Strain

Deformation

Whenever a force is applied to a body, it will tend to change the body's shape and size. These changes are referred to as ***deformation,*** and they may be either highly visible or practically unnoticeable. In a general sense, the deformation of a body will not be uniform throughout its volume, and so the change in geometry of any line segment within the body may vary substantially along its length. These changes will also depend on the orientation of the line segment at the point. For example, a line segment may elongate if it is oriented in one direction, whereas it may contract if it is oriented in another direction.

Strain

In order to describe the deformation of a body by changes in length of line segments and the changes in the angles between them, we will develop the concept of strain. Strain is actually measured by experiments.

Normal Strain. The change in length of a line per unit length, then we will not have to specify the ***actual length*** of any particular line segment. Consider, for example, the line ***AB***, which is contained within the undeformed body shown in Fig. (1-a). This line lies along the ***n*** axis and has an original length of Δs. After deformation, points *A* and *B* are displaced to ***A'*** and ***B'***, and the line becomes a curve having a length of $\Delta s'$, Fig. (1-b). The change in length of the line is therefore $\Delta s'$ - !s. If we define the ***average normal strain*** using the symbol ϵ_{avg} (epsilon), then

$$\epsilon_{avg} = \frac{\Delta s' - \Delta s}{\Delta s} \qquad \ldots (1)$$

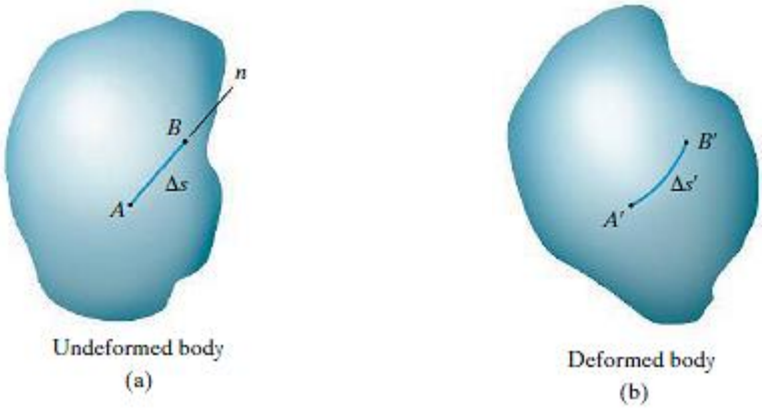


Fig (1)

As point *B* is chosen closer and closer to point *A* , the length of the line will become shorter and shorter, such that $\Delta s \to 0$. Also, this causes *B′* to approach *A′*, such that $\Delta s' \to 0$. Consequently, in the limit the normal strain at *point A* and in the direction of *n* is

$$\epsilon = \lim_{B \to A \text{ along } n} \frac{\Delta s' - \Delta s}{\Delta s} \qquad \ldots (2)$$

Hence, when ϵ (or ϵ_{avg}) is positive the initial line will elongate, whereas if ϵ is negative the line contracts.

Note that normal strain is a ***dimensionless quantity***, since it is a ratio of two lengths. Although this is the case, it is sometimes stated in terms of a ratio of length units. If the SI system is used, then the basic unit for length is the meter (m). Ordinarily, for most engineering applications P will be very small, so measurements of strain are in micrometers per meter (μm/m), where 1 μm = 10^{-6} m. In the Foot Pound-Second system, strain is often stated in units of inches per inch (in. /in.). Sometimes for experimental work, strain is expressed as a percent (e.g., 0.001 m/m = 0.1%). As an example, a normal strain of $480(10^{-6})$ can be reported as $480(10^{-6})$ in./in., 480 μm/m, or 0.0480%. Also, one can state this answer as simply 480 μ (480 “micros”).

Shear Strain. Deformations not only cause line segments to elongate or contract, but they also cause them to change direction. If we select two line segments that are originally perpendicular to one another, then the change in angle that occurs between them is referred to as ***shear strain.*** This angle is denoted by γ (gamma) and is always measured in radians (rad), which are dimensionless. For example, consider the line segments *AB* and *AC* originating from the same point *A* in a body, and directed along the perpendicular ***n*** and ***t*** axes, Fig. (2-a). After deformation, the ends of both lines are displaced, and the lines themselves become curves, such that the angle between them at *A* is θ', Fig. (2–b). Hence the shear strain at point *A* associated with the ***n*** and ***t*** axes becomes

$$\gamma_{nt} = \frac{\pi}{2} - \lim_{\substack{B \to A \text{ along } n \\ C \to A \text{ along } t}} \theta' \qquad \dots (3)$$

Notice that if θ' is smaller than $\pi/2$ the shear strain is positive, whereas if θ' is larger than $\pi/2$ the shear strain is negative.

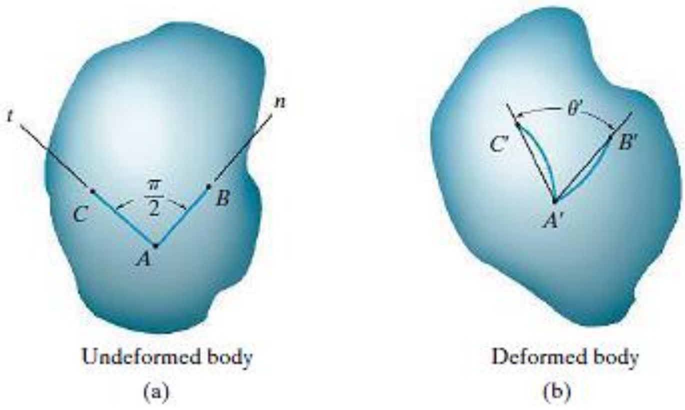


Fig (2)

Cartesian Strain Components. Using the definitions of normal and shear strain, we will now show how these components can be used to describe the deformation of the body in Fig. (3–a). To do so, imagine the body is subdivided into small elements such as the one shown in Figs. (3–a) and (3–b). This element is rectangular, has undeformed dimensions Δx, Δy, and Δz, and is located in the neighborhood of a point in the body, Fig. (3–a). If the element's dimensions are very small, then its deformed shape will be a parallelepiped, Fig. (3–c), since very small line segments will remain approximately straight after the body is deformed. In order to achieve this deformed shape, we will first consider how the normal strain changes the lengths of the sides of the rectangular element, and then how the shear strain changes the angles of each side. For example, Δx elongates $\epsilon_x \Delta x$, so its new length is $\Delta x + \epsilon_x \Delta x$. Therefore, the approximate lengths of the three sides of the parallelepiped are

$$(1 + \epsilon_x)\,\Delta x \qquad (1 + \epsilon_y)\,\Delta y \qquad (1 + \epsilon_z)\,\Delta z$$

And the approximate angles between these sides are

$$\frac{\pi}{2} - \gamma_{xy} \qquad \frac{\pi}{2} - \gamma_{yz} \qquad \frac{\pi}{2} - \gamma_{xz}$$

Notice that the ***normal strains cause a change in volume*** of the element, whereas the ***shear strains cause a change in its shape.*** Of course, both of these effects occur simultaneously during the deformation. In summary, then, the *state of strain* at a point in a body requires specifying three normal strains, Δx, Δy, Δz, and three shear strains, γ_{xy}, γ_{yz}, γ_{xz}. These strains completely describe the deformation of a rectangular volume element of material located at the point and oriented so that its sides are originally parallel to the *x*, *y*, *z* axes. Provided these strains are defined at all points in the body, then the deformed shape of the body can be determined.

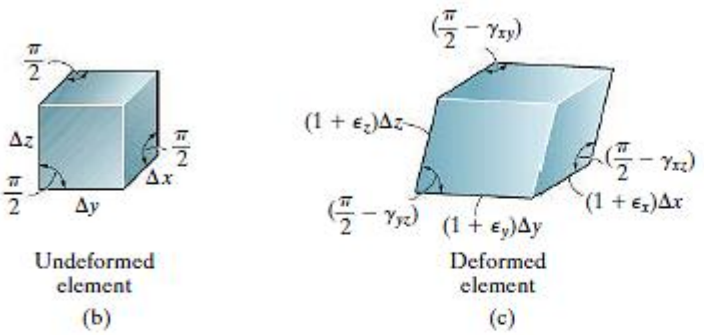


Fig (3)

Important Points

- Loads will cause all material bodies to deform and, as a result, points in a body will undergo *displacements or changes in position.*
- *Normal strain* is a measure per unit length of the elongation or contraction of a small line segment in the body, whereas *shear strain* is a measure of the change in angle that occurs between two small line segments that are originally perpendicular to one another.
- The state of strain at a point is characterized by six strain components: three normal strains ϵ_x, ϵ_y, ϵ_z and three shear strains γ_{xy}, γ_{yz}, γ_{xz}. These components depend upon the original orientation of the line segments and their location in the body.
- Strain is the geometrical quantity that is measured using experimental techniques. Once obtained, the stress in the body can then be determined from material property relations, as discussed in the next chapter.
- Most engineering materials undergo very small deformations, and so the normal strain $\epsilon \ll 1$. This assumption of "small strain analysis" allows the calculations for normal strain to be simplified, since first-order approximations can be made about their size.

EXAMPLE (1): The slender rod shown in Fig. (4) is subjected to an increase of temperature along its axis, which creates a normal strain in the rod of $\boldsymbol{\epsilon_z = 40(10^{-3})z^{1/2}}$, where z is measured in meters. Determine (a) the displacement of the end *B* of the rod due to the temperature increase, and (b) the average normal strain in the rod.

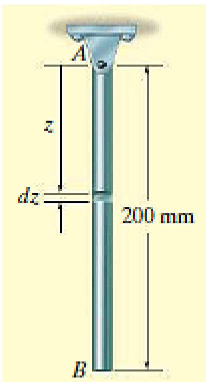


Fig (4)

SOLUTION

Part (a). Since the normal strain is reported at each point along the rod, a differential segment *dz* , located at position *z* , Fig. (4) , has a deformed length that can be determined from Eq. (1) ; that is,

$$dz' = dz + \epsilon_z\, dz$$
$$dz' = [1 + 40(10^{-3})z^{1/2}]dz$$

The sum of these segments along the axis yields the *deformed length* of the rod, i.e.

$$z' = \int_0^{0.2\text{ m}} [1 + 40(10^{-3})z^{1/2}]dz$$
$$= [z + 40(10^{-3})\tfrac{2}{3}z^{3/2}]\big|_0^{0.2\text{ m}}$$
$$= 0.20239\text{ m}$$

The displacement of the end of the rod is therefore

$$\Delta_B = 0.20239\text{ m} - 0.2\text{ m} = 0.00239\text{ m} = 2.39\text{ mm}\downarrow \quad \textit{Ans.}$$

Part (b). The average normal strain in the rod is determined from Eq. (1), which assumes that the rod or "line segment" has an original length of 200 mm and a change in length of 2.39 mm. Hence,

$$\epsilon_{avg} = \frac{\Delta s' - \Delta s}{\Delta s} = \frac{2.39\text{ mm}}{200\text{ mm}} = 0.0119\text{ mm/mm} \quad \textit{Ans.}$$

This strain is called a thermal strain, caused by temperature, *not* by any load.

EXAMPLE (2): When force **P** is applied to the rigid lever arm *ABC* in Fig. (5–a), the arm rotates counterclockwise about pin *A* through an angle of 0.05°. Determine the normal strain developed in wire *BD*.

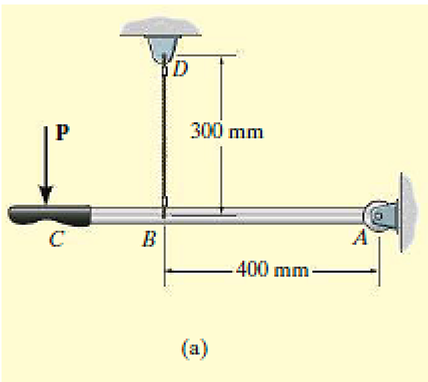


Fig (5)

SOLUTION I

Geometry. The orientation of the lever arm after it rotates about point *A* is shown in Fig. (5–b). From the geometry of this figure,

$$\alpha = \tan^{-1}\left(\frac{400\text{ mm}}{300\text{ mm}}\right) = 53.1301°$$

Then

$$\phi = 90° - \alpha + 0.05° = 90° - 53.1301° + 0.05° = 36.92°$$

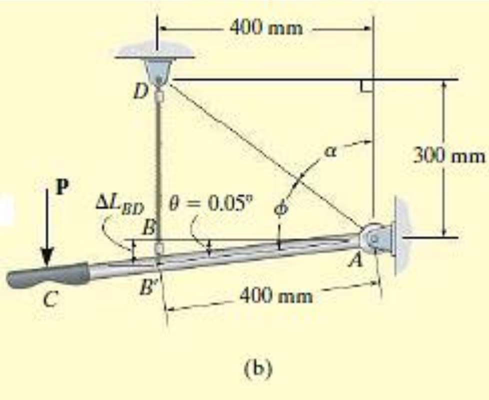


Fig (5)

For triangle *ABD* the Pythagorean Theorem gives

$$L_{AD} = \sqrt{(300\text{ mm})^2 + (400\text{ mm})^2} = 500\text{ mm}$$

Using this result and applying the law of cosines to triangle *AB'D*,

$$L_{B'D} = \sqrt{L_{AD}^2 + L_{AB'}^2 - 2(L_{AD})(L_{AB'})\cos\phi}$$
$$= \sqrt{(500\text{ mm})^2 + (400\text{ mm})^2 - 2(500\text{ mm})(400\text{ mm})\cos 36.92°}$$
$$= 300.3491\text{ mm}$$

Normal Strain.

$$\epsilon_{BD} = \frac{L_{B'D} - L_{BD}}{L_{BD}}$$

$$= \frac{300.3491\text{ mm} - 300\text{ mm}}{300\text{ mm}} = 0.00116\text{ mm/mm} \qquad \textit{Ans.}$$

SOLUTION II

Since the strain is small, this same result can be obtained by approximating the elongation of wire *BD* as ΔL_{BD}, shown in Fig. (5–*b)* . Here,

$$\Delta L_{BD} = \theta L_{AB} = \left[\left(\frac{0.05°}{180°}\right)(\pi\text{ rad})\right](400\text{ mm}) = 0.3491\text{ mm}$$

Therefore,

$$\epsilon_{BD} = \frac{\Delta L_{BD}}{L_{BD}} = \frac{0.3491\text{ mm}}{300\text{ mm}} = 0.00116\text{ mm/mm} \qquad \textit{Ans.}$$

EXAMPLE (3): The plate shown in Fig. (6–*a)* is fixed connected along *AB* and held in the horizontal guides at its top and bottom, *AD* and *BC*. If its right side *CD* is given a uniform horizontal displacement of 2 mm, determine (a) the average normal strain along the diagonal *AC*, and (b) the shear strain at *E* relative to the *x*, *y* axes.

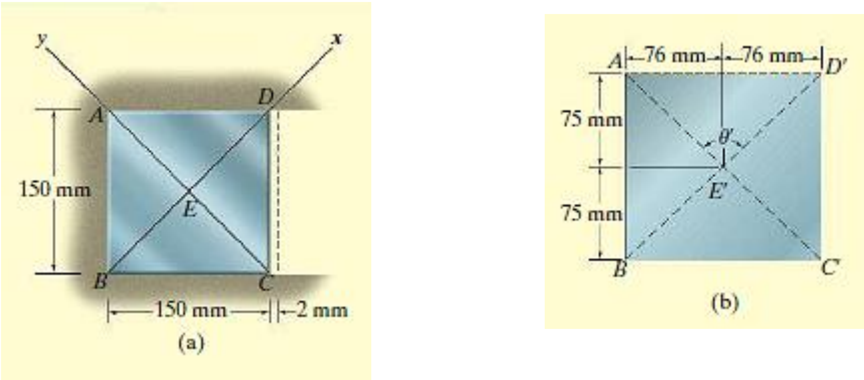


Fig (6)

SOLUTION

Part (a). When the plate is deformed, the diagonal *AC* becomes *AC*′, Fig. (6–b). The lengths of diagonals *AC* and *AC*′ can be found from the Pythagorean Theorem. We have

$$AC = \sqrt{(0.150\text{ m})^2 + (0.150\text{ m})^2} = 0.21213\text{ m}$$

$$AC' = \sqrt{(0.150\text{ m})^2 + (0.152\text{ m})^2} = 0.21355\text{ m}$$

Therefore the average normal strain along the diagonal is

$$(\epsilon_{AC})_{avg} = \frac{AC' - AC}{AC} = \frac{0.21355\text{ m} - 0.21213\text{ m}}{0.21213\text{ m}}$$

$$= 0.00669\text{ mm/mm} \qquad \textit{Ans.}$$

Part (b). To find the shear strain at *E* relative to the *x* and *y* axes, it is first necessary to find the angle θ' after deformation, Fig. (6–b). We have

$$\tan\left(\frac{\theta'}{2}\right) = \frac{76 \text{ mm}}{75 \text{ mm}}$$

$$\theta' = 90.759° = \left(\frac{\pi}{180°}\right)(90.759°) = 1.58404 \text{ rad}$$

Applying Eq. (3), the shear strain at *E* is therefore

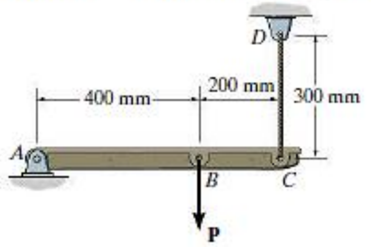
$$\gamma_{xy} = \frac{\pi}{2} - 1.58404 \text{ rad} = -0.0132 \text{ rad} \qquad \textit{Ans.}$$

The *negative sign* indicates that the angle θ' is *greater than* 90°.

NOTE: If the *x* and *y* axes were horizontal and vertical at point *E* , then the 90° angle between these axes would not change due to the deformation, and so $\gamma_{xy} = 0$ at point *E* .

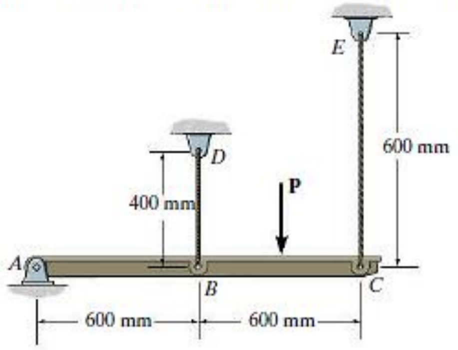
FUNDAMENTAL PROBLEMS

F 1-1 When force **P** is applied to the rigid arm *ABC*, point *B* displaces vertically downward through a distance of 0.2 mm. Determine the normal strain developed in wire *CD*.



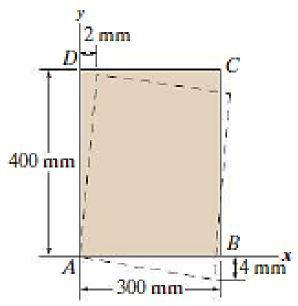
F 1-1

F 1-2 If the applied force **P** causes the rigid arm *ABC* to rotate clockwise about pin *A* through an angle of 0.02°, determine the normal strain developed in wires *BD* and *CE*.



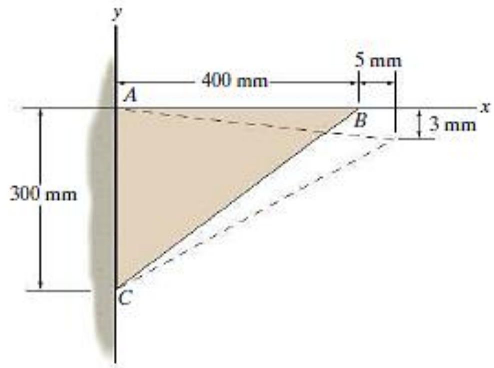
F 1-2

F 1-3 The rectangular plate is deformed into the shape of a parallelogram shown by the dashed line. Determine the average shear strain at corner *A* with respect to the *x* and *y* axes.



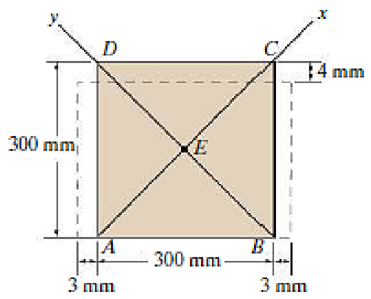
F 1-3

F 1-4 The triangular plate is deformed into the shape shown by the dashed line. Determine the normal strain developed along edge *BC* and the average shear strain at corner *A* with respect to the *x* and *y* axes.



F 1-4

F 1-5 The square plate is deformed into the shape shown by the dashed line. Determine the average normal strain along diagonal *AC* and the shear strain of point *E* with respect to the *x* and *y* axes.



F 1-5