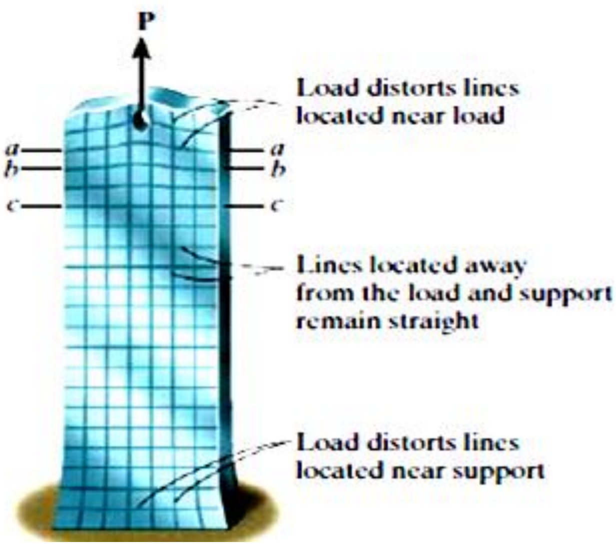
Axial Load

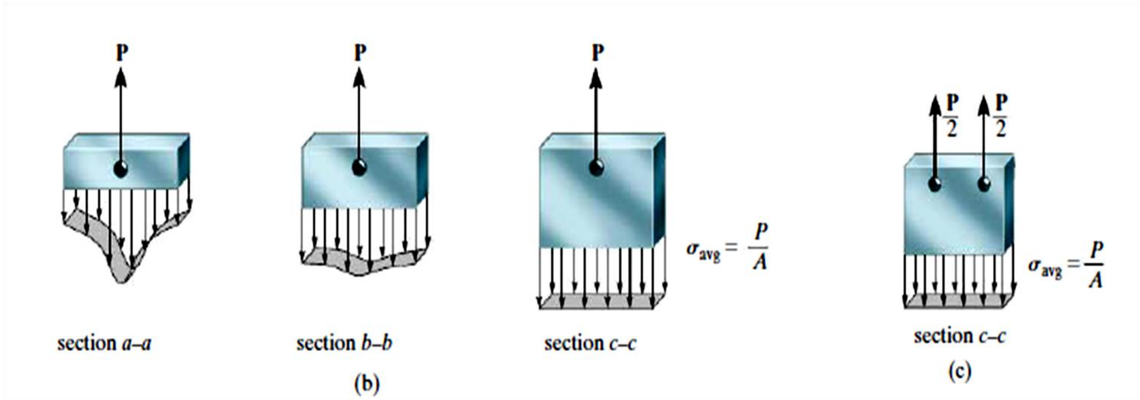
Saint-Venant's Principle

The mathematical relationship between stress and strain depends on the type of material from which the body is made. In particular, if the material behaves in a linear elastic manner, then Hooke's law applies, and there is a proportional relationship between stress and strain.



Using this idea, consider the manner in which a rectangular bar will deform elastically when the bar is subjected to a force ***P*** applied along its Centroid axis, Fig.. Here the bar is fixed connected at one end, with the force applied through a hole at its other end. Due to the loading, the bar deforms as indicated by the once horizontal and vertical grid lines drawn on the bar. Notice how the ***localized deformation*** that occurs at each end tends to even out and become uniform throughout the midsection of the bar. If the material remains elastic then the strains caused by this deformation are directly related to the stress in the bar. As a result, the stress will be distributed more uniformly throughout the cross-sectional area when the section is taken farther and farther from the point where any external load is applied. For example, consider a profile of the variation of the stress distribution acting at sections $a - a$, $b - b$, and $c - c$, each of which is shown in Fig. *b* . By comparison, the stress tends to reach a uniform value at section $c - c$, which is sufficiently removed from the end since the localized deformation caused by ***P*** ***vanishes***. The minimum distance from the bar's end where this occurs

can be determined using a mathematical analysis based on the theory of elasticity.
It has been found that this distance should at least be equal to the ***largest dimension*** of the loaded cross section. Hence, section *c – c* should be located at a distance at least equal to the width (not the thickness) of the bar.



In the same way, the stress distribution at the support will also even out and become uniform over the cross section located the same distance away from the support. The fact that stress and deformation behave in this manner is referred to as ***Saint-Venant's principle***, since it was first noticed by the French scientist Barré de Saint-Venant in 1855. Essentially it states that the ***stress and strain produced at points in a body sufficiently removed from the region of load application will be the same as the stress and strain produced by any applied loadings that have the same statically equivalent resultant, and are applied to the body within the same region.*** For example, if two symmetrically applied forces *P*/2 acts on the bar, Fig. *c*, the stress distribution at section *c – c* will be uniform and therefore equivalent to
$\boldsymbol{\sigma_{avg} = P/A}$,as in Fig. 4–1 *c* .

Elastic Deformation of an Axially Loaded Member

Using the method of sections, a differential element (or wafer) of length *dx* and cross-sectional area *A* (x*)* is isolated from the bar at the arbitrary position *x*. The free-body diagram of this element is shown in Fig. *b*. The resultant internal axial force will be a function of *x* since the external distributed loading will cause it to vary along the length of the bar. This load, *P* (x*)*, will deform the element into the shape indicated by the dashed outline, and therefore the displacement of one end of the element with respect to the other end is *d*δ. The stress and strain in the element are

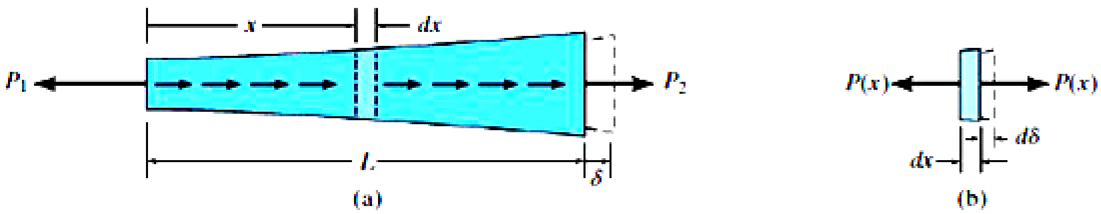
$$\sigma = \frac{P(x)}{A(x)} \text{ and } \epsilon = \frac{d\delta}{dx}$$

Provided the stress does not exceed the proportional limit, we can apply Hooke's law; i.e,

$$\sigma = E(x)\epsilon$$

$$\frac{P(x)}{A(x)} = E(x)\left(\frac{d\delta}{dx}\right)$$

$$d\delta = \frac{P(x)dx}{A(x)E(x)}$$



For the entire length *L* of the bar, we must integrate this expression to find δ. This yield

$$\delta = \int_0^L \frac{P(x)dx}{A(x)E(x)} \quad (1)$$

Where

δ = displacement of one point on the bar relative to the other point
L = original length of bar
P(*x*) = internal axial force at the section, located a distance *x* from one end
A(*x*) = cross-sectional area of the bar expressed as a function of *x*
E(*x*) = modulus of elasticity for the material expressed as a function of *x*.

Constant Load and Cross-Sectional Area

In many cases the bar will have a constant cross-sectional area A; and the material will be homogeneous, so E is constant. Furthermore, if a constant external force is applied at each end, then the internal force P throughout the length of the bar is also constant. As a result, Eq. 1 can be integrated to yield

$$\delta = \frac{PL}{AE} \qquad (2)$$

If the bar is subjected to several different axial forces along its length, or the cross-sectional area or modulus of elasticity changes abruptly from one region of the bar to the next, the above equation can be applied to each *segment* of the bar where these quantities remain *constant*. The displacement of one end of the bar with respect to the other is then found from the *algebraic addition* of the relative displacements of the ends of each segment. For this general case,

$$\delta = \sum \frac{PL}{AE} \qquad (3)$$

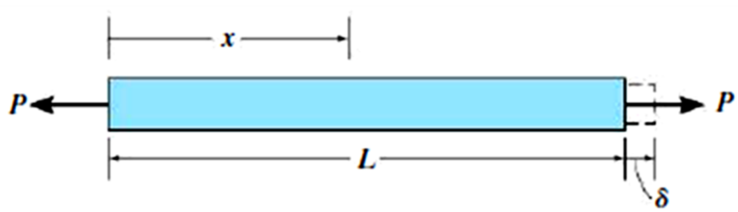
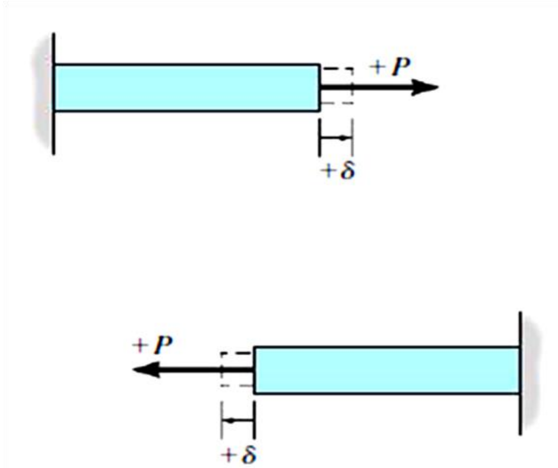


Fig. 3

Sign Convention:

In order to apply Eq. 3, we must develop a sign convention for the internal axial force and the displacement of one end of the bar with respect to the other end. To do so, we will consider both the force and displacement to be ***positive*** if they cause ***tension and elongation,*** respectively, Fig. 4; whereas a

negative force and displacement will cause ***compression* and *contraction***, respectively.

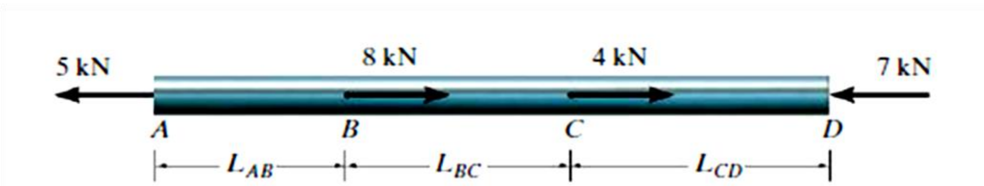


Positive sign convention for P and δ

Fig. 4

Example,

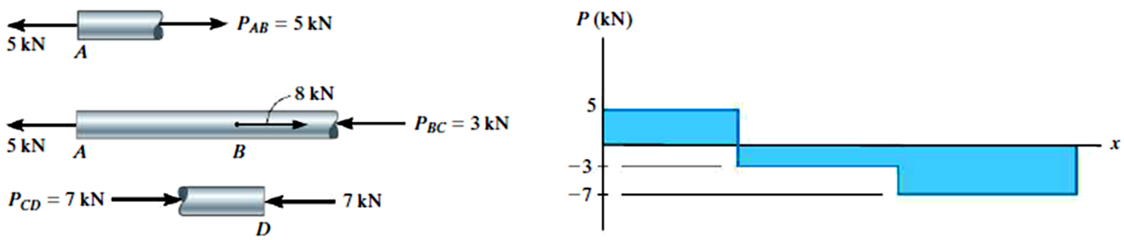
Consider the bar shown in Fig. *a*. Determine the displacement of end *A* relative to end *D*.



(a)

Solution

The *internal axial forces* "P,"are determined by the method of sections for each segment, Fig. 5 *b*. They are P_{AB} = +5 KN, P_{BC} = -3 KN, P_{CD} = -7 KN. This variation in axial load is shown on the axial or ***normal force diagram*** for the bar, Fig. 5 *c*.



(b) (c)

Fig. 5

Since we now know how the *internal* force varies throughout the bar's length, the displacement of end *A* relative to end *D* is determined from

$$\delta_{A/D} = \sum \frac{PL}{AE} = \frac{(5\text{ kN})L_{AB}}{AE} + \frac{(-3\text{ kN})L_{BC}}{AE} + \frac{(-7\text{ kN})L_{CD}}{AE}$$

Important Points

- *Saint-Venant's principle* states that both the localized deformation and stress which occur within the regions of load application or at the supports tend to "even out" at a distance sufficiently removed from these regions.
- The displacement of one end of an axially loaded member relative to the other end is determined by relating the applied *internal* load to the stress using $\sigma = P/A$ and relating the displacement to the strain using $\epsilon = d\delta/dx$. Finally these two equations are combined using Hooke's law, $\sigma = E\epsilon$, which yields Eq. 4–1.
- Since Hooke's law has been used in the development of the displacement equation, it is important that no internal load causes yielding of the material, and that the material behaves in a linear elastic manner.

Example

The assembly shown in Fig. consists of an aluminum tube *AB* having a cross-sectional area of 400 mm2. A steel rod having a diameter of 10 mm is attached to a rigid collar and passes through the tube. If a tensile load of 80 KN is applied to the rod, determine the displacement of the end *C* of the rod. Take E_{st} = 200 GPa, E_{al} = 70 GPa.

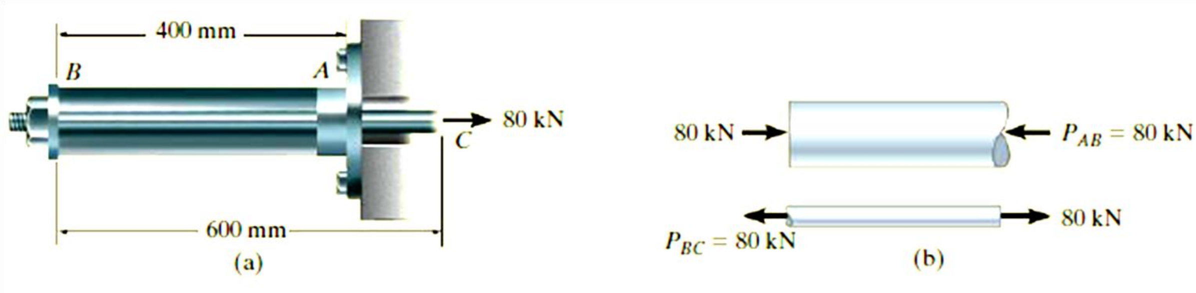


Fig.

SOLUTION

Internal Force

The free-body diagram of the tube and rod segments in Fig. 6 *b* , shows that the rod is subjected to a tension of 80 KN and the tube is subjected to a compression of 80 KN.

Displacement

We will first determine the displacement of end *C* with respect to end *B*. Working in units of Newton and meters, we have

$$\delta_{C/B} = \frac{PL}{AE} = \frac{[+80(10^3)\text{ N}]\,(0.6\text{ m})}{\pi\,(0.005\text{ m})^2[200\,(10^9)\text{ N/m}^2]} = +0.003056\text{ m} \rightarrow$$

The *positive sign* indicates that end *C* moves *to the right* relative to end *B*, since the bar elongates.

The displacement of end *B* with respect to the *fixed* end *A* is

$$\delta_B = \frac{PL}{AE} = \frac{[-80(10^3)\text{ N}](0.4\text{ m})}{[400\text{ mm}^2(10^{-6})\text{ m}^2/\text{mm}^2][70(10^9)\text{ N/m}^2]}$$
$$= -0.001143\text{ m} = 0.001143\text{ m} \rightarrow$$

Here the *negative sign* indicates that the tube shortens, and so *B* moves to the *right* relative to *A*.

Since both displacements are to the right, the displacement of *C* relative to the fixed end *A* is therefore

$$(\overset{+}{\rightarrow}) \qquad \delta_C = \delta_B + \delta_{C/B} = 0.001143\text{ m} + 0.003056\text{ m}$$
$$= 0.00420\text{ m} = 4.20\text{ mm} \rightarrow \qquad \textit{Ans.}$$