-plane with r = 1, Thus $X(z)|_{z\,=\,e^{j\Omega}} = X(\Omega)$

$z = e^{j\Omega}$ represents a unit circle z-plane as shown in the figure.

It is concluded that DTFT corresponds to z-transform evaluated on unit circle.

4. Poles and Zeros

Consider a ratio of two polynomials, say F(z)

$$F(z) = \frac{N(z)}{D(z)}$$

for above function, poles and zeros are defined as

Zeros: Roots of N(z) = 0 are defined as zeros when function N(z) vanishes. It is represented as zi,

$$\text{i.e.} \lim_{z \to zi} F(z) = 0$$

Poles: Roots of D(z) = 0 are defined as poles when function A(z) becomes infinity. It is represented as pi.

$$\text{i.e.} \lim_{z \to pi} F(z) = \infty$$

Example 4.1: Which one of the following is the correct statement? The region of convergence of z-transform of x[n] consists of the values of z for which $x[n].r^{-n}$ is

(a) absolutely integrable

(b) absolutely summable

(c) unity

(d) < 1

Example 4.2: Find the z-transform of the following signals and comment on ROC of finite duration signals.

(1) $x_1[n] = \{\underset{\uparrow}{1}, 1, 1\}$

(2) $x_2[n] = \{1, 1, \underset{\uparrow}{1}\}$

(3) $x_3[n] = \{1, \underset{\uparrow}{1}, 1\}$

Solution:

(1) $x_1[n] = \{\underset{\uparrow}{1}, 1, 1\}$ is a right-sided signal.

By definition, $$X_1(z) = \sum_{n=0}^{2} x_1[n].Z^{-n}$$

$$X_1(z) = x_1[0].Z^{-0} + x_1[1].Z^{-1} + x_1[2].Z^{-2}$$

$$X_1(z) = 1 + Z^{-1} + Z^{-2}$$

Since $x_1[n]$ is finite duration signal hence ROC is entire z-plane, but $X_1(z)$ becomes unbounded as z→ 0 so ROC will not include z = 0.

Comment: For $x_1[n]$, a right sided finite duration signal ROC is entire z-plane except z = 0.

(2) $x_2[n] = \{1, 1, \underset{\uparrow}{1}\}$ is a left-sided signal.

By definition, $$X_2(z) = \sum_{n=-2}^{0} x_2[n].Z^{-n}$$

$$X_2(z) = x_2[-2].Z^{+2} + x_2[-1].Z^{+1} + x_2[0].Z^{-0}$$

$$X_2(z) = Z^2 + Z + 1$$

Since $x_2[n]$ is a finite duration signal hence ROC is entire z-plane but $X_2(z)$ becomes unbounded as z→ ∞ so ROC will not include z = ∞.

Comment: For $x_2[n]$, a left sided finite duration signal, the ROC is entire z-plane except z = ∞,

(3) $x_3[n] = \{1, \underset{\uparrow}{1}, 1\}$ is a two-sided signal.

By definition, $X_3(z) = \sum_{n=-1}^{1} x_3[n].Z^{-n}$

$$X_3(z) = x_3[-1].Z^{+1} + x_3[0].Z^{-0} + x_3[1].Z^{-1}$$

$$X_3(z) = Z + 1 + Z^{-1}$$

Since $x_3[n]$ is finite duration signal.Hence ROC is entire z-plane but $X_3(z)$ becomes unbounded when z→ 0 and z→ ∞, so ROC will not include z = 0 and z = ∞.

Comment: For $x_3[n]$, a two-sided finite duration signal the ROC is entire z-plane except z = 0 and z = ∞.

Table 1 z-transform conversion

$x[n]$	$X(z)$	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}, \frac{z}{z-1}$	$\lvert z\rvert > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}, \frac{z}{z-1}$	$\lvert z\rvert < 1$
$\delta[n-m]$	z^{-m}	All z except 0 if $(m>0)$ or ∞ if $(m<0)$
$a^n u[n]$	$\frac{1}{1-az^{-1}}, \frac{z}{z-a}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}, \frac{z}{z-a}$	$\lvert z < \lvert a\rvert$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}, \frac{az}{(z-a)^2}$	$\lvert z\rvert > \lvert a\rvert$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}, \frac{az}{(z-a)^2}$	$\lvert z\rvert < \lvert a\rvert$
$(n+1)a^n u[n]$	$\frac{1}{(1-az^{-1})^2}, \left[\frac{z}{z-a}\right]^2$	$\lvert z\rvert > \lvert a\rvert$
$(\cos \Omega_0 n)u[n]$	$\frac{z^2-(\cos \Omega_0)z}{z^2-(2\cos \Omega_0)z+1}$	$\lvert z\rvert > 1$
$(\sin \Omega_0 n)u[n]$	$\frac{(\sin \Omega_0)z}{z^2-(2\cos \Omega_0)z+1}$	$\lvert z\rvert > 1$
$(r^n \cos \Omega_0 n)u[n]$	$\frac{z^2-(r\cos \Omega_0)z}{z^2-(2r\cos \Omega_0)z+r^2}$	$\lvert z\rvert > r$
$(r^n \sin \Omega_0 n)u[n]$	$\frac{(r\sin \Omega_0)z}{z^2-(2r\cos \Omega_0)z+r^2}$	$\lvert z\rvert > r$
$\begin{cases} a^n & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$