University of Al Maarif

SIGNAL PROCESSING

Department of Medical Instruments Techniques Engineering

Class: 3rd

Lecture 2

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Signal Sampling and Reconstruction

The process of sampling is a bridge between continuous-time and discrete-time systems figure 1. Sampling is a process of converting a continuous-time signal to discrete-time signal and under certain condition the continuous-time signal can be completely recovered from its sampled sequence.

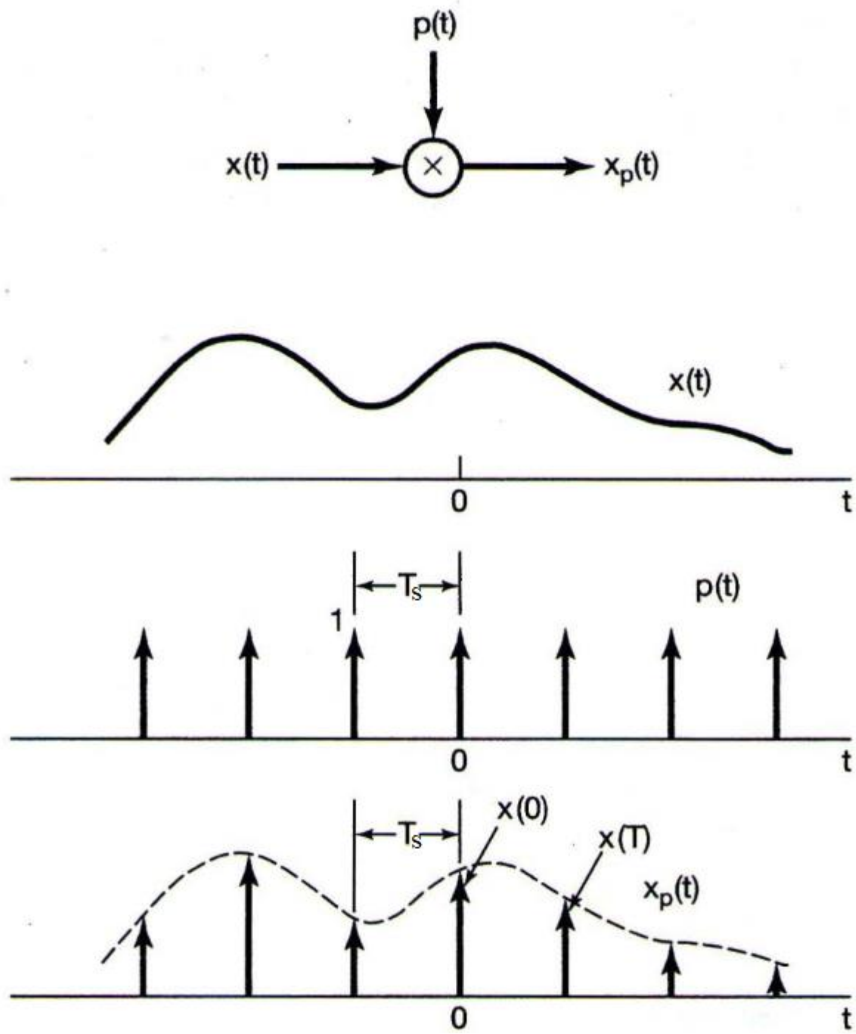


Figure 1 Impulse-train sampling

1. The Sampling Theorem

According to sampling theorem "A band limited signal of finite energy can be completely reconstructed from its samples taken uniformly at a rate of **(fs ≥ 2fm sample/sec)**.

In other words, the minimum sampling rate is **fs = 2fm Hz**.

To prove sampling theorem, let x(t) be continuous-time finite energy signal whose spectrum is band limited to $\mathbf{f_m}$ Hz (i.e. X(f) = 0 for | f | > fm).

The input x(t) is sampled at the rate of **fs** Hz by multiplying x(t) by a periodic impulse train p(t) with period $\mathbf{T_s = 1/f_s}$.

The sampled signal '$x_p(t)$' consists of impulses spaced every $\mathbf{T_s}$ second whose strength (area) is equal to instantaneous value of input signal x(t):

$$x_p(t) = x(t).p(t) \quad (1)$$

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s) \quad (2)$$

Where: x(t) continuous-time signal, p(t) periodic impulse train, $x_p(t)$ sampled signal, T_s sampling period.

$$x_p(t) = x(t).\sum_{n=-\infty}^{+\infty} \delta(t - nT_s) \quad (3)$$

$$x_p(t) = \sum_{n=-\infty}^{+\infty} x(nT_s)\delta(t - nT_s) \quad \text{(4) [Sampling property of impulse]}$$

Taking Fourier transform of equation (2) is

$$p(f) = \frac{1}{T_s}\sum_{n=-\infty}^{+\infty} \delta(f - nf_s) \quad (5)$$

Let's evaluate the Fourier transform of the output signal $x_p(t)$

$$x(t).p(t) \overset{FT}{\Leftrightarrow} x(f) * p(f) \quad (6)$$

Multiplication in the time domain is convolution in the frequency domain

$$x_p(f) = x(f) * p(f) = x(f) * \frac{1}{T_s}\sum_{n=-\infty}^{+\infty} \delta(f - nf_s) \quad (7)$$

Where * denotes convolution. Next, we apply simple properties of convolution to movie x(f) inside the sum; that is, $x_p(f)$ becomes

$$x_p(f) = \frac{1}{T_s}\sum_{n=-\infty}^{+\infty} x(f - nf_s) \quad (8)$$

This result means that the spectrum $x_p(f)$ consists of x(f) repeating periodically with period $\mathbf{T_s=1/f_s}$. We have assumed spectrum x(t) is band limited to $\mathbf{f_m}$.

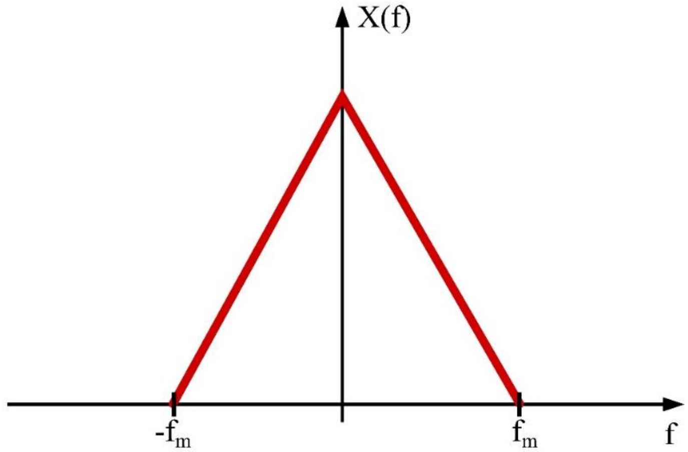


Figure 2 X(f) spectrum

The sampling frequency can take three possible values with respect to spectrum width ($2f_m$)

(i) $f_s > 2f_m$, (ii) $f_s = 2f_m$, (iii) $f_s < 2f_m$

Case-1 Oversampling: $f_s > 2f_m$

Let $\mathbf{f_s = 3f_m}$, the spectrum of sampled signal will be

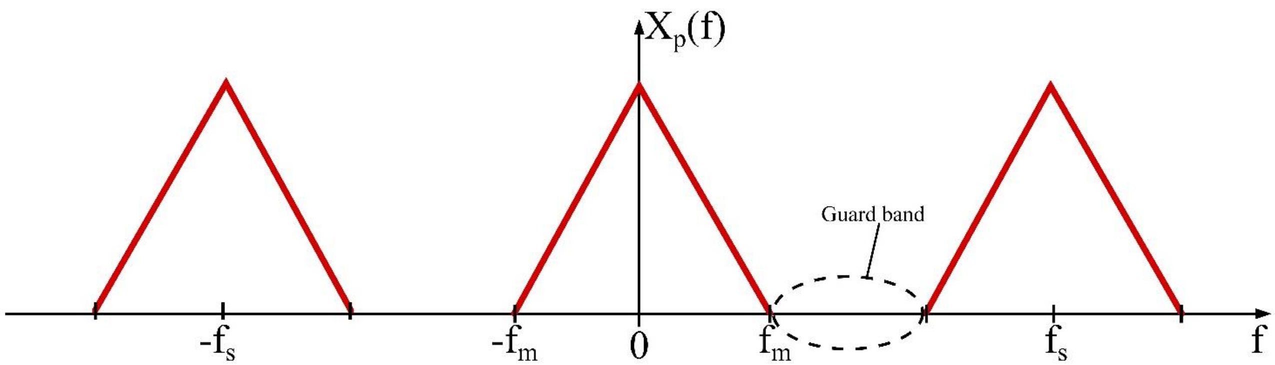


Figure 3 Spectrum of sampled signal with $f_s > 2f_m$

We see for $\mathbf{f_s > 2f_m}$, there is no overlap between the shifted spectrum of X(f). Thus, as long as the sampling frequency $\mathbf{f_s}$ is greater than the twice the signal bandwidth ($2\mathbf{f_m}$), x(t) can be recovered by passing the sampled signal x(t) through ideal or practical low pass filter having cutoff frequency between $\mathbf{f_m}$ and $\mathbf{(f_s - f_m)}$.

$$Guard\ band = (f_s - f_m) - f_m = f_s - 2f_m \qquad (9)$$

Case-2 Nyquist Rate: $f_s = 2f_m$

The spectrum of sampled signal will be

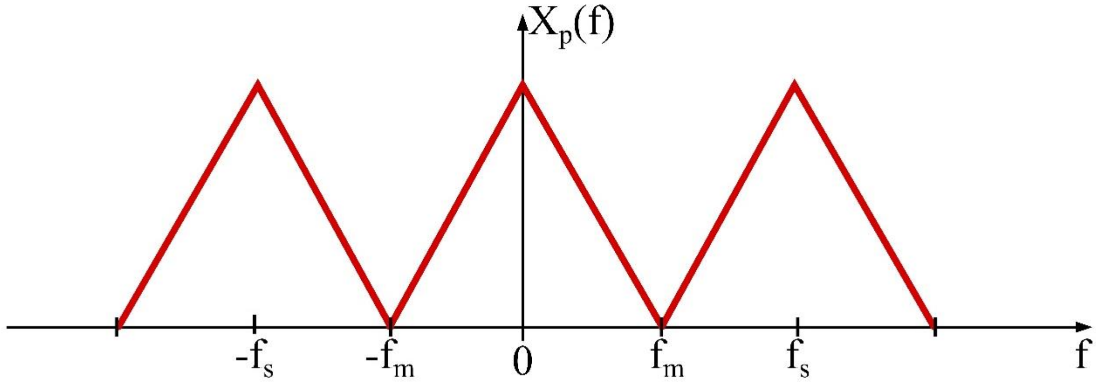


Figure 4 Spectrum of sampled signal with $f_s = 2f_m$

We see for $\mathbf{f_s = 2f_m}$ there is no overlap between shifted spectrum of X(f). Consequently x(t) can be recovered from its sampled signal by means of an ideal low pass filter.

Case-3 Undersampling: $f_s < 2f_m$

Let $\mathbf{f_s = f_m}$, the spectrum of sampled signal will be

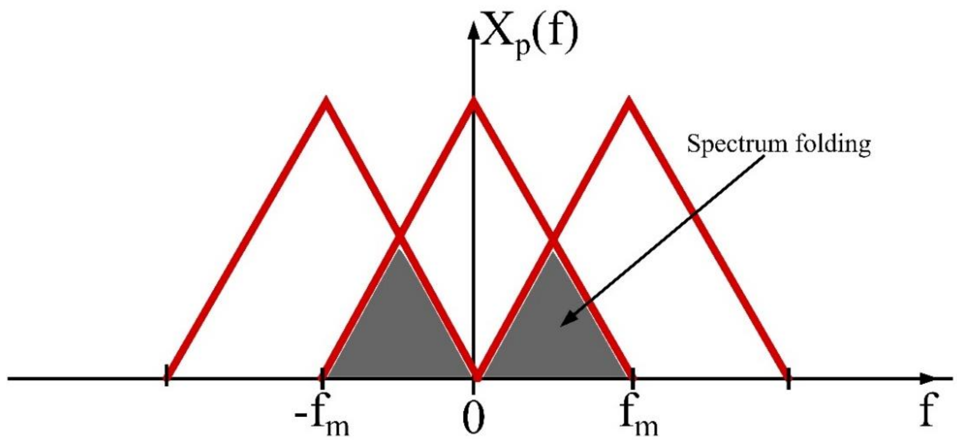


Figure 5 Spectrum of sampled signal with $f_s < 2f_m$

We see for $\mathbf{f_s < 2f_m}$ there is overlap between shifted spectrum of X(f). Consequently, the signal x(t) cannot be exactly recovered from its sampled signal.

2. Aliasing or Spectrum Folding

When $\mathbf{f_s < 2f_m}$, the copies overlap, and hence, the perfect reconstruction becomes impossible. If this Nyquist criterion is not considered, the folded back portion overlaps the original spectrum. This results a new shape of the reconstructed spectrum filtered by LPF. And this, undoubtedly, gives a signal different from X(t). This problem is called "aliasing". Two conditions are necessary to avoid the aliasing:

(1) The input signal must be limited according the Nyquist condition, i.e. $\mathbf{f_m \leq f_s/2}$.

(2) The sampling frequency must sufficiently greater than the maximum frequency component of the signal, i.e. $\mathbf{f_s \geq 2f_m}$. This condition is called "Nyquist Rate".

Note: Aliasing is an irreversible process once aliasing has occurred then signal cannot be recovered back.

Example 2.1: Statement (I): Aliasing occurs when the sampling frequency is less than twice the maximum frequency in the signal. Statement (II): Aliasing is a reversible process.

(a) Both Statement (I) and Statement (II) are individually true and Statement (II) is the correct explanation of Statement (I).

(b) Both Statement (I) and Statement (II) are individually true but Statement (II) is not the correct explanation of Statement (I).

(c) Statement (I) is true but Statement (II) is false.

(d) Statement (I) is false but Statement (II) is true.

Example 2.2: Specify the Nyquist rate for following signals

(1) $x_1(t) = cos\,(2\pi \times 10^3 t)$

(2) $x_2(t) = cos\,(2\pi \times 10^3 t) + cos\,(6\pi \times 10^3 t)$

Solution:

(1)

$$x_1(t) = cos\,(2\pi \times 10^3 t)$$

$$\omega_m = 2\pi \times 10^3 rad/sec$$

$$f_m = 10^3\ Hz$$

$$f_s = 2f_m = 2 \times 10^3\ Hz$$

$$Nyquist\ rate = 2\ kHz$$

(2)

$$x_2(t) = cos\,(2\pi \times 10^3 t) + cos\,(6\pi \times 10^3 t)$$

$$f_{m1} = 10^3\ Hz$$

$$f_{m2} = 3 \times 10^3\ Hz$$

$$f_m = \max(f_{m1}, f_{m2}) = 3\ kHz$$

$$f_s = 2f_m = 2 \times 3\ kHz$$

$$Nyquist\ rate = 6\ kHz$$

3. Reconstruction

To perform successful sampling, we keep $\mathbf{f_s \geq 2f_m}$. This ensures the signal X(f) always contained perfectly in Xp(f). As a result, X(f) can be recovered exactly by simply passing Xp(f) through a LPF that with cutoff frequency ($\mathbf{f_c}$) between $\mathbf{f_m}$ and $\mathbf{(f_s - f_m)}$.

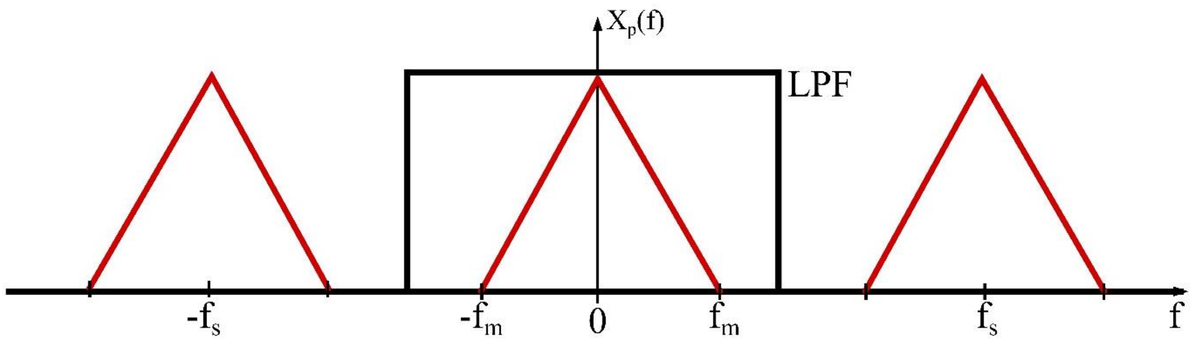
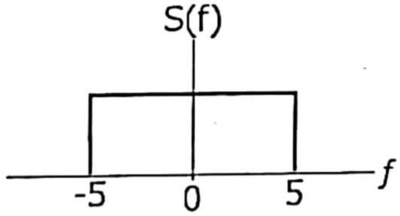
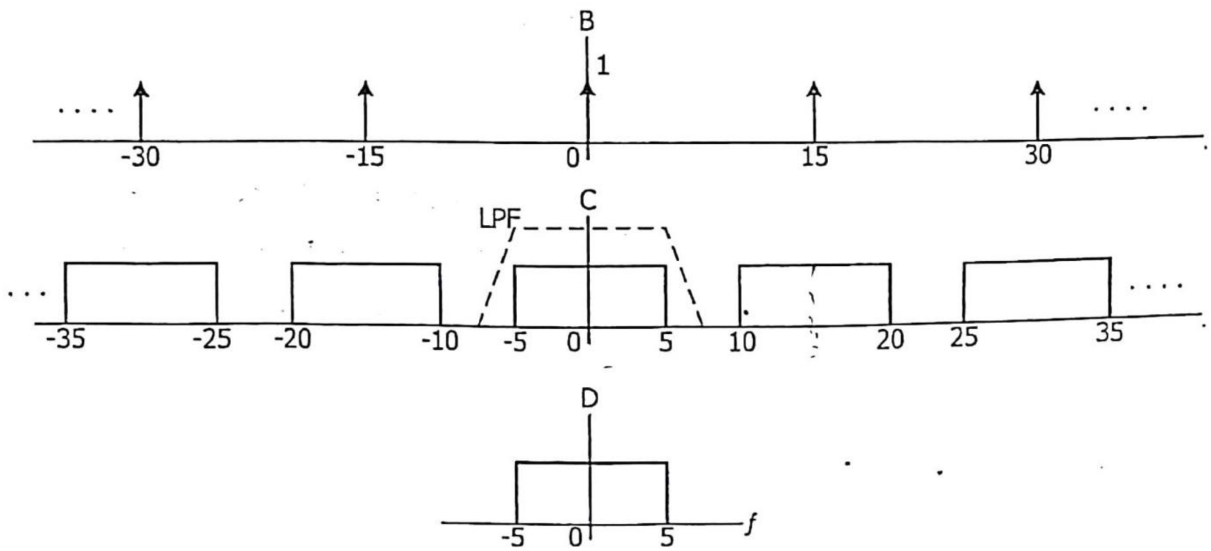


Figure 6 Low Pass Filter

Example 2.3: A signal s(t) is ideally sampled at a sampling rate (a) 15Hz, (b) 9Hz, (c) 10Hz. If the sampled signal, then passed through an ideal LPF with cutoff frequency equal 6Hz. Plot the spectrum of the signal at each step. State if there is any aliasing.



Solution:



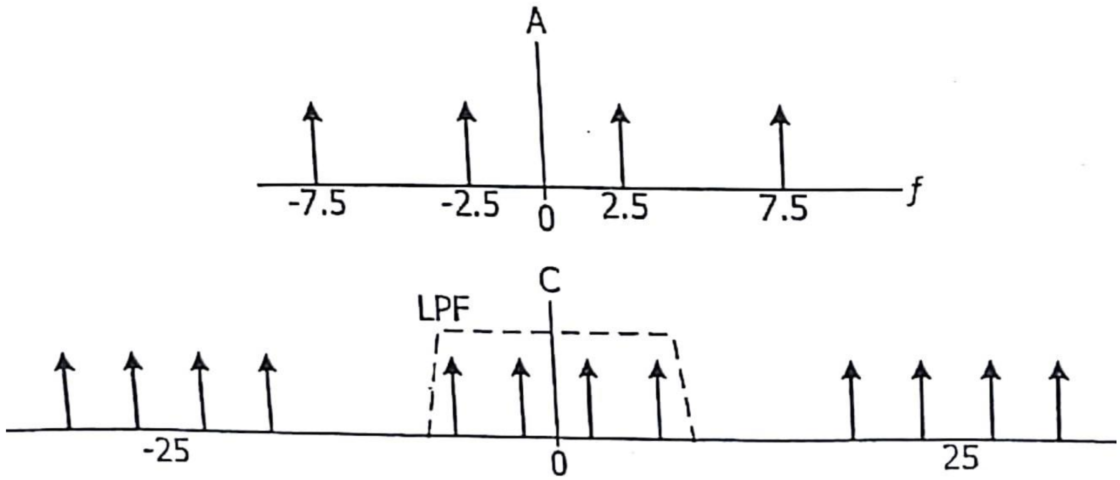
Now work for: (b) and (c) and compare.

Example 2.4: Repeat the above example if the cutoff frequency of the filter is 10Hz and fs =25Hz and the analog signal is $s(t) = 2cos(5\pi t) - 2cos(15\pi t)$.

Solution: $s(t) = 2cos(5\pi t) - 2cos(15\pi t)$ and its frequency components can be written as:

$$f_{m1} = 2.5\ Hz$$

$$f_{m2} = 7.5\ Hz$$



Example 2.5: We can hear sounds with frequency components between 20 Hz and 20 kHz. What is the maximum sampling interval T_s that can be used to sample a signal without loss of audible information?

(a) 100 µs, (**b)** 50 µs, (**c)** 25 µs, (**d)** 100π µs, (**e)** 50π µs, (**f)** 25π µs

Solution:

$$f_s = 2f_m = 2 \times 20\ kHz = 40kHz$$

$$T_s = \frac{1}{f_s} = 25\mu s$$

4. Practical Sampling

A Sampling procedure discussed in the class using Impulse train is only theoretically possible. In reality, Sampling is done by using a pulse train in two different ways:

a. Natural Sampling

Out of the two practical procedures this is the one easy to realize mathematically but relatively difficult to implement practically. The resulting sampled signal will sustain the natural variation of the signal in the pulse duration and hence the name Natural Sampling also known as zero-order hold sampling, involves taking discrete interval samples of a continuous signal, similar to uniform sampling.

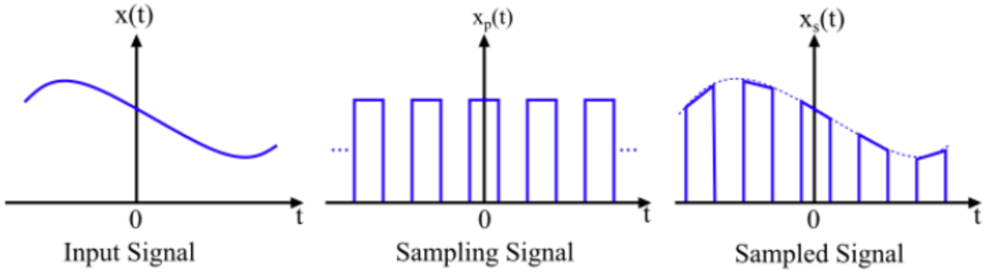


Figure 7 Natural Sampling

b. Flattop Sampling

Out of the two practical Sampling procedures this is difficult to realize mathematically but relatively easy to implement practically. The resultant sampled signal will hold the instantaneous values of the signal for the entire pulse duration resulting in a flat-top waveform.

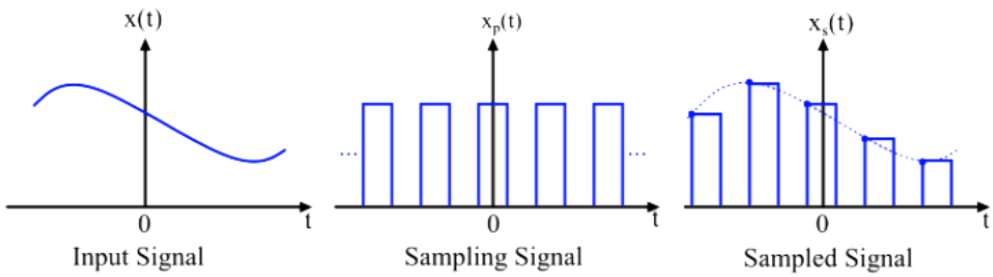


Figure 8 Flattop Sampling

Convolution

Two most important attributes of systems are linearity and time-invariance. In this lecture we develop the fundamental input-output relationship for systems having these attributes. It will be shown that the input-output relationship for LTI systems is described in terms of a convolution operation. The importance of the convolution operation in LTI systems stems from the fact that knowledge of the response of an LTI system to the unit impulse input allows us to find its output to any input signals.

The impulse response h (t) of a continuous-time LTI system (represented by **T**) is defined to be the response of the system when the input is δ (t), that is,

$$h(t) = \boldsymbol{T}\{\delta(t)\} \qquad (10)$$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \qquad (11)$$

Equation (2) is commonly called the convolution integral. Thus, we have the fundamental result that the output of any continuous-time **LTI** system is the convolution of the input x (t) with the impulse response h (t) of the system. Figure 9 illustrates the definition of the impulse response h (t) and the relationship of Eq. (2).

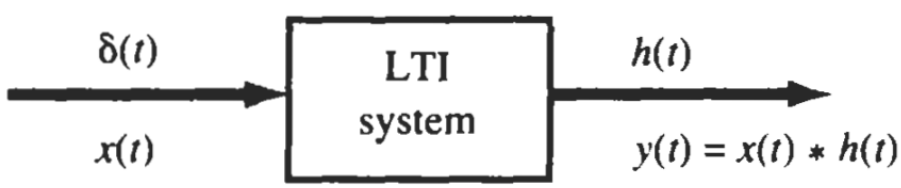


Figure 9 Continuous-time LTl system

Convolution Operation

The convolution integral operation involves the following four steps:

1. The impulse response h (τ) is time-reversed (that is, reflected about the origin) to obtain h (-τ) and then shifted by t to form h (t - τ) = h [- (τ - t)] which is a function of τ with parameter t.

2. The signal x (τ) and h (t - τ) are multiplied together for all values of τ with t fixed at some value.
3. The product x(τ)h(t - τ) is integrated over all τ to produce a single output value y (t).
4. Steps 1 to 3 are repeated as t varies over -∞ to ∞ to produce the entire output y(t)

Example 2.6: Evaluate y (t) = x (t)* h (t), where x (t) and h (t) are shown in Figure 10, by a graphical method.

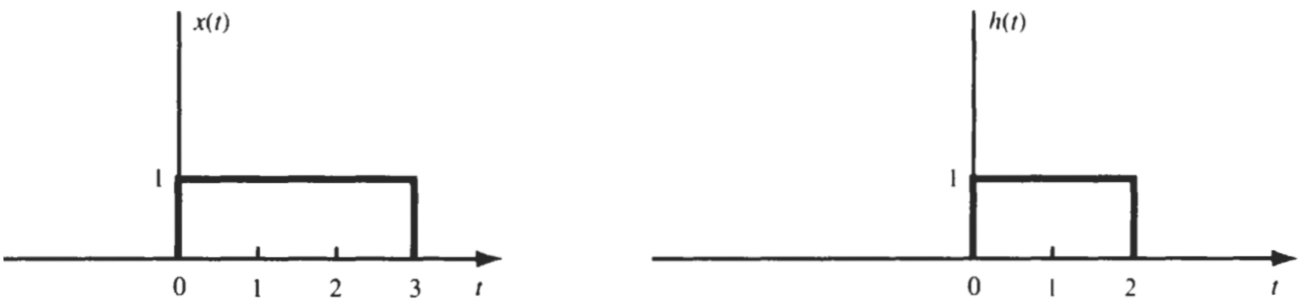


Figure 10

Functions h(τ), x(τ) and h(t – τ), x(τ)h(t – τ) for different values of t are sketched in Figure 11. From Figure 11 we see that x(τ) and h(t - τ) do not overlap for t < 0 and t > 5, and hence y(t) = 0 for t < 0 and t > 5. For the other intervals, x(τ) and h(t – τ) overlap. Thus, computing the area under the rectangular pulses for these intervals, we obtain

$$y(t) = \begin{cases} 0 & t < 0 \\ t & 0 < t \leq 2 \\ 2 & 2 < t \leq 3 \\ 5 - t & 3 < t \leq 5 \\ 0 & 5 < t \end{cases}$$

which is plotted in Figure 12.

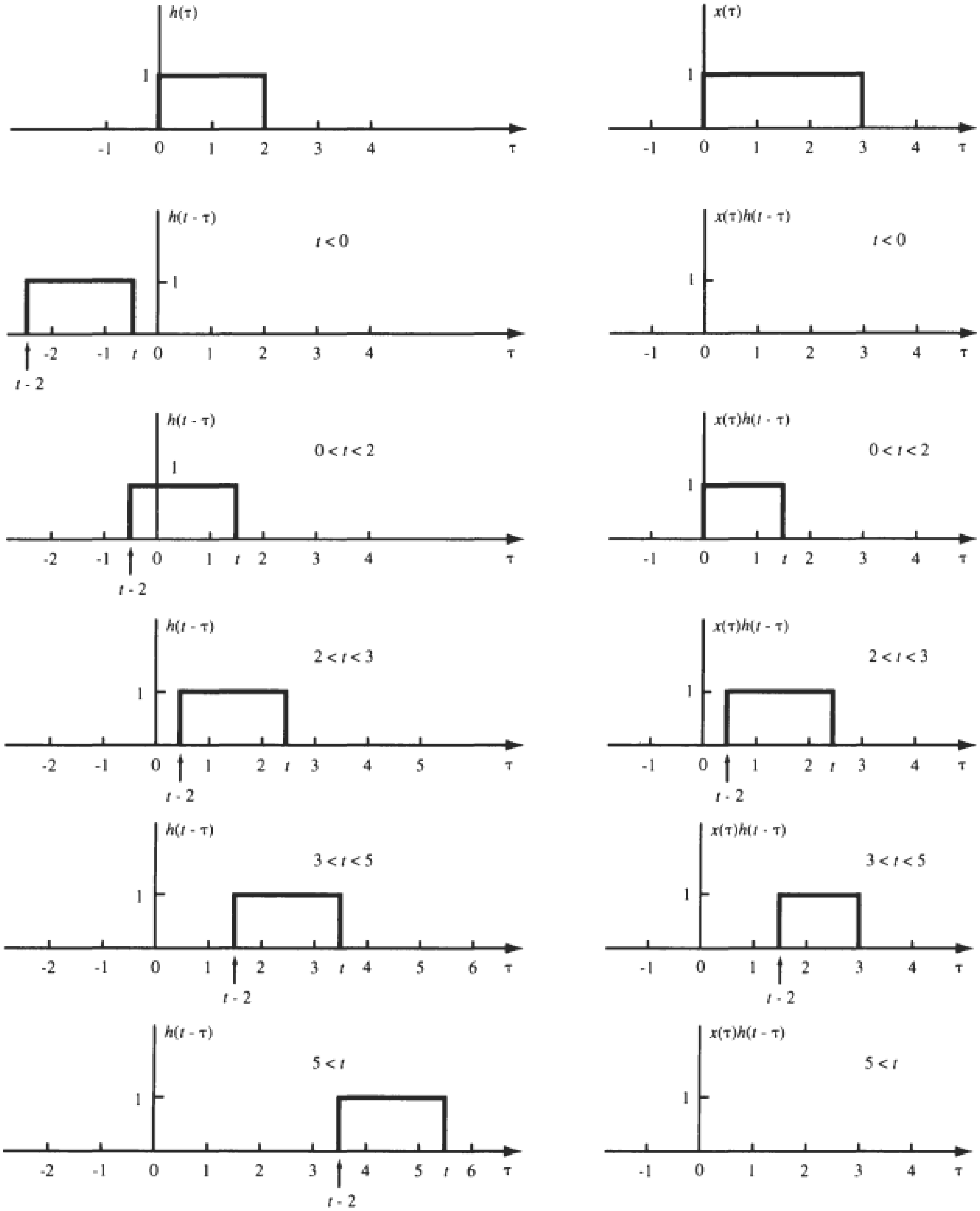


Figure 11

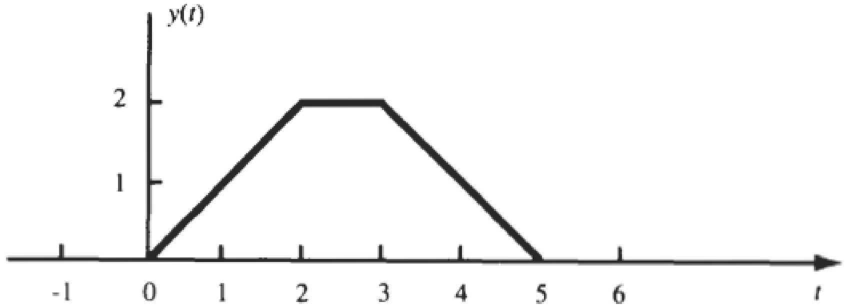


Figure 12

Properties of the Convolution Integral

The convolution integral has the following properties.

Commutative:

$$x(t) * h(t) = h(t) * x(t) \qquad (12)$$

Associative:

$$\{x(t) * h_1(t)\} * h_2(t) = x(t) * \{h_1(t) * h_2(t)\} \qquad (13)$$

Distributive:

$$x(t) * \{h_1(t) + h_2(t)\} = x(t) * h_1(t) + x(t) * h_2(t) \qquad (14)$$

Convolution Sum (Discrete Convolution)

The following equation defines the convolution of two sequences x[n] and h[n] denoted by

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \qquad (15)$$

Equation (15) is commonly called the convolution sum. Thus, again, we have the fundamental result that the output of any discrete-time **LTI** system is the convolution of the input x[n] with the impulse response h[n] of the system. Figure 13 illustrates the definition of the impulse response h[n] and the relationship of Eq. (15).

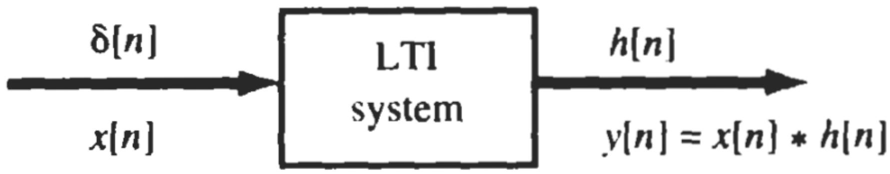


Figure 13 Discrete-time LTI system

Properties of the Convolution Sum

The following properties of the convolution sum are analogous to the convolution integral properties shown previously.

Commutative:

$$x[n] * h[n] = h[n] * x[n] \qquad (16)$$

Associative:

$$\{x[n] * h_1[n]\} * h_2[n] = x[n] * \{h_1[n] * h_2[n]\} \qquad (17)$$

Distributive:

$$x[n] * \{h_1[n] + h_2[n]\} = x[n] * h_1[n] + x[n] * h_2[n] \qquad (18)$$

Convolution Sum Operation

Again, applying the commutative property (16) of the convolution sum to Eq. (15) we obtain

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \qquad (18)$$

which may at times be easier to evaluate than Eq. (15). Similar to the continuous-time case, the convolution sum [Eq. (15)] operation involves the following four steps:

1. The impulse response h [k] is time-reversed (that is, reflected about the origin) to obtain h [– k] and then shifted by n to form h[n – k] = h[– (k – n)] which is a function of k with parameter n.
2. Two sequences x[k] and h[n – k] are multiplied together for all values of k with n fixed at some value.

3. The product x[k]h[n – k] is summed over all k to produce a single output sample y[n].
4. Steps 1 to 3 are repeated as n varies over -∞ to ∞ to produce the entire output y[n].

Example 2.7: Evaluate y[n] = x[n]*h[n], where x[n] and h [n] are shown in Fig. 2-23, (a) by an analytical technique, and (b) by a graphical method.

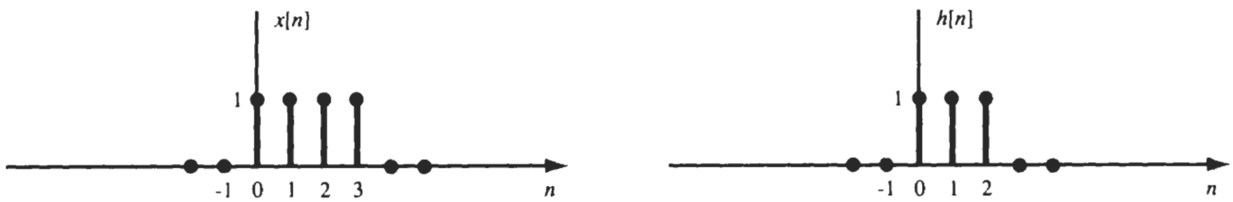


Figure 14

(*a*) Note that $x[n]$ and $h[n]$ can be expressed as

$$x[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3]$$
$$h[n] = \delta[n] + \delta[n-1] + \delta[n-2]$$

Now, using Eqs. (*2.38*), (*2.130*), and (*2.131*), we have

$$\begin{aligned} x[n]*h[n] &= x[n]*\{\delta[n] + \delta[n-1] + \delta[n-2]\} \\ &= x[n]*\delta[n] + x[n]*\delta[n-1] + x[n]*\delta[n-2]\} \\ &= x[n] + x[n-1] + x[n-2] \end{aligned}$$

Thus,
$$\begin{aligned} y[n] = &\ \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] \\ &+ \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4] \\ &+ \delta[n-2] + \delta[n-3] + \delta[n-4] + \delta[n-5] \end{aligned}$$

or
$$y[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + 3\delta[n-3] + 2\delta[n-4] + \delta[n-5]$$

or
$$y[n] = \{1, 2, 3, 3, 2, 1\}$$

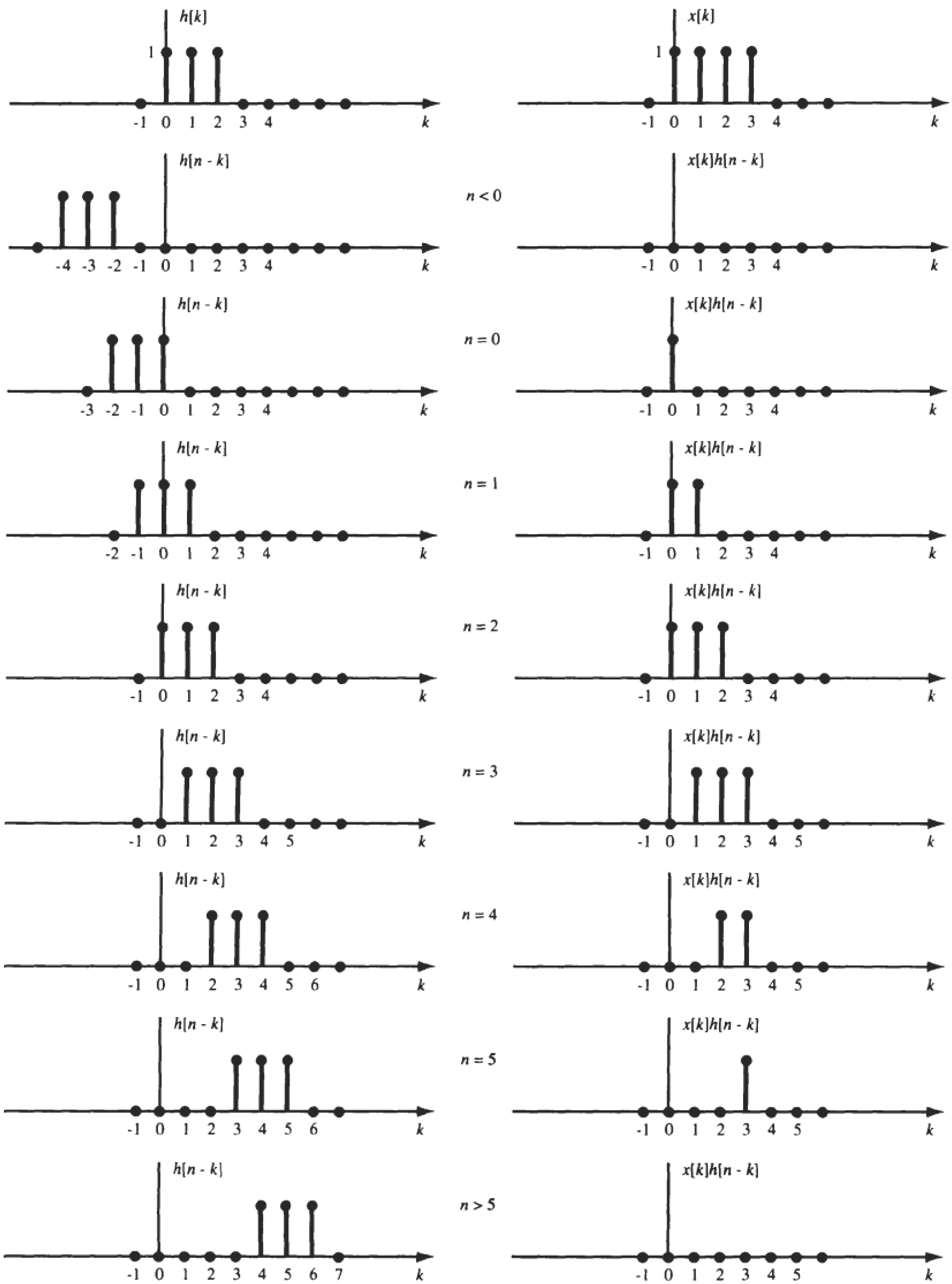


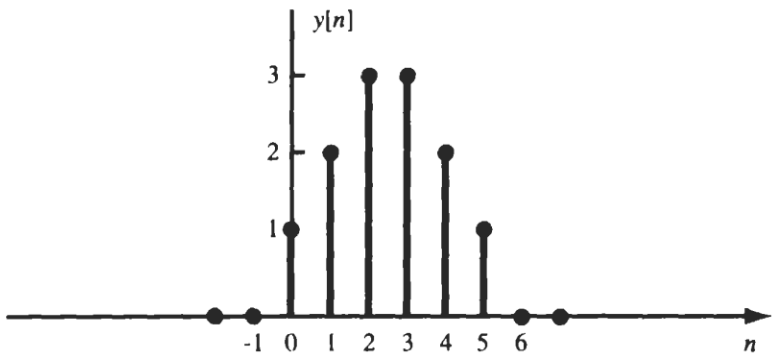
Figure 15

(*b*) Sequences $h[k]$, $x[k]$ and $h[n-k]$, $x[k]h[n-k]$ for different values of n are sketched in Fig. 15 From Fig. 15 we see that $x[k]$ and $h[n-k]$ do not overlap for $n < 0$ and $n > 5$, and hence $y[n] = 0$ for $n < 0$ and $n > 5$. For $0 \le n \le 5$, $x[k]$ and $h[n-k]$ overlap. Thus, summing $x[k]h[n-k]$ for $0 \le n \le 5$, we obtain

$y[0] = 1$ $y[1] = 2$ $y[2] = 3$ $y[3] = 3$ $y[4] = 2$ $y[5] = 1$

or

$$y[n] = \{1, 2, 3, 3, 2, 1\}$$



Example 2.8: Let x[n] = {−1, 1, −1, 1} and h[n] = {−1, 0, 2, 3, 4} be two discrete time signals. What is the signal obtained when x[n] is linearly convolved with h[n]?

(a) {1, -1, -1, -2, -3, 3, -1, 4}

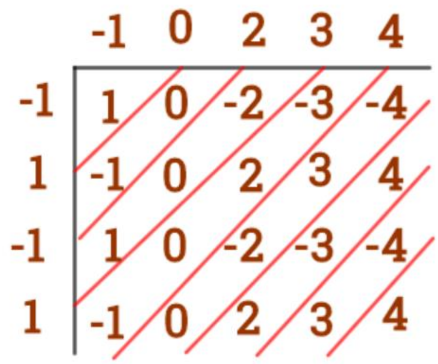
(b) {-4, 1, -3, 3, 2, 1, 1, -1}

(c) {1, 0, -2, 3, 0}

(d) {-8, 8, -8, 8}

Solution:

Let the given two signals x[n] and h[n] of length l_1 and l_2 be linearly convolved to obtain signal $y[n] = x[n] * h[n]$, where $*$ depicts convolution. The length of $y[n] = l_1+l_2-1 = 4 + 5 - 1 = 8$. Due to this, we can eliminate the last two options. To compute the convolution, we use the **tabular method**,



From the table, we obtain, $y[n] = \{1, -1, -1, -2, -3, 3, -1, 4\}$.