University of Al Maarif

SIGNAL PROCESSING

Department of Medical Instruments Techniques Engineering

Class: 3rd

Lecture 3

Lecturer: Mohammed Jabal

2024-2025

Continuous-time Fourier analysis

Fourier series is an approximation process where any general (periodic or aperiodic) signal is expressed as sum of harmonically related sinusoids. It gives us frequency domain (or spectral) representation. If the signal is periodic Fourier series represents the signal in the entire interval (-∞, ∞). i.e. Fourier series can be generalized for periodic signals only.

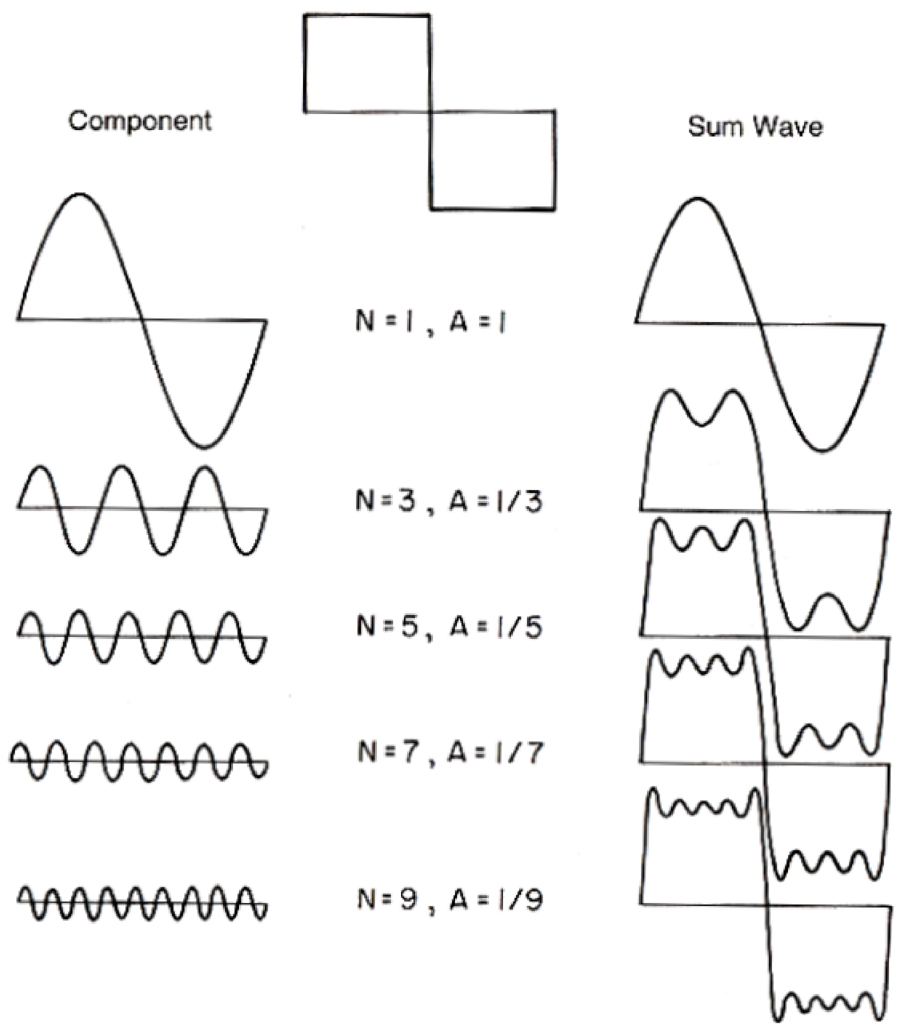

Figure 1 Fourier analysis of a square wave. At the left are the successive harmonics; at the right are the sum waves including each successive harmonic. The graph at the top is the wave being synthesized.

It has been found that square waves are mathematically equivalent to the sum of a sine wave at that same frequency, plus an infinite series of odd-multiple frequency sine waves.

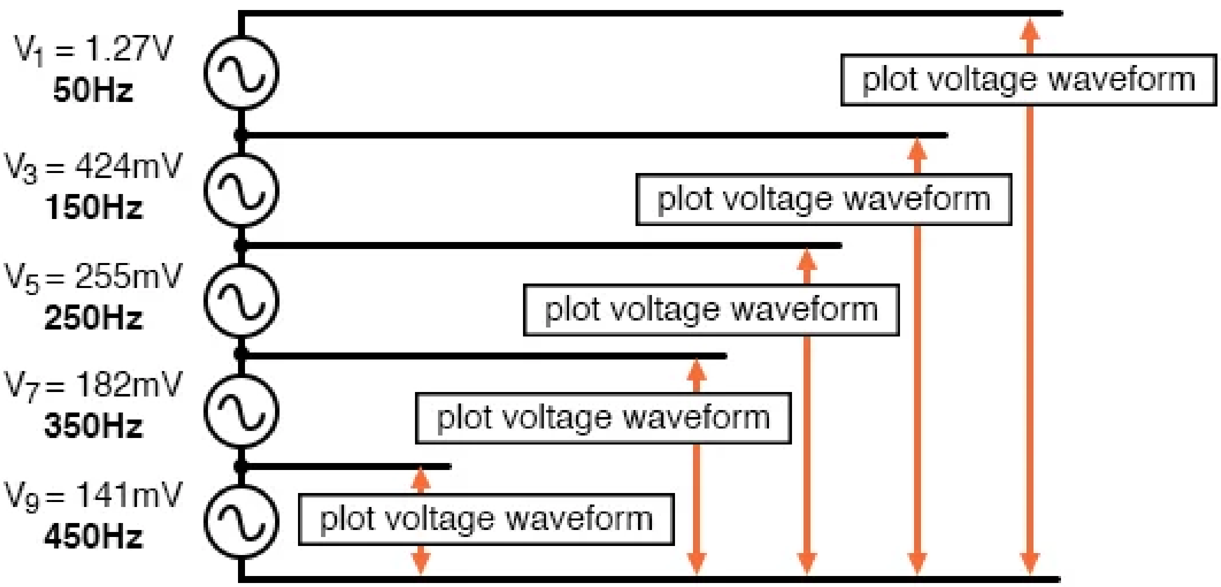

Figure 2 A square wave is approximated by the sum of harmonics

Different Forms of Fourier Series

1. Trigonometric Fourier series
2. Polar form
3. Exponential Fourier series

Trigonometric Fourier Series

A periodic signal x(t) can be expressed as infinite sum of sine or cosine functions that are integral multiples of ω_o

$$x(t) = \boldsymbol{a_o} + \sum_{n=1}^{\infty}[\boldsymbol{a_n} Cos(n\omega_o t) + \boldsymbol{b_n} Sin(n\omega_o t)] \qquad (1)$$

Where **$\omega o = 2\pi/T$** is known as fundamental frequency (rad/sec) and the constant a_o, a_n, and b_n are the Fourier coefficients. The coefficient 'a_o' is the dc component. The process of determining the coefficients is called Fourier analysis. The following trigonometric integrals are very useful in Fourier analysis.

$$\boldsymbol{a_o} = \frac{1}{T}\int_0^T x(t)dt \qquad (2)$$

$$\boldsymbol{a_n} = \frac{2}{T}\int_0^T x(t).Cos(n\omega_o t)dt \qquad (3)$$

$$\boldsymbol{b_n} = \frac{2}{T}\int_0^T x(t).Sin(n\omega_o t)dt \qquad (4)$$

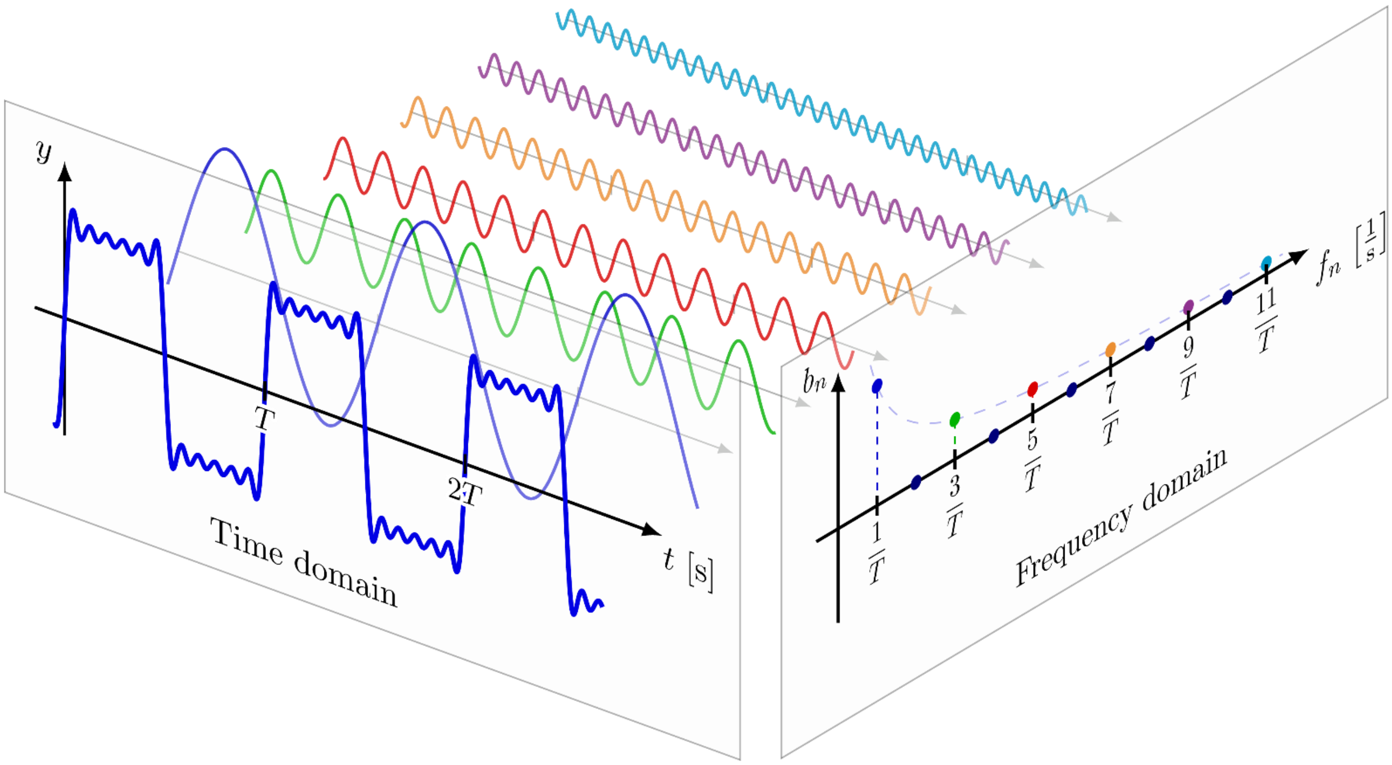

Figure 3 Fourier expansion of square wave time and frequency domains

Continuous-Time Fourier Transform (CTFT)

In the previous chapter, we saw that Fourier series can be generalized only for periodic signals not for aperiodic one i.e. any periodic signal is represented as linear combination of hormonally related complex exponentials. This limitation is overcome through Fourier transform which is applicable for both aperiodic and periodic signals. The Fourier transform gives a frequency domain description of time domain signal. An aperiodic signal is one which is periodic with an infinite period. As the period increases the fundamental frequency decreases which makes aperiodic signal infinitesimally close in frequency and the representation in terms of linear combination takes the form of an integral rather than sum. The resulting spectrum is called Fourier transform. Thus, Fourier transform is extension of Fourier series for aperiodic signals.

Consider a continuous time signal x(t). Its Fourier transform is defined as

$$FT[x(t)] = X(\omega) = \int_{-\infty}^{\infty} x(t).e^{-j\omega t}dt \qquad (5)$$

The transform X(ω) is frequency domain description of x(t) and right side of equation is known as Fourier integral. Thus x(t) and X(ω) makes Fourier transform pair as

$$x(t) \overset{FT}{\leftrightarrow} X(\omega) \qquad (6)$$

Fourier Transform of Some Basic Signals

(i) Unit impulse: $\boldsymbol{x(t) = \delta(t)}$

By definition

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t).e^{-j\omega t}dt = 1$$

Therefore, $$\delta(t) \overset{FT}{\leftrightarrow} 1 \qquad (7)$$

Figure 4 Unit impulse and its Fourier transform

(ii) Exponential Signal: $\boldsymbol{x(t) = e^{-at}u(t), a > 0}$

By definition

$$X(\omega) = \int_{-\infty}^{\infty} e^{-at}u(t).e^{-j\omega t}dt = \int_{0}^{\infty} e^{-(a+j\omega)t}dt$$

$$X(\omega) = \frac{-1}{a+j\omega}\left[e^{-(a+j\omega)t}\right]_0^{\infty} = \frac{1}{a+j\omega}, a > 0$$

$$e^{-at}u(t) \overset{FT}{\leftrightarrow} \frac{1}{a+j\omega}, a > 0 \qquad (8)$$

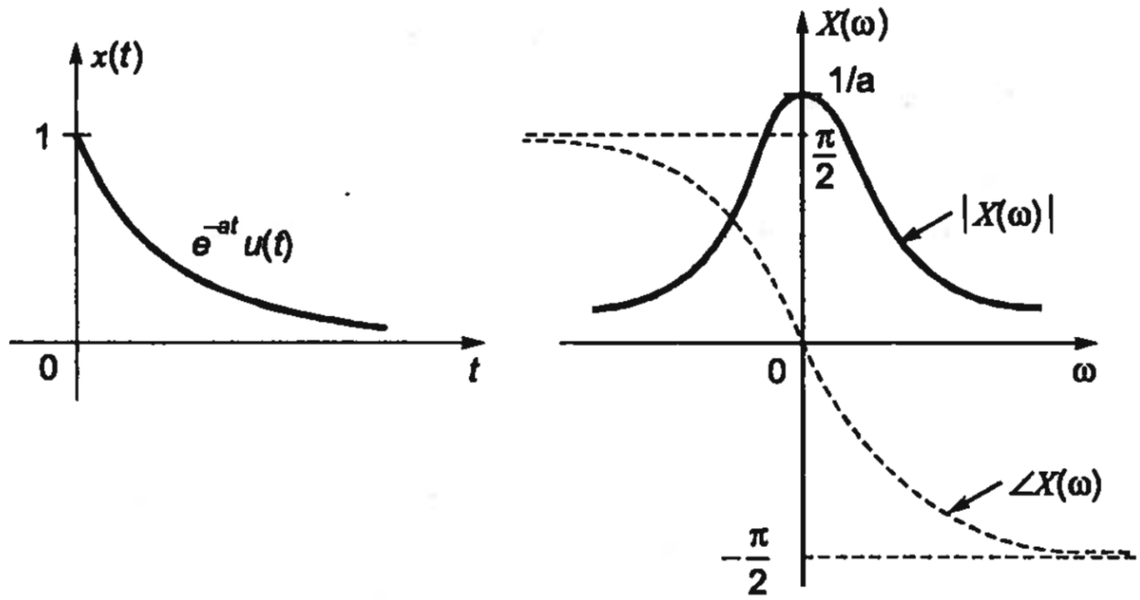

Figure 5 Exponential Signal and its Fourier transform

(iii) Signum Function: $\boldsymbol{x(t) = sgn(t)}$

$$x(t) = sgn(t) = \begin{cases} 1, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0 \end{cases}$$

Also
$$sgn(t) = u(t) - u(-t)$$

This signal is not absolutely integrable so we calculate Fourier transform of sgn(t) as limiting case of sum of exponential $[e^{-at}.u(t) - e^{at}u(-t)]$, $a \to 0$.

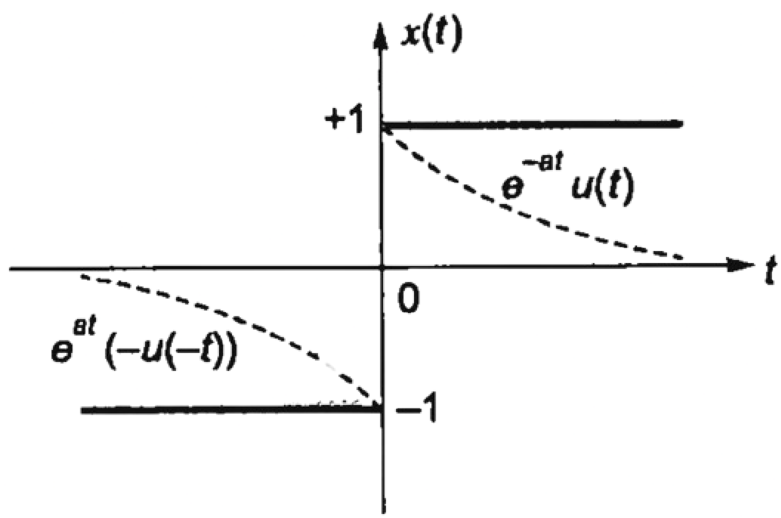

Figure 6 sgn function as limiting case of sum of exponential

$$x(t) = sgn(t) = \lim_{a\to 0}[e^{-at}.u(t) - e^{at}.u(-t)]$$

Taking Fourier transform of this equation

$$X(\omega) = \lim_{a\to 0}\left[\frac{1}{a+j\omega} - \frac{1}{a-j\omega}\right] = \lim_{a\to 0}\left[\frac{-2j\omega}{a^2+\omega^2}\right] = -\frac{2j\omega}{\omega^2} = \frac{2}{j\omega}$$

$$sgn(t) \overset{FT}{\leftrightarrow} \frac{2}{j\omega} \qquad (9)$$

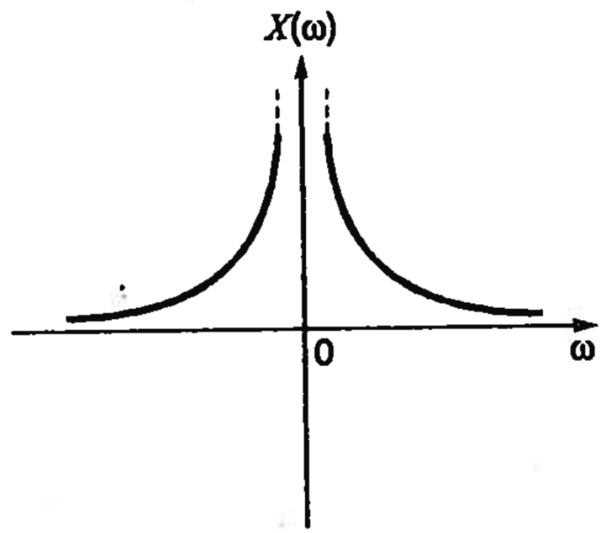

Figure 7 Fourier transform of Signum Function

(iv) Unit Step: $\boldsymbol{x(t) = u(t)}$

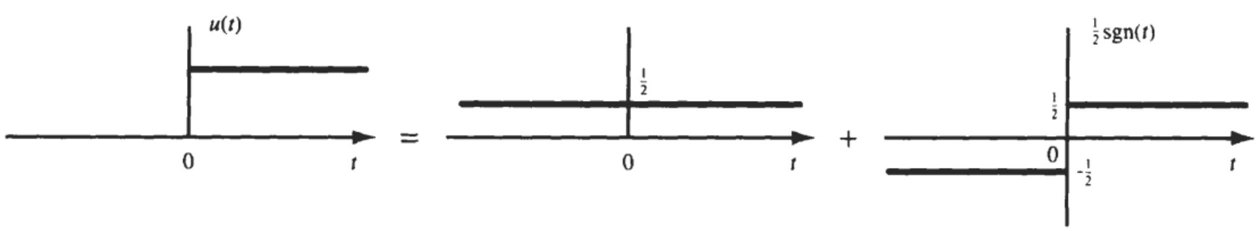

Figure 8 Relation between Signum function and Unit step function

$$u(t) = \frac{1 + sgn(t)}{2}$$

Note: $\boldsymbol{A_o \overset{FT}{\leftrightarrow} 2\pi\, A_o \delta(\omega)}$

$$FT[u(t)] = FT\left[\frac{1}{2} + \frac{1}{2}sgn(t)\right] = \frac{1}{2}FT[1] + \frac{1}{2}FT[sgn(t)]$$

$$X(\omega) = \frac{1}{2}2\pi\delta(\omega) + \frac{1}{2}\frac{2}{j\omega} = \pi\delta(\omega) + \frac{1}{j\omega}$$

$$u(t) \overset{FT}{\leftrightarrow} \pi\delta(\omega) + \frac{1}{j\omega} \qquad (10)$$

(v) $\boldsymbol{x(t) = Cos(\omega_o t)}$

$$Cos(\omega_o t) = \frac{e^{j\omega_o t} + e^{-j\omega_o t}}{2} \qquad \text{Euler's formula}$$

$$X(\omega) = FT[Cos(\omega_o t)] = \frac{1}{2}FT\left[e^{j\omega_o t} + e^{-j\omega_o t}\right]$$

Note: $\boldsymbol{e^{\pm j\omega_o t} \overset{FT}{\leftrightarrow} 2\pi\delta(\omega \mp \omega_o)}$

$$X(\omega) = \frac{1}{2}[2\pi\delta(\omega - \omega_o) + 2\pi\delta(\omega + \omega_o)]$$

$$X(\omega) = \pi[\delta(\omega - \omega_o) + \delta(\omega + \omega_o)]$$

$$Cos(\omega_o t) \overset{FT}{\leftrightarrow} \pi[\delta(\omega - \omega_o) + \delta(\omega + \omega_o)] \qquad (11)$$

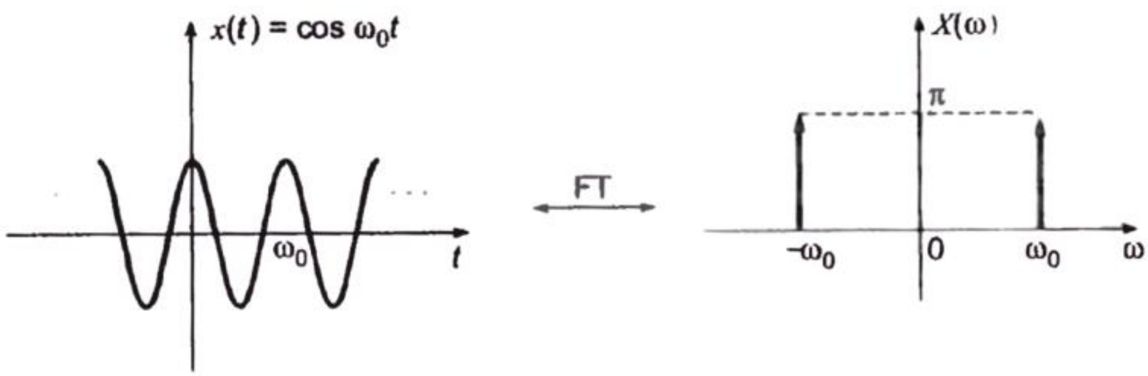

Figure 9 $Cos(\omega_o t)$ Signal and its Fourier transform

(vi) $\boldsymbol{x(t) = Sin(\omega_o t)}$

$$Sin(\omega_o t) = \frac{e^{j\omega_o t} - e^{-j\omega_o t}}{2j} \qquad \text{Euler's formula}$$

$$X(\omega) = FT[Sin(\omega_o t)] = \frac{1}{2j} FT\left[e^{j\omega_o t} - e^{-j\omega_o t}\right]$$

Note: $\boldsymbol{e^{\pm j\omega_o t} \overset{FT}{\leftrightarrow} 2\pi\delta(\omega \mp \omega_o)}$

$$X(\omega) = \frac{1}{2j}[2\pi\delta(\omega - \omega_o) - 2\pi\delta(\omega + \omega_o)]$$

$$X(\omega) = \frac{\pi}{j}[\delta(\omega - \omega_o) - \delta(\omega + \omega_o)]$$

$$Sin(\omega_o t) \overset{FT}{\leftrightarrow} \frac{\pi}{j}[\delta(\omega - \omega_o) - \delta(\omega + \omega_o)] \qquad (12)$$

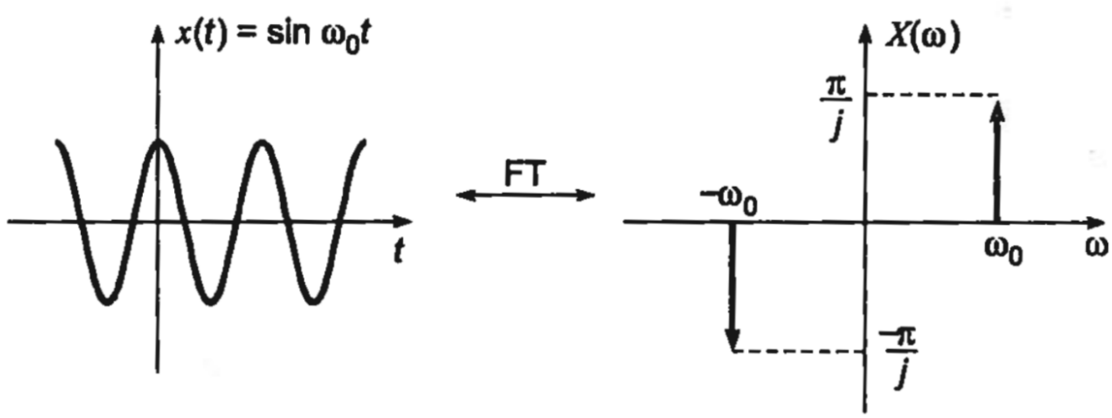

Figure 10 $Sin(\omega_o t)$ Signal and its Fourier transform

Inverse Fourier Transform

The inverse Fourier transform maps the frequency domain description X(ω) back into time domain is given as

$$IFT = FT^{-1}[X(\omega)] = x(t) \qquad (13)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega).\, e^{j\omega t} d\omega \qquad (14)$$

(i) $\boldsymbol{X(\omega) = \delta(\omega)}$

By definition of IFT

$$FT^{-1}[X(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega).\, e^{j\omega t} d\omega = \frac{1}{2\pi}$$

Therefore, $$\delta(\omega) \overset{IFT}{\longleftrightarrow} \frac{1}{2\pi}$$

Thus, $$A_o \overset{FT}{\leftrightarrow} 2\pi\, A_o\delta(\omega) \qquad (15)$$

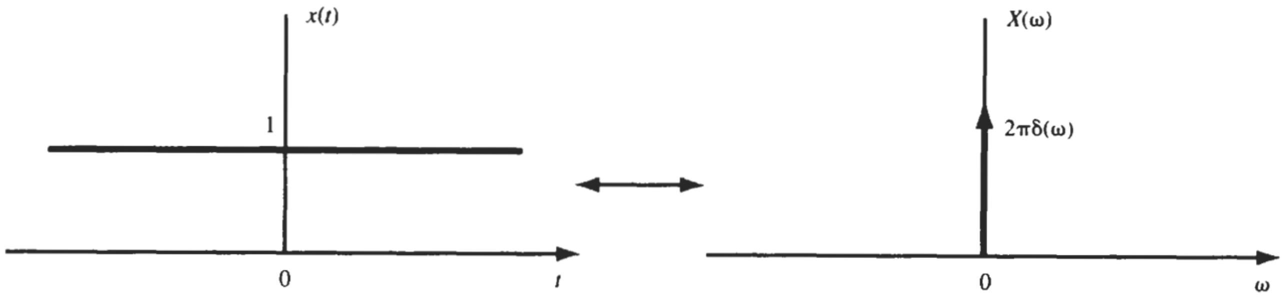

Figure 11 DC Signal and its Fourier transform

(ii) $\boldsymbol{X(\omega) = \delta(\omega - \omega_o)}$

By definition of IFT

$$FT^{-1}[X(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(\omega-\omega_o).\,e^{j\omega t}d\omega = \frac{1}{2\pi}.\,e^{j\omega_o t}$$

Therefore, $$\delta(\omega-\omega_o) \overset{IFT}{\longleftrightarrow} \frac{1}{2\pi}.\,e^{j\omega_o t}$$

Thus, $$e^{j\omega_o t} \overset{FT}{\leftrightarrow} 2\pi\delta(\omega-\omega_o) \qquad (17)$$

Fourier Transform Table

No.	x(t)	X(ω)
1	$\delta(t)$	1
2	$e^{-at}u(t)$	$\frac{1}{a+j\omega}\ ,a>0$
3	$sgn(t)$	$\frac{2}{j\omega}$
4	$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$
5	$Cos(\omega_o t)$	$\pi[\delta(\omega-\omega_o)+\delta(\omega+\omega_o)]$
6	$Sin(\omega_o t)$	$\frac{\pi}{j}[\delta(\omega-\omega_o)-\delta(\omega+\omega_o)]$
7	A_o	$2\pi\, A_o\delta(\omega)$
8	$e^{j\omega_o t}$	$2\pi\delta(\omega-\omega_o)$

Fourier Transform Properties

1. **Linearity:**

If $x_1(t) \overset{FT}{\leftrightarrow} X_1(\omega)$ and $x_2(t) \overset{FT}{\leftrightarrow} X_2(\omega)$

Then $\alpha x_1(t)+\beta x_2(t) \overset{FT}{\leftrightarrow} \alpha X_1(\omega)+\ \beta X_2(\omega)$

Meaning: The FT of linear combination of the signals is equal to linear combination of their Fourier transforms.

2. **Time shifting:**

If $$x(t) \overset{FT}{\leftrightarrow} X(\omega)$$

Then $$x(t - t_o) \overset{FT}{\leftrightarrow} e^{-j\omega t_o} X(\omega)$$

Meaning: A shift of '$\boldsymbol{t_o}$' in time domain is equivalent to introducing a phase shift of **–** $\boldsymbol{\omega t_o}$. But amplitude remains same.

3. **Frequency shifting:**

If $$x(t) \overset{FT}{\leftrightarrow} X(\omega)$$

Then $$x(t).\,e^{j\omega_o t} \overset{FT}{\leftrightarrow} X(\omega - \omega_o)$$

Meaning: Shifting the frequency by '$\boldsymbol{\omega_o}$' in frequency domain is equivalent to multiplying the time domain signal by $\boldsymbol{e^{j\omega_o t}}$**.**

4. **Time scaling:**

If $$x(t) \overset{FT}{\leftrightarrow} X(\omega)$$

Then $$x(at) \overset{FT}{\leftrightarrow} \frac{1}{|a|} X(\frac{\omega}{a})$$, 'a' any real constant.

Meaning: Compression of a signal in time domain is equivalent to expansion in frequency domain and vice versa.

5. **Time reversal:**

If $$x(t) \overset{FT}{\leftrightarrow} X(\omega)$$

Then $$x(-t) \overset{FT}{\leftrightarrow} X(-\omega)$$

6. **Time Convolution:**

If $$x_1(t) \overset{FT}{\leftrightarrow} X_1(\omega) \text{ and } x_2(t) \overset{FT}{\leftrightarrow} X_2(\omega)$$

Then $$x_1(t) * x_2(t) \overset{FT}{\leftrightarrow} X_1(\omega).X_2(\omega)$$

Meaning: The convolution in time domain is equivalent to multiplication in frequency domain.

7. Duality:

If $$x(t) \overset{FT}{\leftrightarrow} X(\omega)$$

Then $$X(t) \overset{FT}{\leftrightarrow} 2\pi x(-\omega)$$

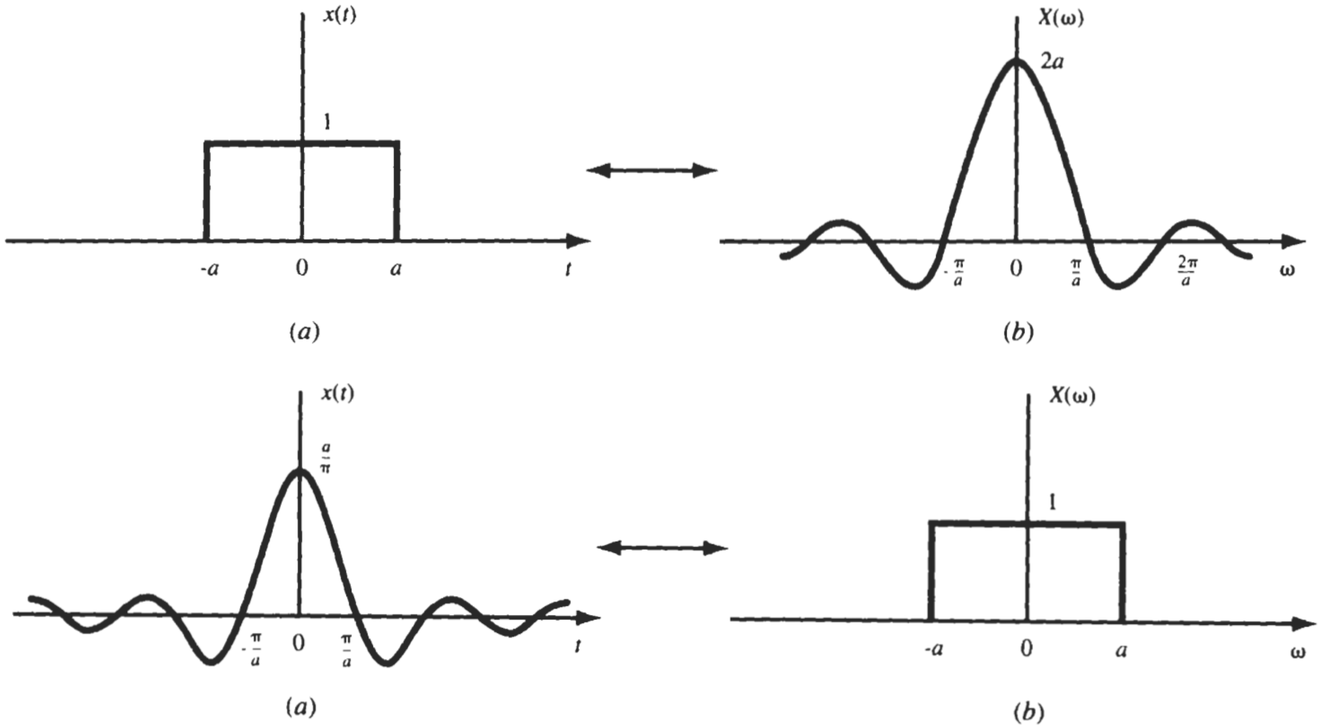

Figure 12 Duality property