Chapter Six

Methods of Integration

6.1. Integration by Parts:

The formula for integration by parts comes from the product rule:

$$d(u.v) = u.dv + v.du \Rightarrow u.dv = d(u.v) - v.du$$

and integrated to give: $\int u.dv = \int d(u.v) - \int v.du$

then the integration by parts formula is:-

$$\int u.dv = u.v - \int v.du$$

Rule for choosing u and dv is:

For u: choose something that becomes simpler when differentiated.

For dv: choose something whose integral is simple.

It is not always possible to follow this rule, but when we can.

Example 1: Evaluate the following integrals:

$$1)\ \int xe^x\,dx \qquad 2)\ \int x\cdot \cos x\,dx$$

Solution.

$$1)\quad let\quad \left.\begin{array}{l} u = x \Rightarrow du = dx \\ dv = e^x dx \Rightarrow v = e^x \end{array}\right\} \Rightarrow \int udv = u\cdot v - \int vdu$$

$$\int x\cdot e^x dx = x\cdot e^x - \int e^x dx = x\cdot e^x - e^x + c$$

$$2)\quad let\quad \left.\begin{array}{l} u = x \quad \Rightarrow \quad du = dx \\ dv = \cos x\, dx \Rightarrow v = \sin x \end{array}\right\} \Rightarrow \quad \int u dv = u \cdot v - \int v du$$

$$\int x \cdot \cos x\, dx = x \cdot \sin x - \int \sin x\, dx = x \cdot \sin x + \cos x + c$$

Example 2 : Evaluate the following integrals:

$$3)\ \int e^x Sinx\, dx \qquad\qquad 4)\ \int lnx^x\ dx$$

Solution.

3) Let $u = Sinx \Rightarrow du = Cosxdx$

$dv = e^x dx \Rightarrow v = e^x$

$$\int udv = uv - \int vdu \Rightarrow \int e^x Sinx\, dx = e^x Sinx - \int e^x Cosx\, dx$$

Let $u = Cosx \Rightarrow du = -Sinxdx$

$dv = e^x dx \Rightarrow v = e^x$

$$\int e^x Sinx\, dx = e^x Sinx - \left[e^x Cosx - \int -e^x Sinx\, dx\right]$$

$$\Rightarrow \int e^x Sinx\, dx = e^x Sinx - e^x Cosx - \int e^x Sinx\, dx$$

$$\Rightarrow 2\int e^x Sinx\, dx = e^x Sinx - e^x Cosx + C$$

$$\Rightarrow \int e^x Sinx\, dx = \frac{e^x Sinx - e^x Cosx + C}{2}$$

$$4)\ \int lnx^x\ dx = \int x\ lnx\ dx = \int\ lnx\ xdx$$

Let $u = lnx \Rightarrow du = \frac{1}{x}dx$

$$dv = xdx \Rightarrow v = \frac{x^2}{2}$$

$$\int udv = uv - \int vdu \Rightarrow \int lnx^x\ dx = \frac{x^2}{2}lnx - \int \frac{x^2}{2}\frac{1}{x}dx$$

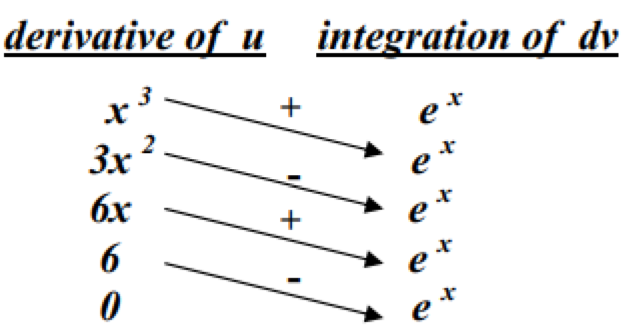
$$= \frac{x^2}{2}lnx - \frac{1}{2}\int x\,dx$$

$$= \frac{x^2}{2}lnx - \frac{1}{2}\frac{x^2}{2} + C$$

$$= \frac{x^2}{2}lnx - \frac{x^2}{4} + C$$

Example 3 : Evaluate the following integral:

$$\int x^3 e^x dx$$



$$\int x^3 e^x dx = x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x + C$$

6.2. Odd and Even Powers of Sine and Cosine

To integrate an odd positive power of $sinx$ (say $sin^{2n+1}x$) we split off a factor of $sinx$ and rewrite the remaining even power in terms of the cosine. We write:-

$$\int sin^{2n+1} x \cdot dx = \int (1-cos^2 x)^n \cdot sin\, x\ dx$$

$$and\ \int cos^{2n+1} x \cdot dx = \int (1-sin^2 x)^n \cdot cos\, x\ dx$$

Example : **Evaluate:**

$$1)\ \int sin^3 x\, dx \qquad\qquad 2)\ \int cos^5 x\, dx$$

Solution.

$$1)\ \int sin^3 x dx = \int sin^2 x\, sinx dx = \int (1 - cos^2 x)\, sinx\, dx$$

$$= \int (sinx - sinx\, cos^2 x)\, dx = \int sinx\, dx - \int sinx\, (cosx)^2\, dx$$

$$= \int sinx\, dx + \int -sinx\, (cosx)^2\, dx = -cosx + \frac{(cosx)^3}{3} + c = -cosx + \frac{cos^3 x}{3} + c$$

$$2)\ \int cos^5 x dx = \int cos^4 x\, cosx\, dx = \int (cos^2 x)^2\, cosx\, dx = \int (1 - sin^2 x)^2\, cosx\, dx$$

$$= \int (1 - 2sin^2 x + sin^4 x)\, cosx\, dx = \int (cosx\ - 2cosx\, sin^2 x + cosx\, sin^4 x)\, dx$$

$$= \int cosx\, dx - 2\int cosx\ sin^2 x dx + \int cosx\, sin^4 x\ dx$$

$$= sinx - 2\frac{sin^3 x}{3} + \frac{sin^5 x}{5} + c$$

To integrate an even positive power of $sinx$ (say $sin^{2n}x$) we use the relations:

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad or \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

then we can write:-

$$\int \sin^{2n} x \cdot dx = \int \left(\frac{1-\cos 2x}{2}\right)^n dx$$

$$and \int \cos^{2n} x \cdot dx = \int \left(\frac{1+\cos 2x}{2}\right)^n dx$$

Example : Evaluate:

$$1)\ \int \cos^2\theta\, d\theta \qquad\qquad 2)\ \int \sin^4\theta\, d\theta$$

Solution.

$$1)\ \int cos^2\theta\, d\theta = \int \left(\frac{1+cos2\theta}{2}\right) d\theta = \frac{1}{2}\int (1+cos2\theta) d\theta = \frac{1}{2}\left[\int d\theta + \int cos2\theta d\theta\right]$$

$$= \frac{1}{2}\left[\theta + \frac{sin2\theta}{2}\right] + c = \frac{1}{2}\theta + \frac{sin2\theta}{4} + c$$

$$2)\ \int sin^4\theta\, d\theta = \int (sin^2\theta)^2\, d\theta = \int \left(\frac{1-cos2\theta}{2}\right)^2 d\theta = \frac{1}{4}\int (1-cos2\theta)^2 d\theta$$

$$= \frac{1}{4}\int (1-2cos2\theta + cos^2 2\theta)\, d\theta = \frac{1}{4}\left[\int d\theta - 2\int cos2\theta\, d\theta + \int cos^2 2\theta\, d\theta\right]$$

$$= \frac{1}{4}\left[\int d\theta - 2\int cos2\theta\, d\theta + \int \left(\frac{1+cos4\theta}{2}\right) d\theta\right]$$

$$= \frac{1}{4}\left[\theta - 2\frac{sin2\theta}{2} + \frac{1}{2}\left(\theta + \frac{sin4\theta}{4}\right)\right] + c$$

$$= \frac{1}{4}\theta - \frac{sin2\theta}{4} + \frac{1}{8}\theta + \frac{sin4\theta}{32} + c$$

$$= \frac{3}{8}\theta - \frac{sin2\theta}{4} + \frac{sin4\theta}{32} + c$$

To integrate the following identities:-

$$\int SinAx.SinBx\,dx\ ,\quad \int SinAx.CosBx\,dx\ ,\quad \int CosAx.CosBx\,dx$$

we use the following formulas:-

$$SinAx.SinBx = \frac{1}{2}[Cos(A-B)x - Cos(A+B)x]$$

$$SinAx.CosBx = \frac{1}{2}[Sin(A-B)x + Sin(A+B)x]$$

$$CosAx.CosBx = \frac{1}{2}[Cos(A-B)x + Cos(A+B)x]$$

Example : **Evaluate:**

1) $\int \sin 3x \cdot \cos 5x\,dx$ *2)* $\int \cos x \cdot \cos 7x\,dx$ *3)* $\int \sin x \cdot \sin 2x\,dx$

Solution.

$$1)\ \int sin3x.cos5xdx = \frac{1}{2}\int (sin(3-5)x + sin(3+5)x)\,dx$$

$$= \frac{1}{2}\int (\sin(-2)\,x + sin8x)dx = \frac{1}{2}\int (-sin2x + sin8x)\,dx$$

$$= \frac{1}{2}\left[\frac{cos2x}{2} - \frac{cos8x}{8}\right] + c = \left[\frac{cos2x}{4} - \frac{cos8x}{16}\right] + c$$

$$2)\ \int cosx.cos7xdx = \frac{1}{2}\int (cos(1-7)x + cos(1+7)x)\,dx$$

$$= \frac{1}{2}\int (\cos(-6)\,x + cos8x)dx = \frac{1}{2}\int (cos6x + cos8x)\,dx$$

$$= \frac{1}{2}\left[\frac{sin6x}{6} + \frac{sin8x}{8}\right] + c = \left[\frac{sin6x}{12} + \frac{sin8x}{16}\right] + c$$

3) $\int sinx.\,sin2xdx = \frac{1}{2}\int (cos(1-2)x - cos(1+2)x)\,dx$

$$= \frac{1}{2}\int (\cos(-1)\,x - cos3x)dx = \frac{1}{2}\int (cosx - cos3x)\,dx$$

$$= \frac{1}{2}\left[sinx - \frac{sin3x}{3}\right] + c = \left[\frac{sinx}{2} - \frac{sin3x}{6}\right] + c$$

6.3. Trigonometric substitutions:

Trigonometric substitutions enable us to replace the binomials $a^2 - u^2$, $a^2 + u^2$, and $u^2 + a^2$ be single square terms. We can use:

$$u = a\sin\theta \quad for \quad a^2 - u^2 = a^2 - a^2\sin^2\theta = a^2(1-\sin^2\theta) = a^2\cos^2\theta$$
$$u = a\tan\theta \quad for \quad a^2 + u^2 = a^2 + a^2\tan^2\theta = a^2(1+\tan^2\theta) = a^2\sec^2\theta$$
$$u = a\sec\theta \quad for \quad u^2 - a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2\tan^2\theta$$

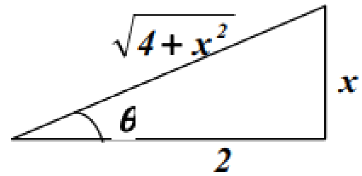
Example : **Evaluate the following integrals:**

2) $\int \frac{dx}{\sqrt{4+x^2}}$ 3) $\int \frac{dx}{4-x^2}$ 5) $\int \frac{dt}{\sqrt{25t^2-9}}$

Solution.

2) let $x = 2\tan\theta \Rightarrow dx = 2\sec^2\theta \cdot d\theta \qquad \tan\theta = \frac{x}{2}$

$$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2\sec^2\theta\ d\theta}{\sqrt{4+4\tan^2\theta}} = \int \sec\theta\ d\theta = \ln\left|\sec\theta + \tan\theta\right| + c$$



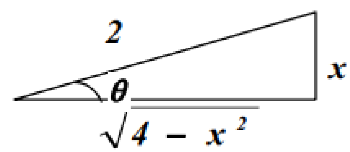
$$= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + c$$

$$= \ln\left|\sqrt{4+x^2} + x\right| + c' \qquad \text{where } c' = c - \ln 2$$

3) let $x = 2\sin\theta \Rightarrow dx = 2\cos\theta \cdot d\theta$

$$\int \frac{dx}{4-x^2} = \int \frac{2\cos\theta\ d\theta}{4-4\sin^2\theta} = \frac{1}{2}\int \frac{d\theta}{\cos\theta} = \frac{1}{2}\int \sec\theta\ d\theta$$

$$= \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right| + c$$



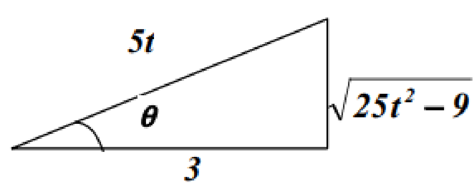
$$= \frac{1}{2}\ln\left|\frac{2}{\sqrt{4-x^2}} + \frac{x}{\sqrt{4-x^2}}\right| + c$$

$$= \frac{1}{2}\ln\left|\frac{2+x}{\sqrt{(2-x)(2+x)}}\right| + c = \frac{1}{2}\ln\left|\sqrt{\frac{2+x}{2-x}}\right| + c = \frac{1}{4}\ln\left|\frac{2+x}{2-x}\right| + c$$

5) let $5t = 3\sec\theta \Rightarrow 5dt = 3\sec\theta \cdot \tan\theta\ d\theta$

$$\int \frac{dt}{\sqrt{25t^2-9}} = \int \frac{3/5\ \sec\theta \cdot \tan\theta\ d\theta}{\sqrt{9\sec^2\theta - 9}} = \frac{1}{5}\int \sec\theta\ d\theta$$

$$= \frac{1}{5}\ln\left|\sec\theta + \tan\theta\right| + c$$



$$= \frac{1}{5}\ln\left|\frac{5t}{3} + \frac{\sqrt{25t^2-9}}{3}\right| + c$$

$$= \frac{1}{5}\ln\left|5t + \sqrt{25t^2-9}\right| + c' \qquad \text{where } c' = c - \frac{1}{5}\ln 3$$