Chapter Seven

Application of Integrals (تطبيقات التكامل)

7.1. Definite integrals: (التكاملات المحددة)

If $f(x)$ is continuous in the interval $a \le x \le b$ and it is integrable in the interval then the area under the curve:-

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

where $F(x)$ is any function such that $F'(x) = f(x)$ in the interval.

Some of the more useful properties of the definite integral are:-

1) $\int_a^b c\ f(x)\,dx = c\int_a^b f(x)\,dx$, *where c is constant.*

2) $\int_a^b \big(f(x) \mp g(x)\big)dx = \int_a^b f(x)dx \mp \int_a^b g(x)dx$

3) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

4) *Let* $a < c < b$ *then* $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

5) $\int_a^a f(x)\,dx = 0$

6) *If* $f(x) \ge 0$ *for* $a \le x \le b$ *then* $\int_a^b f(x)\,dx \ge 0$

7) *If* $f(x) \le g(x)$ *for* $a \le x \le b$ *then* $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$

Example 1. **Evaluate the following definite integrals:**

$$1)\ \int_{2}^{6} \frac{dx}{x+2} \qquad 2)\ \int_{\pi/2}^{3\pi/2} \cos x\, dx$$

$$5)\ \int_{-2}^{4} e^{-\frac{x}{2}}\, dx \qquad 6)\ \int_{0}^{\pi} (\pi - x)\cdot \cos x\, dx$$

Solution.

$$1)\ \int_{2}^{6} \frac{dx}{x+2} = \ln(x+2)\Big|_{2}^{6} = \ln(6+2) - \ln(2+2) = \ln 8 - \ln 4 = 3\ln 2 - 2\ln 2 = \ln 2$$

$$2)\ \int_{\pi/2}^{3\pi/2} \cos x\, dx = \sin x\Big|_{\pi/2}^{3\pi/2} = \sin(\frac{3}{2}\pi) - \sin(\frac{\pi}{2}) = -1 - 1 = -2$$

$$5)\ \int_{-2}^{4} e^{-\frac{x}{2}}\, dx = -2e^{-\frac{x}{2}}\Big|_{-2}^{4} = -2(e^{-2} - e) = 2(e - e^{-2})$$

$$6)\ \text{Let} \quad u = \pi - x \quad \Rightarrow \quad du = -dx \quad \& \quad dv = \cos x\, dx \quad \Rightarrow \quad v = \sin x$$

$$\int_{0}^{\pi} (\pi - x)\cdot \cos x\, dx = (\pi - x)\sin x\Big|_{0}^{\pi} + \int_{0}^{\pi} \sin x\, dx = (\pi - x)\sin x - \cos x\Big|_{0}^{\pi}$$

$$= (\pi - \pi)\sin\pi - \cos\pi - \left((\pi - 0)\sin 0 - \cos 0\right) = 0 - (-1) - (0 - 1) = 2$$

7.2. Area Between Two Curves: (المساحة المحددة بين منحنيين)

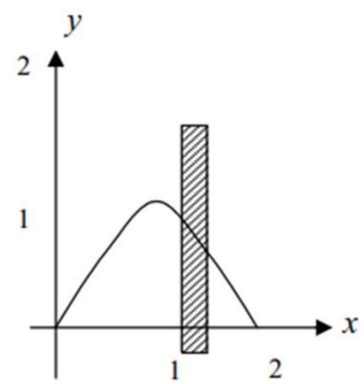
Suppose that $y_1 = f_1(x)$ and $y_2 = f_2(x)$ define two functions of x that are continuous for $a \le x \le b$ then the area bounded above by the y_1 curve, below by y_2 curve and on the sides by the vertical lines $x = a$ and $x = b$ is:

$$A = \int_a^b [f_1(x) - f_2(x)]\,dx$$

Example 2. **Find the area bounded by the *x*-axis and the curve:**

$$y = 2x - x^2$$

Solution.



$$y = 0 \quad \ldots(1)$$

$$y = 2x - x^2 \quad \ldots(2)$$

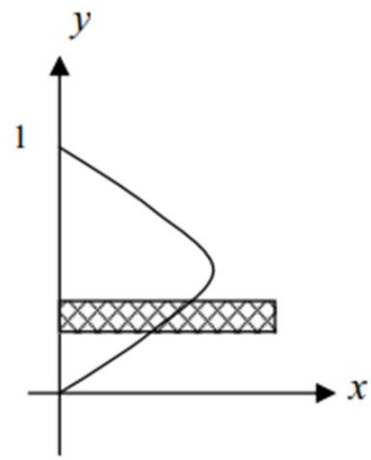
$$2x - x^2 = 0 \Rightarrow x(2 - x) = 0 \Rightarrow either\ x = 0\ \ or\ x = 2$$

The intersection points (0,0), (2,0)

$$Area = \int_0^2 (2x - x^2)dx = 2\frac{x^2}{2} - \frac{x^3}{3}\Big|_0^2 = 2^2 - \frac{2^3}{3} = 4 - \frac{8}{3} = \frac{4}{3}\ unit^2$$

***Example 3*. Find the area bounded by the *y*-axis and the curve:**

$$\boldsymbol{x = y^2 - y^3}$$



Solution.

$$\boldsymbol{x = 0} \quad \dots(\boldsymbol{1})$$

$$\boldsymbol{x = y^2 - y^3} \quad \dots(\boldsymbol{2})$$

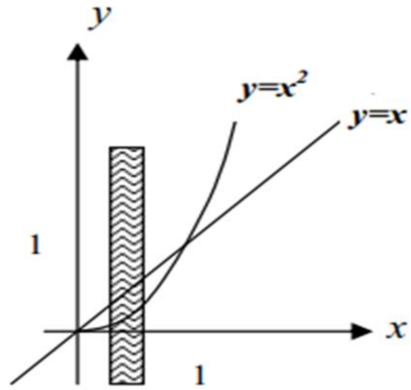
$$\boldsymbol{y^2 - y^3 = 0 \Rightarrow y^2(1 - y) = 0 \Rightarrow either\ y = 0\ \ or\ y = 1}$$

The intersection points (0,0), (0,1)

$$Area = \int_0^1 (\boldsymbol{y^2 - y^3})dy = \frac{y^3}{3} - \frac{y^4}{4}\Big|_0^1 = \frac{1^3}{3} - \frac{1^4}{4} = \frac{1}{12}\ \ unit^2$$

Example 4. **Find the area bounded by the curve $y = x^2$ and the line $y = x$.**

Solution.



$$y = x^2 \quad \dots(1)$$

$$y = x \quad \dots(2)$$

$$x = x^2 \Rightarrow x^2 - x = 0 \Rightarrow x(x-1) = 0 =$$

The intersection points (0,0), (1,1)

$$Area = \int_0^1 (x - x^2)dx = \frac{x^2}{2} - \frac{x^3}{3}\bigg|_0^1 = \frac{1^2}{2} - \frac{1^3}{3} = \frac{1}{6} \;\; unit^2$$

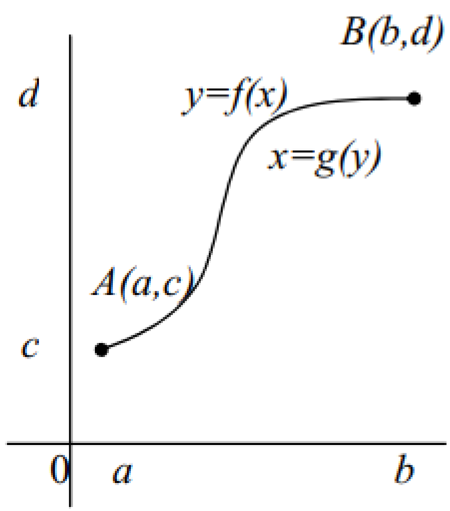
7.3. The Length of a Plane Curve: (طول المنحنى)

The length of the curve $y = f(x)$ from point A(a,c) to B(b,d) is:-

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

If x can be expressed as a function of y then the length is:-

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$



Example 5. **Find the length of the curve: $\boldsymbol{y = 3x + 4}$ from $\boldsymbol{x = 1}$ to $\boldsymbol{x = 5}$.**

Solution.

$$\boldsymbol{y = 3x + 4 \implies \frac{dy}{dx} = 3}$$

$$L = \int_a^b \sqrt{1 + \left(\boldsymbol{\frac{dy}{dx}}\right)^2}\, dx \implies \boldsymbol{L} = \int_1^5 \sqrt{1 + (\boldsymbol{3})^2}\, dx$$

$$\implies \boldsymbol{L} = \int_1^5 \sqrt{10}\, dx \implies \boldsymbol{L} = \sqrt{10}\, x \Big|_{\boldsymbol{1}}^{\boldsymbol{5}} = \sqrt{10}\,(5 - 1) = 4\sqrt{10}\ unit$$

Example 6. **Find the length of the curve:** $y = \frac{1}{3}(x^2+2)^{\frac{3}{2}}$ **from** $x = 0$ **to** $x = 3$.

Solution.

$$y = \frac{1}{3}(x^2+2)^{\frac{3}{2}} \Rightarrow \frac{dy}{dx} = \frac{1}{3} \times \frac{3}{2}(x^2+2)^{\frac{1}{2}} \times 2x \Rightarrow \frac{dy}{dx} = x(x^2+2)^{\frac{1}{2}}$$

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \Rightarrow L = \int_0^3 \sqrt{1+\left(x(x^2+2)^{\frac{1}{2}}\right)^2}\, dx$$

$$\Rightarrow L = \int_0^3 \sqrt{1+x^2(x^2+2)}dx \Rightarrow L = \int_0^3 \sqrt{1+x^4+2x^2}dx$$

$$\Rightarrow L = \int_0^3 \sqrt{x^4+2x^2+1}dx \Rightarrow L = \int_0^3 \sqrt{(x^2+1)^2}dx \Rightarrow L = \frac{x^3}{3} + x\Big|_0^3$$

$$\Rightarrow L = \frac{3^3}{3} + 3 = 9 + 3 = 12 \ \ unit$$

Example 7. **Find the length of the curve:** $\boldsymbol{y = \frac{x^3}{12} + \frac{1}{x}}$ **from** $\boldsymbol{x = 1}$ **to** $\boldsymbol{x = 4}$**.**

Solution.

$$y = \frac{x^3}{12} + \frac{1}{x} \Rightarrow \frac{dy}{dx} = \frac{3x^2}{12} - x^{-2} \Rightarrow \frac{dy}{dx} = \frac{x^2}{4} - \frac{1}{x^2}$$

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \Rightarrow L = \int_1^4 \sqrt{1 + \left(\frac{x^2}{4} - \frac{1}{x^2}\right)^2}\, dx$$

$$\Rightarrow L = \int_1^4 \sqrt{1 + \frac{x^4}{16} - \frac{1}{2} + \frac{1}{x^4}}\, dx \Rightarrow L = \int_1^4 \sqrt{\frac{x^4}{16} + \frac{1}{2} + \frac{1}{x^4}}\, dx$$

$$\Rightarrow L = \int_1^4 \sqrt{\left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2}\, dx \Rightarrow L = \int_1^4 \left(\frac{x^2}{4} + x^{-2}\right) dx$$

$$\Rightarrow L = \frac{x^3}{12} + \frac{x^{-1}}{-1}\Big|_1^4 \Rightarrow L = \frac{x^3}{12} - \frac{1}{x}\Big|_1^4 \Rightarrow L = \left(\frac{4^3}{12} - \frac{1}{4}\right) - \left(\frac{1^3}{12} - \frac{1}{1}\right)$$

$$= \left(\frac{64 - 4}{12}\right) - \left(\frac{1 - 12}{12}\right) = \left(\frac{61}{12}\right) - \left(\frac{-11}{12}\right) = \frac{72}{12} = 6\ unit$$

7.4. Volumes (الحجوم)

المجسم الدوراني هو كل جسم ينشأ من دوران منطقة مستوية حول محور دوران مستقيم ثابت دورة كاملة ويسمى الخط المستقيم بمحور الدوران

وهناك طريقتان لحساب الحجوم:-

1- Disk Method: (طريقة القرص)

اذا كان المجسم الدوراني ينتج عن دوران منطقة مستوية حول *x-axis* او مستقيم موازي له فان حجمه يكون

$$V = \pi \int_{a}^{b} [R(x)]^2 \, dx$$

حيث $R(x)$ هو نصف قطر القرص ويحسب من نقطة قطع الشريحة لمنطقة التكامل الى محور الدوران

اما اذا كان المجسم الدوراني ينتج عن دوران منطقة مستوية حول *y-axis* او مستقيم موازي له فان حجمه يكون

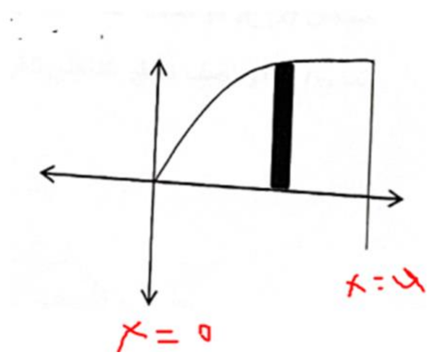
$$V = \pi \int_{a}^{b} [R(y)]^2 \, dy$$

ملاحظة: تكون الشريحة عمودية على محور التدوير دائما في هذه الطريقة.

Example 8. ***Find the volume generated by revolving the region bounded by:-***

a) $y = \sqrt{x}$, $x = 0$, ***and*** $x = 4$ ***about x-axis?***

Solution.

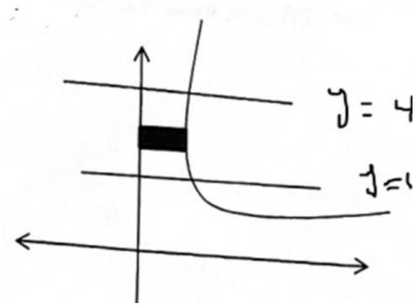


$$V = \pi \int_a^b [R(x)]^2\, dx = \pi \int_0^4 \left[\sqrt{x}\right]^2 dx = \pi \frac{x^2}{2}\Big|_0^4$$

$$\Rightarrow V = \pi \frac{4^2}{2} = 8\pi\ unit^3$$

b) $x = \frac{2}{y}$, $y = 1$, ***and*** $y = 4$ ***about y-axis?***

Solution.



$$V = \pi \int_a^b [R(y)]^2\, dy = \pi \int_1^4 \left[\frac{2}{y}\right]^2 dx = 4\pi \int_1^4 y^{-2} dy = 4\pi \frac{y^{-1}}{-1}\Big|_1^4$$

$$\Rightarrow V = -4\pi \frac{1}{y}\Big|_1^4 = -4\pi\left(\frac{1}{4} - \frac{1}{1}\right) = -4\pi\left(\frac{-3}{4}\right) = 3\pi\ \ unit^3$$

Example 9. Find the volume generated by revolving the region bounded by:

$y = \sqrt{x}\,, x = 0,$ ***and*** $x = 4$ ***about*** $y = 1$***?***

Solution.

$$V = \pi \int_a^b [R(x)]^2\, dx = \pi \int_0^4 \left[\sqrt{x} - 1\right]^2 dx == \pi \int_0^4 \left(x - 2\sqrt{x} + 1\right) dx$$

$$\Rightarrow V = \pi \left(\frac{x^2}{2} - 2\,\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + x \right)\Bigg|_0^4 \Rightarrow V = \pi \left(\frac{4^2}{2} - \frac{4}{3}\sqrt{4^3} + 4 \right)$$

$$\Rightarrow V = \pi \left(12 - \frac{32}{3} \right) = \frac{4\pi}{3} unit^3$$

2- Washer's Method (طريقة القرص المجوف)

هذه الطريقة تكتب كتقرير من قبل الطالب

7.5. Surface Area of Solid of Revolution*:* (المساحات السطحية لجسم دوراني)

اذا كانت $f(x) \geq 0$ مستمرة وقابلة للاشتقاق وعلى الفترة المغلقة $[a, b]$ فان المساحة السطحية الناتجة من دوران المنحني $y = f(x)$ حول *x-axis* تكون

$$S = 2\pi \int_{a}^{b} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

وكذلك اذا كانت $g(x) \geq 0$ مستمرة وقابلة للاشتقاق وعلى الفترة المغلقة $[a, b]$ فان المساحة السطحية الناتجة من دوران المنحني $x = g(y)$ حول *y-axis* تكون

$$S = 2\pi \int_{a}^{b} x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Example 10. ***Find the area of the surface generated by revolving by the curve***

$x = 1 - y$***, and*** $0 \le y \le 1$***, about y-axis?***

Solution.

$$\frac{dx}{dy} = -1$$

We will use the formula

$$S = 2\pi \int_a^b x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \Rightarrow S = 2\pi \int_0^1 x\sqrt{1 + (-1)^2}\, dy$$

$$\Rightarrow S = 2\sqrt{2}\pi \int_0^1 (1 - y)dy = 2\sqrt{2}\pi \left(y - \frac{y^2}{2}\right)\Bigg|_0^1$$

$$= 2\sqrt{2}\pi \left(1 - \frac{1}{2}\right) = 2\sqrt{2}\pi \left(\frac{1}{2}\right) = \sqrt{2}\pi$$

Example 11. ***Find the area of the surface generated by revolving by the curve***

$y = 2\sqrt{x}$***, and*** $1 \leq x \leq 2$***, about x-axis?***

Solution.

$$\frac{dy}{dx} = 2 \times \frac{1}{2} x^{-\frac{1}{2}} \quad \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{x}}$$

We will use the formula

$$S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \Rightarrow S = 2\pi \int_1^2 y \sqrt{1 + \left(\frac{1}{\sqrt{x}}\right)^2}\, dx$$

$$\Rightarrow S = 2\pi \int_1^2 2\sqrt{x} \sqrt{1 + \frac{1}{x}}\, dx = 4\pi \int_1^2 \sqrt{x} \sqrt{\frac{x+1}{x}}\, dx = 4\pi \left(1 - \frac{1}{2}\right)$$

$$= 4\pi \int_1^2 \sqrt{x+1}\, dx = 4\pi \frac{(x+1)^{\frac{3}{2}}}{\frac{3}{2}} = \frac{8\pi}{3}\left(\sqrt{3^3} - \sqrt{2^3}\right) = \left(3\sqrt{3} - 2\sqrt{2}\right)\frac{8\pi}{3}$$