Stress

We begin by considering the sectioned area to be subdivided into small areas, such as ΔA shown in Fig. (1) *a*. As we reduce ΔA to a smaller and smaller size, we will consider the material to be ***continuous,*** that is, to consist of a *continuum* or uniform distribution of matter having no voids. Also, the material must be ***cohesive,*** meaning that all portions of it are connected together, without having breaks, cracks, or separations. A typical finite yet very small force! F, acting on ΔA, is shown in Fig. (1) *a*. This force, like all the others, will have a unique direction, but for further discussion we will replace it by its *three components*, namely ΔF_x, ΔF_y, and! ΔF_z, which are taken tangent, tangent, and normal to the area, respectively. As ΔA approaches zero, so do! F and its components; however, the quotient of the force and area will, in general, approach a finite limit. This quotient is called *stress* , and as noted, it describes the *intensity of the internal force* acting on a *specific plane* (area) passing through a point.

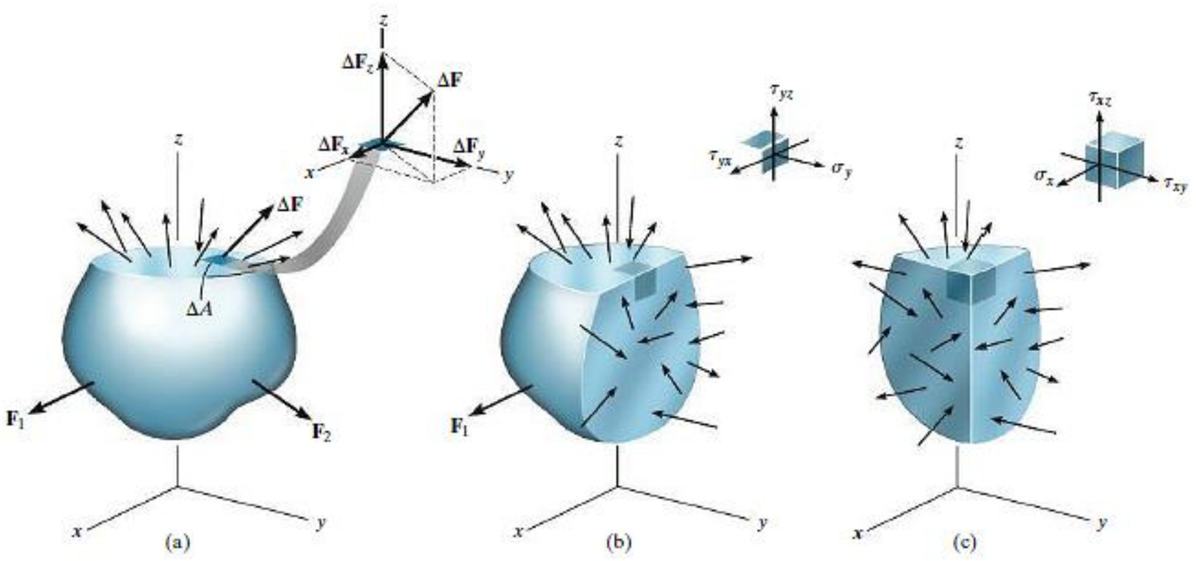


Fig. 1

Normal Stress. The *intensity* of the force acting normal to Δ*A* is defined as the ***normal stress,*** s (sigma). Since Δ**F**z is normal to the area then

$$\sigma_z = \lim_{\Delta A \to 0} \frac{\Delta F_z}{\Delta A} \qquad (1)$$

If the normal force or stress "pulls" on ΔA as shown in Fig. (1) *a*, it is referred to as *tensile stress* , whereas if it "pushes" on ΔA it is called *compressive stress* .

Shear Stress. The intensity of force acting tangent to ΔA is called the ***shear stress***, τ (tau). Here we have shear stress components,

$$\tau_{zx} = \lim_{\Delta A \to 0} \frac{\Delta F_x}{\Delta A}$$

$$\tau_{zy} = \lim_{\Delta A \to 0} \frac{\Delta F_y}{\Delta A} \qquad (2)$$

Note that in this subscript notation z specifies the orientation of the area ΔA, Fig. (2), and x and y indicate the axes along which each shear stress acts.

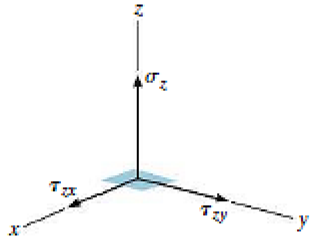


Fig. 2

General State of Stress. If the body is further sectioned by planes parallel to the $x-z$ plane, Fig. (1)*b* , and the $y-z$ plane, Fig. (1) *c*, we can then "cut out" a cubic volume element of material that represents the ***state of stress*** acting around the chosen point in the body. This state of stress is then characterized by three components acting on each face of the element, Fig. (3).

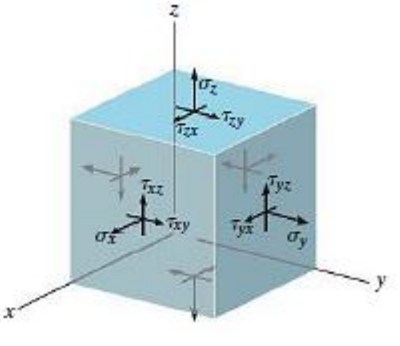


Fig. 3

Units. Since stress represents a force per unit area, in the International Standard or SI system, the magnitudes of both normal and shear stress are specified in the basic units of newtons per square meter (N/m2). This unit, called a Pascal (1 Pa = 1 N/m2) is rather small, and in engineering work prefixes such as kilo- (10^3), symbolized by k, mega- (10^6), symbolized by M, or giga- (10^9), symbolized by G, are used to represent larger, more realistic values of stress. Likewise, in the Foot-Pound-Second system of units, engineers usually express stress in pounds per square inch (psi) or kilopounds per square inch (ksi), where 1 kilopound (kip) = 1000 lb.

Average Normal Stress in an Axially Loaded Bar

We will determine the average stress distribution acting on the cross-sectional area of an axially loaded bar such as the one shown in Fig. (4) *a*. This bar is **prismatic** since all cross sections are the same throughout its length. When the load *P* is applied to the bar through the centroid of its cross-sectional area, then the bar will deform uniformly throughout the central region of its length, as shown in Fig. (4) *b*, provided the material of the bar is both homogeneous and isotropic. ***Homogeneous material*** has the same physical and mechanical properties throughout its volume, and ***isotropic material*** has these same properties in all directions. Many engineering materials may be approximated as being both homogeneous and isotropic as assumed here. Steel, for example, contains thousands of randomly oriented crystals in each cubic millimeter of its volume, and since most problems involving this material have a physical size that is very much larger than a single crystal, the above assumption regarding its material composition is quite realistic. Note that anisotropic materials such as wood have different properties in different directions, and although this is the case, if the anisotropy is oriented along the bar’s axis (as for instance in a typical wood rod), then the bar will also deform uniformly when subjected to the axial load *P*.

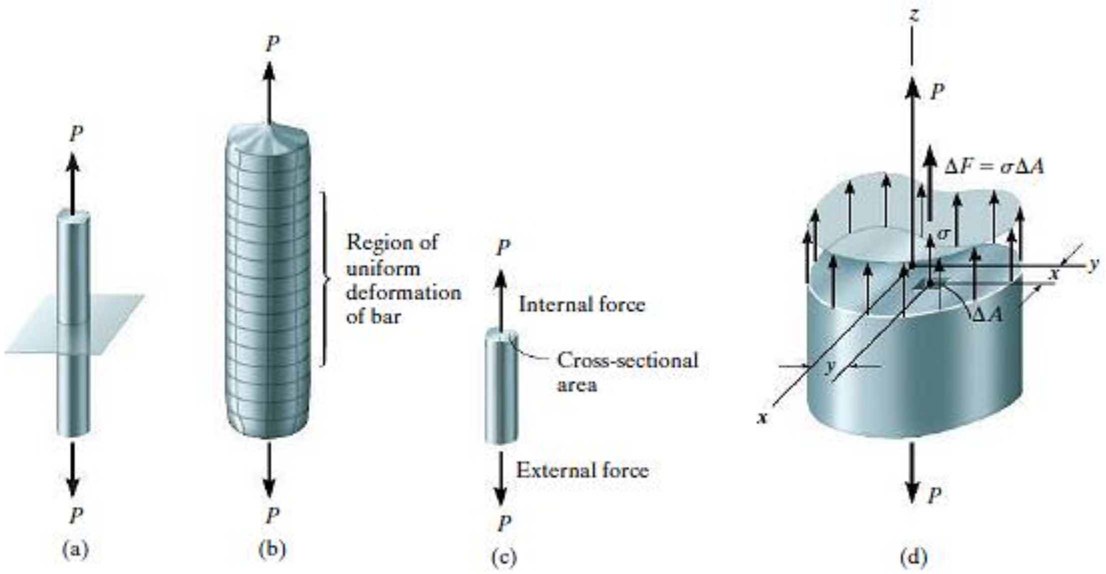


Fig. 4

Average Normal Stress Distribution. If we pass a section through the bar, and separate it into two parts, then equilibrium requires the resultant normal force at the section to be *P*, Fig. (4) *c*. Due to the *uniform* deformation of the material, it is necessary that the cross section be subjected to a *constant normal stress distribution*, Fig. (4) *d*.

A result, each small area ΔA on the cross section is subjected to a force $\Delta F = \sigma \Delta A$, and the *sum* of these forces acting over the entire cross sectional area must be equivalent to the internal resultant force **P** at the section. If we let $\Delta A \rightarrow dA$ and therefore $\Delta F \rightarrow dF$, then, recognizing s is *constant*, we have

$$+\uparrow F_{Rz} = \Sigma F_z; \qquad \int dF = \int_A \sigma \, dA$$

$$P = \sigma A$$

$$\boxed{\sigma = \frac{P}{A}} \qquad (3)$$

s = average normal stress at any point on the cross-sectional area.
P = internal resultant normal force, which acts through the *centroid* of the cross-sectional area. *P* is determined using the method of sections. and the equations of equilibrium.
A = cross-sectional area of the bar where s is determined.

Since the internal load P passes through the centroid of the cross section, the uniform stress distribution will produce zero moments about the x and y axes passing through this point, Fig. (4) *d*. To show this, we require the moment of P about each axis to be equal to the moment of the stress distribution about the axes, namely,

$$(M_R)_x = \Sigma M_x; \qquad 0 = \int_A y\, dF = \int_A y\sigma\, dA = \sigma \int_A y\, dA$$

$$(M_R)_y = \Sigma M_y; \qquad 0 = -\int_A x\, dF = -\int_A x\sigma\, dA = -\sigma \int_A x\, dA$$

Equilibrium. It should be apparent that only a normal stress exists on any small volume element of material located at each point on the cross section of an axially loaded bar. If we consider vertical equilibrium of the element, Fig. (5), then apply the equation of force equilibrium,

$$\Sigma F_z = 0; \qquad \sigma(\Delta A) - \sigma'(\Delta A) = 0$$

$$\sigma = \sigma'$$

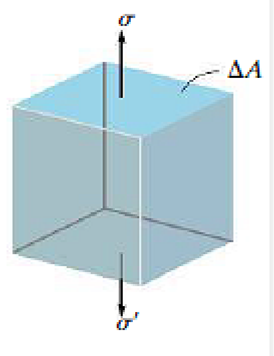


Fig. 5

In other words, the two normal stress components on the element must be equal in magnitude but opposite in direction. This is referred to as *uniaxial stress*. The previous analysis applies to members subjected to either tension or compression, as shown in Fig. (6). As a graphical interpretation, the ***magnitude*** of the internal resultant force **P** is ***equivalent*** to the ***volume*** under the stress diagram; that is, $P = \sigma\, A$

(volume = height × base). Furthermore, as a consequence of the balance of moments, ***this resultant passes through the centroid of this volume.***

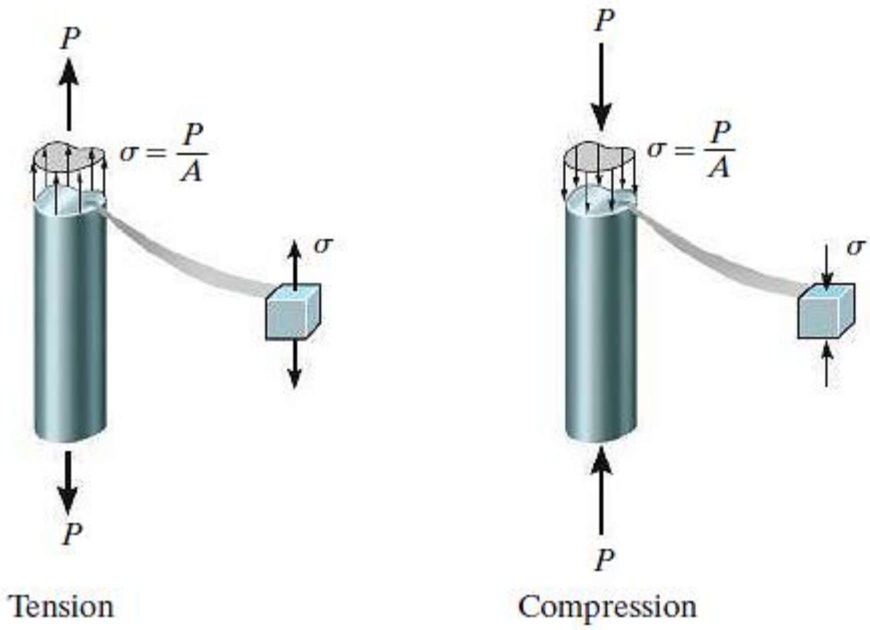


Fig. 6

Maximum Average Normal Stress. The normal stress within the bar could be different from one section to the next, and, if the *maximum* average normal stress is to be determined, then it becomes important to find the location where the ratio P/A is a *maximum*. To do this it is necessary to determine the internal force P at various sections along the bar. Here it may be helpful to show this variation by drawing an ***axial or normal force diagram.*** Specifically, this diagram is a plot of the normal force P versus its position x along the bar's length. As a sign convention, P will be positive if it causes tension in the member, and negative if it causes compression. Once the internal loading throughout the bar is known, the maximum ratio of P / A can then be identified.

Important Points

- When a body subjected to external loads is sectioned, there is a distribution of force acting over the sectioned area which holds each segment of the body in equilibrium. The intensity of this internal force at a point in the body is referred to as *stress*.
- Stress is the limiting value of force per unit area, as the area approaches zero. For this definition, the material is considered to be continuous and cohesive.
- The magnitude of the stress components at a point depends upon the type of loading acting on the body, and the orientation of the element at the point.
- When a prismatic bar is made from homogeneous and isotropic material, and is subjected to an axial force acting through the centroid of the cross-sectional area, then the center region of the bar will deform uniformly. As a result, the material will be subjected *only to normal stress*. This stress is uniform or *averaged* over the cross-sectional area.

Procedure for Analysis

The equation $\sigma = P/A$ gives the *average* normal stress on the cross-sectional area of a member when the section is subjected to an internal resultant normal force **P**. For axially loaded members, application of this equation requires the following steps.

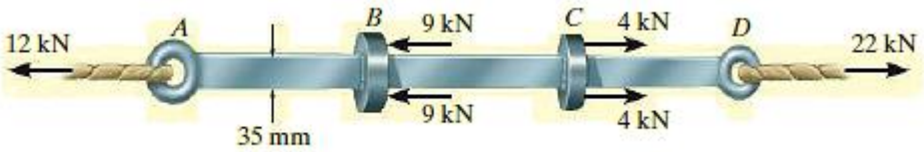
Internal Loading.

- Section the member *perpendicular* to its longitudinal axis at the point where the normal stress is to be determined and use the necessary free-body diagram and force equation of equilibrium to obtain the *internal axial force* **P** at the section.

Average Normal Stress.

- Determine the member's cross-sectional area at the section and calculate the average normal stress $\sigma = P/A$.
- It is suggested that σ be shown acting on a small volume element of the material located at a point on the section where stress is calculated. To do this, first draw σ on the face of the element coincident with the sectioned area A. Here σ acts in the *same direction* as the internal force **P** since all the normal stresses on the cross section develop this resultant. The normal stress σ on the other face of the element acts in the opposite direction.

EXAMPLE (1): The bar in Fig. (7- a) has a constant width of 35 mm and a thickness of 10 mm. Determine the maximum average normal stress in the bar when it is subjected to the loading shown.



(a)

Fig. (7)

SOLUTION

Internal Loading. By inspection, the internal axial forces in regions *AB*, *BC*, and *CD* are all constant yet have different magnitudes. Using the method of sections, these loadings are determined in Fig. (7- b). The normal force diagram, which represents these results graphically, is shown in Fig. (7- c). The largest loading is in region *BC*, where *PBC* = 30 kN. Since the cross-sectional area of the bar is *constant*, the largest average normal stress also occurs within this region of the bar.

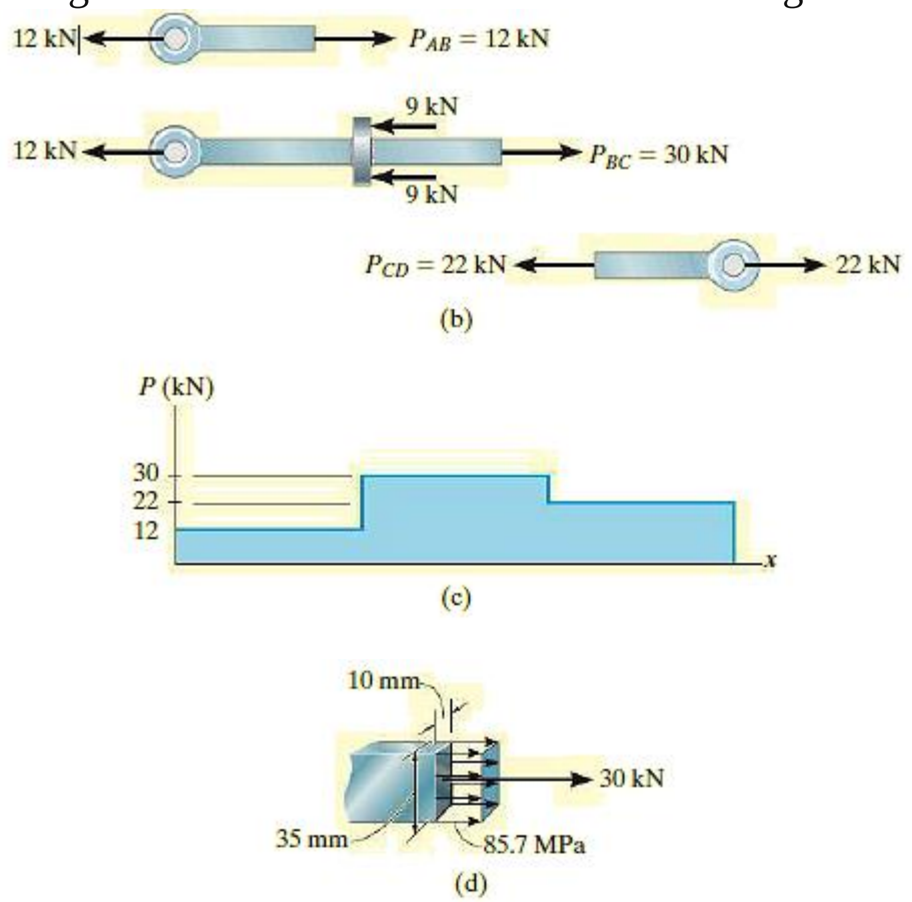


Fig. (7)

Average Normal Stress. Applying Eq. 3, we have

$$\sigma_{BC} = \frac{P_{BC}}{A} = \frac{30(10^3)\text{ N}}{(0.035\text{ m})(0.010\text{ m})} = 85.7\text{ MPa} \qquad \textit{Ans.}$$

NOTE: The stress distribution acting on an arbitrary cross section of the bar within region *BC* is shown in Fig. (7- d) .

EXAMPLE 2: The 80-kg lamp is supported by two rods *AB* and *BC* as shown in Fig. (8-a). If *AB* has a diameter of 10 mm and *BC* has a diameter of 8 mm, determine the average normal stress in each rod.

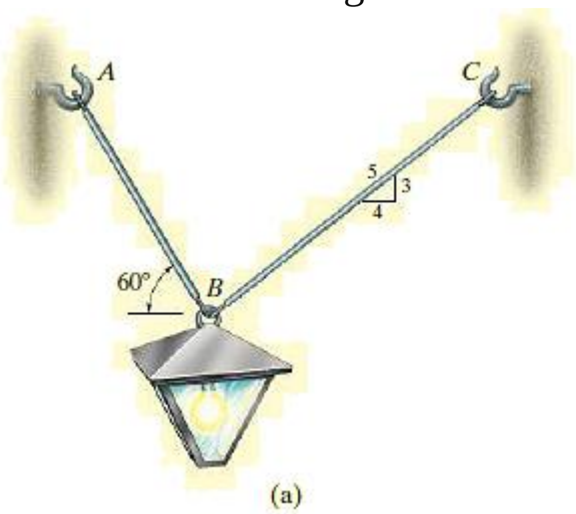


Fig. (8)

SOLUTION

Internal Loading. We must first determine the axial force in each rod. A free-body diagram of the lamp is shown in Fig. (8-b). Applying the equations of force equilibrium,

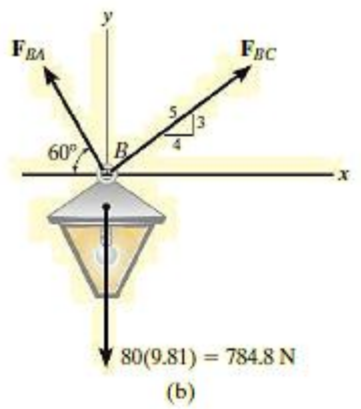


Fig. (8)

$$\xrightarrow{+} \Sigma F_x = 0; \qquad F_{BC}\left(\tfrac{4}{5}\right) - F_{BA}\cos 60^\circ = 0$$

$$+\uparrow \Sigma F_y = 0; \qquad F_{BC}\left(\tfrac{3}{5}\right) + F_{BA}\sin 60^\circ - 784.8\text{ N} = 0$$

$$F_{BC} = 395.2\text{ N}, \qquad F_{BA} = 632.4\text{ N}$$

Average Normal Stress. Applying Eq. (3) .

$$\sigma_{BC} = \frac{F_{BC}}{A_{BC}} = \frac{395.2\ \text{N}}{\pi(0.004\ \text{m})^2} = 7.86\ \text{MPa} \qquad \textit{Ans.}$$

$$\sigma_{BA} = \frac{F_{BA}}{A_{BA}} = \frac{632.4\ \text{N}}{\pi(0.005\ \text{m})^2} = 8.05\ \text{MPa} \qquad \textit{Ans.}$$

NOTE: The average normal stress distribution acting over a cross section of rod *AB* is shown in Fig. (8-c), and at a point on this cross section, an element of material is stressed as shown in Fig. (8-d).

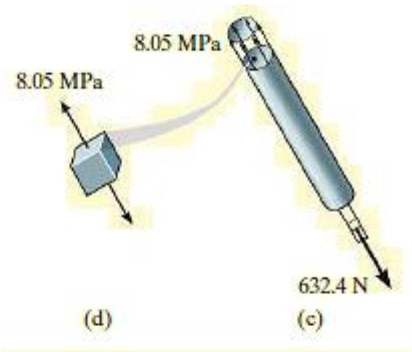


Fig. (8)

EXAMPLE 3: The casting shown in Fig. (9-a) is made of steel having a specific weight of γ_{st} = 490 lb/ft3. Determine the average compressive stress acting at points A and B.

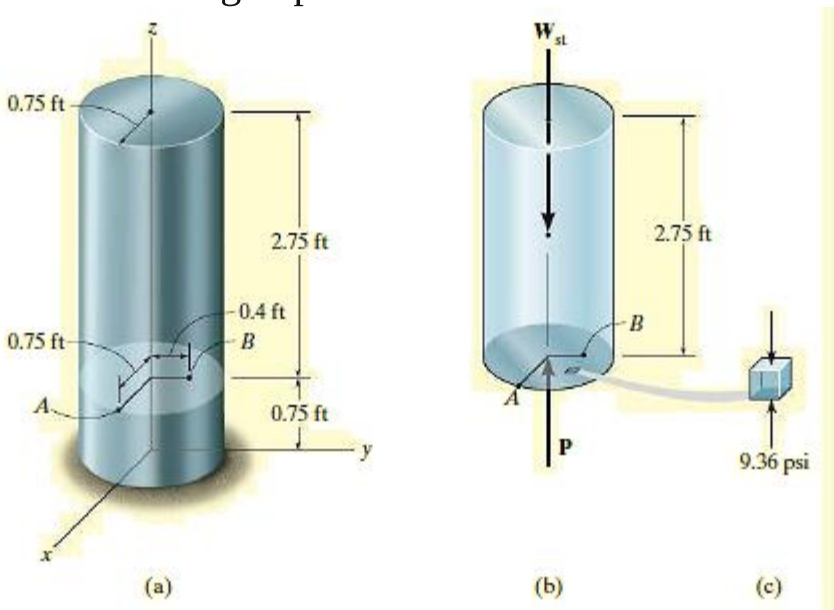


Fig. (9)

SOLUTION

Internal Loading. A free-body diagram of the top segment of the casting where the section passes through points A and B is shown in Fig. (9-b). The weight of this segment is determined from $W_{st} = \gamma_{st} V_{st}$. Thus the internal axial force P at the section is

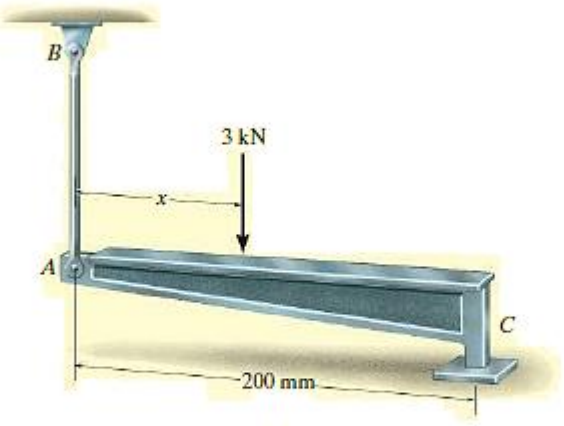
$$+\uparrow \Sigma F_z = 0; \qquad P - W_{st} = 0$$
$$P - (490\ \text{lb/ft}^3)(2.75\ \text{ft})\left[\pi(0.75\ \text{ft})^2\right] = 0$$
$$P = 2381\ \text{lb}$$

Average Compressive Stress. The cross-sectional area at the section is $A = \pi(0.75\ \text{ft})^2$, and so the average compressive stress becomes

$$\sigma = \frac{P}{A} = \frac{2381\ \text{lb}}{\pi(0.75\ \text{ft})^2} = 1347.5\ \text{lb/ft}^2$$
$$\sigma = 1347.5\ \text{lb/ft}^2(1\ \text{ft}^2/144\ \text{in}^2) = 9.36\ \text{psi} \qquad \textit{Ans.}$$

NOTE: The stress shown on the volume element of material in Fig (9-c) is representative of the conditions at either point A or B . Notice that this stress acts ***upward*** on the bottom or shaded face of the element since this face forms part of the bottom surface area of the section, and on this surface, the resultant internal force **P** is pushing upward.

EXAMPLE 4: Member AC shown in Fig. (10-a) is subjected to a vertical force of 3 kN. Determine the position x of this force so that the average compressive stress at the smooth support C is equal to the average tensile stress in the tie rod AB. The rod has a cross-sectional area of 400 mm^2 and the contact area at C is 650 mm^2.



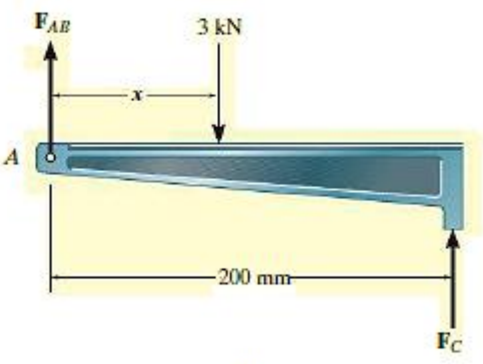


Fig. (10)

SOLUTION

Internal Loading. The forces at *A* and *C* can be related by considering the free-body diagram for member *AC*, Fig. (10-b). There are three unknowns, namely, $\boldsymbol{F_{AB}}$, $\boldsymbol{F_C}$, and $\boldsymbol{x}$. To solve this problem we will work in units of newton’s and millimeters.

$$+\uparrow \Sigma F_y = 0; \qquad F_{AB} + F_C - 3000\text{ N} = 0 \qquad (1)$$

$$\circlearrowleft + \Sigma M_A = 0; \qquad -3000\text{ N}(x) + F_C(200\text{ mm}) = 0 \qquad (2)$$

Average Normal Stress. A necessary third equation can be written that requires the tensile stress in the bar *AB* and the compressive stress at *C* to be equivalent, i.e,

$$\sigma = \frac{F_{AB}}{400\text{ mm}^2} = \frac{F_C}{650\text{ mm}^2}$$

$$F_C = 1.625 F_{AB}$$

Substituting this into Eq. 1, solving for $\boldsymbol{F_{AB}}$, then solving for $\boldsymbol{F_C}$, we obtain

$$F_{AB} = 1143\text{ N}$$
$$F_C = 1857\text{ N}$$

The position of the applied load is determined from Eq. 2,

$$x = 124\text{ mm} \qquad \textit{Ans.}$$

NOTE: $0 < x < 200$ mm, as required.