Chapter One
Limits and Continuity
1. Limits

Definition 1.1 (**Limits**): We write

$$\lim_{x \to c} f(x) = L$$

and say " the limit of $f(x)$, as x approaches c, equals L"

Example 1.2: Find the limit of the function $f(x) = x^2 - x + 2$ as x approaches 2.

Solution:

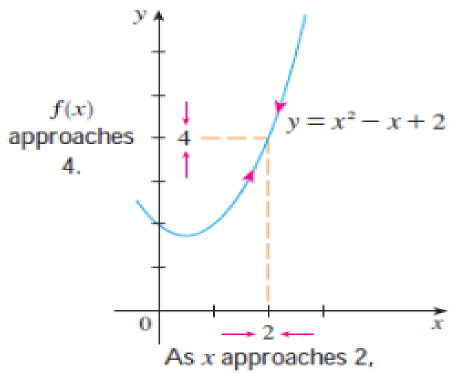

Figure 1

$$\lim_{x \to 2} x^2 - x + 2 = 2^2 - 2 + 2 = 4$$

Example 1.3: Guess the value of $\lim_{x \to 1} \frac{x-1}{x^2-1}$

Solution:

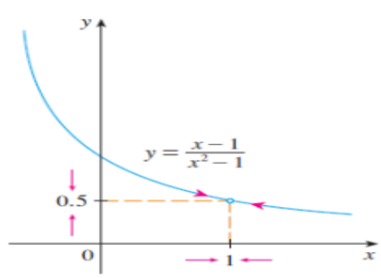

Figure 2

$$\lim_{x \to 1} \frac{x-1}{x^2-1} = \lim_{x \to 1} \frac{x-1}{(x+1)(x-1)} = \lim_{x \to 1} \frac{1}{x+1} = \frac{1}{1+1} = \frac{1}{2} = 0.5$$

The Limit Laws

To calculate limits of functions that are arithmetic combinations of functions having known limits, we can use several easy rules:

THEOREM 1—Limit Laws

If L, M, c, and k are real numbers and

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x\to c}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x\to c}(f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x\to c}(k \cdot f(x)) = k \cdot L$
4. *Product Rule:* $\lim_{x\to c}(f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x\to c}[f(x)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{x\to c}\sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

(If n is even, we assume that $f(x) \geq 0$ for x in an interval containing c.)

Example 1.4: Find the following limits

$$(a) \lim_{x\to 2} x^3 + 4x^2 - 3 \qquad (b) \lim_{x\to -1} \frac{x^4 + x^2 - 1}{x^2 + 5} \qquad (c) \lim_{x\to -2} \sqrt{4x^2 - 3}$$

Solution:

$$(a) \lim_{x\to 2} x^3 + 4x^2 - 3 = \lim_{x\to 2} x^3 + \lim_{x\to 2} 4x^2 - \lim_{x\to 2} 3 = (2)^3 + 4(2)^2 - 3 = 21$$

$$(b) \lim_{x\to -1} \frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{\lim_{x\to -1} x^4 + x^2 - 1}{\lim_{x\to -1} x^2 + 5} = \frac{(-1)^4 + (-1)^2 - 1}{(-1)^2 + 5} = \frac{1}{6}$$

$$(c) \lim_{x\to -2} \sqrt{4x^2 - 3} = \sqrt{\lim_{x\to -2} 4x^2 - 3} = \sqrt{4(-2)^2 - 3} = \sqrt{13}$$

THEOREM 2—Limits of Polynomials

If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then

$$\lim_{x\to c} P(x) = P(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0.$$

ملاحظة : غاية الدالة كثيرة الحدود تحل بالتعويض المباشر.

Example 1.5 : Find $\lim_{x\to 3} x^3 - 3x^2 + 2x - 1$

Solution: $\lim_{x\to 3} x^3 - 3x^2 + 2x - 1 = 3^3 - 3(3)^2 + 2(3) - 1 = 5$

THEOREM 3—Limits of Rational Functions

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x\to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

ملاحظة : غاية الدالة الكسرية اذا كان المقام لا يساوي صفر تحل بالتعويض المباشر اما اذا كان المقام يساوي صفر فتحل بالاستعانة باحدى طرق التحليل المعروفة كـ (الفرق بين مربعين , العامل المشترك, الفرق او مجموع مكعبين او التجربة) او باستخدام العامل المنسب (المرافق).

Example 1.6 : Find the following limits

$$(1)\ \lim_{x\to -1} \frac{x^3 + 4x^2 - 3}{x^2 + 5} \qquad (2)\ \lim_{x\to 1} \frac{x^2 + x - 2}{x^2 - x} \qquad (3)\ \lim_{x\to 0} \frac{\sqrt{x^2 + 100} - 10}{x^2}$$

Solution:

$$(1)\ \lim_{x\to -1} \frac{x^3 + 4x^2 - 3}{x^2 + 5} = \frac{(-1)^3 + 4(-1)^2 - 3}{(-1)^2 + 5} = \frac{-1 + 4 - 3}{6} = \frac{0}{6} = 0$$

$$(2)\ \lim_{x\to 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x\to 1} \frac{(x+2)(x-1)}{x(x-1)} = \lim_{x\to 1} \frac{x+2}{x} = \frac{1+2}{1} = 3$$

$$(3)\ \lim_{x\to 0} \frac{\sqrt{x^2 + 100} - 10}{x^2} = \lim_{x\to 0} \frac{\sqrt{x^2 + 100} - 10}{x^2} \times \frac{\sqrt{x^2 + 100} + 10}{\sqrt{x^2 + 100} + 10}$$

$$\lim_{x\to 0} \frac{x^2 + 100 - 100}{x^2\left(\sqrt{x^2 + 100} + 10\right)} = \lim_{x\to 0} \frac{x^2}{x^2\left(\sqrt{x^2 + 100} + 10\right)} = \lim_{x\to 0} \frac{1}{\sqrt{x^2 + 100} + 10} = \frac{1}{20} = 0.05$$

Properties of Limits: خصائص الغايات

1- غاية الدالة الثابتة تساوي الثابت نفسه

1) $\lim_{x \to c} K = K$

2- غاية الدالة المحايدة x تساوي ثابت الاقتراب c

2) $\lim_{x \to c} x = c$

3) $\lim_{x \to 0^+} \frac{1}{x} = +\infty$ & $\lim_{x \to 0^-} \frac{1}{x} = -\infty$ but $\lim_{x \to 0} \frac{x}{1} = 0$

4) $\lim_{x \to 0} sinx = 0$ & $\lim_{x \to 0} cosx = 1$ & $\lim_{x \to 0} tanx = 0$

5) $\lim_{x \to 0} \frac{sinax}{ax} = 1$ & $\lim_{x \to 0} \frac{ax}{sinax} = 1$

6) $\lim_{x \to 0} \frac{tanax}{ax} = 1$ & $\lim_{x \to 0} \frac{ax}{tanax} = 1$

Indeterminate Forms:

There are seven indeterminate forms in limits:

$$\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty, \infty - \infty, 0^{\infty}, \infty^{0} \text{ and } 1^{\infty}$$

Example 1.7 : Find $\lim_{x \to 0} \frac{1}{x^2}$ if it exists.

Solution: As x becomes close to 0, x^2 also becomes close to 0, and $1/x^2$ becomes very large. (See the table).

x	$\frac{1}{x^2}$
±1	1
±0.5	4
±0.2	25
±0.1	100
±0.05	400
±0.01	10,000
±0.001	1,000,000

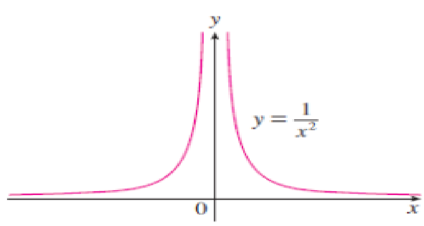

Figure 4

Hence, $\lim_{x \to 0} \frac{1}{x^2} = \frac{1}{0} = \infty$. So, $\lim_{x \to 0} \frac{1}{x^2}$ does not exists.

Example 1.8 : Investigate $\lim_{x \to 0} sin(\frac{\pi}{x})$

Solution: The function $f(x) = sin\ (\pi/x)$ is undefined at 0. The graph of function was shown in Figure 5.

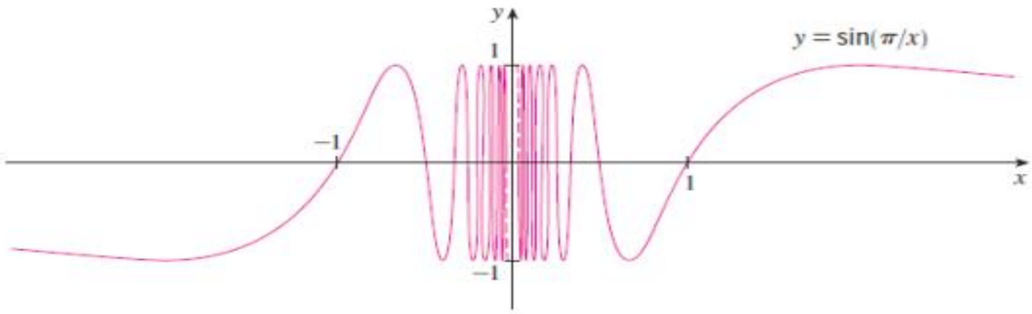

Figure 5

Since, $\lim_{x\to 0} sin(\frac{\pi}{x}) = sin\left(\frac{\pi}{0}\right) = sin\infty = \infty$. So, $\lim_{x\to 0} sin(\frac{\pi}{x})$ does not exists.

THEOREM 4—The Sandwich Theorem
Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L.$$

Then $\lim_{x\to c} f(x) = L$.

The Sandwich Theorem is also called the **Squeeze Theorem**

Example 1.9 : Show that $\lim_{x\to 0} x^2\, sin\frac{1}{x} = 0$.

Solution: First note that we cannot use

$$\lim_{x\to 0} x^2\, sin\frac{1}{x} = \lim_{x\to 0} x^2 \times \lim_{x\to 0} sin\frac{1}{x} = 0 \times \infty$$

because $\lim_{x\to 0} sin\frac{1}{x}$ does not exists. However, since

$$-1 \le sin\frac{1}{x} \le 1$$

we have,

$$-x^2 \le x^2 sin\frac{1}{x} \le x^2$$

$$\lim_{x\to 0} -x^2 \le \lim_{x\to 0} x^2 sin\frac{1}{x} \le \lim_{x\to 0} x^2$$

$$0 \le \lim_{x\to 0} x^2 sin\frac{1}{x} \le 0$$

So, by Squeeze Theorem, we obtain

$$\lim_{x\to 0} x^2 sin\frac{1}{x} = 0.$$

One-Sided Limits

In this section we extend the limit concept to one-sided limits, which are limits as x approaches the number c from the left-hand side (where x < c) or the right-hand side (x > c) only.

> **THEOREM 6** Suppose that a function f is defined on an open interval containing c, except perhaps at c itself. Then $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:
>
> $$\lim_{x\to c} f(x) = L \quad \Longleftrightarrow \quad \lim_{x\to c^-} f(x) = L \quad \text{and} \quad \lim_{x\to c^+} f(x) = L.$$

In another word:

- Right-hand limit is the limit of $f(x)$ as x approaches c from the right, or $\lim_{x\to c^+} f(x) = L$
- Left-hand limit is the limit of $f(x)$ as x approaches c from the left, or $\lim_{x\to c^-} f(x) = L$
- $\lim_{x\to c^+} f(x) = L$ if and only if $\lim_{x\to c^+} f(x) = L$ and $\lim_{x\to c^-} f(x) = L$

Example 1.10 : Let $f(x) = \begin{cases} x^2+1\ , & x \le 1 \\ 3-x\ \ , & x > 1 \end{cases}$

(a) Find $\lim_{x\to 1^+} f(x)$ and $\lim_{x\to 1^-} f(x)$

(b) Does $\lim_{x\to 1} f(x)$ exist ? why ?

ملاحظة :1- تكون غاية الدالة الفرعية موجودة اذا كانت الغاية من اليمين تساوي الغاية من اليسار

2 - علامة اكبر (>) تدل على غاية اليمين وعلامة اصغر (<) تدل على غاية اليسار

Solution:

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 3 - x = 3 - 1 = 2$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 + 1 = 1^2 + 1 = 2$$

Hence, $\lim_{x\to 1} f(x) = 2$, so the limit exist.

Example 1.11: Let $f(x) = \begin{cases} x^2 + 1\,, & x \ge 0 \\ x\,, & x < 0 \end{cases}$

(a) Find $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^-} f(x)$

(b) Does $\lim_{x\to 0} f(x)$ exist ? why ?

Solution:

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 + 1 = 0^2 + 1 = 1$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x = 0$$

So, $\lim_{x\to 0} f(x)$ does not exist.

Piecewise-Defined Functions

Sometimes a function is described in pieces by using different formulas on different parts of its domain. One example is the absolute value function

$$|x| = \begin{cases} x, & x \ge 0 \quad \text{First formula} \\ -x, & x < 0 \quad \text{Second formula} \end{cases}$$

Absolute Value in Limit Problems

Example 1.12: Find

$$\lim_{x\to-2}(x+3)\frac{|x+2|}{x+2}$$

Solution:

$$|x+2| = \begin{cases} x+2 & ,x \geq -2 \\ -x-2 & ,x < -2 \end{cases}$$

$$\lim_{x\to-2^+}(x+3)\frac{x+2}{x+2} = \lim_{x\to-2^+}(x+3) = -2+3 = 1$$

$$\lim_{x\to-2^-}(x+3)\frac{-x-2}{x+2} = \lim_{x\to-2^-}(x+3)\frac{-(x+2)}{x+2} = -\lim_{x\to-2^-}(x+3)\frac{(x+2)}{x+2}$$

$$= -\lim_{x\to-2^-}(x+3) = -(-2+3) = -1$$

Limits Involving $\left(\frac{\sin x}{x}\right)$

THEOREM 7—Limit of the Ratio $\sin\theta/\theta$ as $\theta \to 0$

$$\lim_{\theta\to0}\frac{\sin\theta}{\theta} = 1 \qquad (\theta \text{ in radians}) \qquad (1)$$

Example 1.13: Show that

$$(a)\ \lim_{x\to0}\frac{\sin 2x}{5x} = \frac{2}{5} \qquad and \qquad (b)\ \lim_{x\to0}\frac{\cos x - 1}{x} = 0$$

Solution:

$$(a)\ \lim_{x\to0}\frac{\sin 2x}{5x} = \frac{1}{5}\lim_{x\to0}\frac{\sin 2x}{x} = \frac{1}{5}\lim_{x\to0}\frac{2\sin 2x}{2x} = \frac{2}{5}\lim_{x\to0}\frac{\sin 2x}{2x} = \frac{2}{5}\times 1 = \frac{2}{5}$$

$$(b)\ \lim_{x\to0}\frac{\cos x - 1}{x} = \lim_{x\to0}\frac{\left(1-2\sin^2\left(\frac{x}{2}\right)\right)-1}{x} = \lim_{x\to0}\frac{-2\sin^2\left(\frac{x}{2}\right)}{x}$$

Let $\theta = \frac{x}{2}$

$$= -\lim_{x\to0}\frac{2\sin^2(\theta)}{2\theta} = = -\lim_{x\to0}\frac{\sin\theta}{\theta}\sin\theta = -(1)(0) = 0$$

Example 1.14: Find

$$\lim_{t\to 0}\frac{tant\ sec2t}{3t}$$

Solution :

$$\lim_{t\to 0}\frac{tant\ sec2t}{3t}=\frac{1}{3}\lim_{t\to 0}\frac{\frac{sint}{cost}\ \frac{1}{cos2t}}{t}=\frac{1}{3}\lim_{t\to 0}\frac{sint}{t}\times\frac{1}{cost}\times\frac{1}{cos2t}$$

$$=\frac{1}{3}\lim_{t\to 0}\frac{sint}{t}\times\lim_{t\to 0}\frac{1}{cost}\times\lim_{t\to 0}\frac{1}{cos2t}=\frac{1}{3}\times 1\times\frac{1}{1}\times\frac{1}{1}=\frac{1}{3}$$

Limits at Infinity

Sometimes we need to know what happens to $f(x)$ as x gets large and positive $(x\to\infty)$ or large and negative $(x\to-\infty)$.

Consider a function $f(x)=\frac{1}{x}$ what dose $\lim_{x\to\infty} f(x)$ equals ?

See that

x	1	10	100	1000	10000	100000	…	∞
$f(x)=\frac{1}{x}$	1	0.1	0.01	0.001	0.0001	0.00001	…	0

$f(x)$ gets close to 0, as x gets large and large, This written

$$\lim_{x\to+\infty}\frac{1}{x}=0$$

Or

$$\lim_{x\to-\infty}\frac{1}{x}=0$$

Example 1.15: Show that $\lim_{\theta\to\infty}\frac{sin\theta}{\theta}=0$.

Solution: We know

$$-1\le sin\theta\le 1$$

$$\frac{-1}{\theta} \leq \frac{sin\theta}{\theta} \leq \frac{1}{\theta}$$

$$\lim_{\theta\to\infty}\frac{-1}{\theta} \leq \lim_{\theta\to\infty}\frac{sin\theta}{\theta} \leq \lim_{\theta\to\infty}\frac{1}{\theta}$$

$$0 \leq \lim_{\theta\to\infty}\frac{sin\theta}{\theta} \leq 0$$

Then from Sandwich theorem:

$$\lim_{\theta\to\infty}\frac{sin\theta}{\theta} = 0$$

Limits at Infinity of Rational Functions

For rational function $\frac{f(x)}{g(x)}$

1- Divide both the numerator and the denominator by the highest power of x in denominator.

2- If degree of $f(x)$ less than degree of $g(x)$, then $\lim_{x\to\pm\infty}\frac{f(x)}{g(x)} = 0$

3- If degree of $f(x)$ equals degree of $g(x)$, then $\lim_{x\to\pm\infty}\frac{f(x)}{g(x)} = c$

4- If degree of $f(x)$ greater than degree of $g(x)$, then $\lim_{x\to\pm\infty}\frac{f(x)}{g(x)} = \pm\infty$

ملاحظة : لحساب غاية الدوال الكسرية عند الملانهاية نقسم كل حد موجود فيها على اعلى اس موجود في المقام فاذا كانت درجة البسط اقل من درجة المقام سيكون الناتج صفرا اما اذا كانت درجة البسط تساوي درجة المقام فسيكون الناتج عددا ثابتا اما اذا كانت درجة البسط اكبر من درجة المقام فسيكون الناتج $\pm\infty$

Examples 1.16: Find the following limits if they exist

$$(1)\ \lim_{x\to\infty}\frac{11x+2}{2x^3-1} \qquad (2)\ \lim_{x\to\infty}\frac{5x^2+8x-3}{3x^2+2} \qquad (3)\ \lim_{x\to\infty}\frac{x^3-4x^2+7}{2x^2-3}$$

Solution:

$$(1)\ \lim_{x\to\infty}\frac{11x+2}{2x^3-1} = \lim_{x\to\infty}\frac{\frac{11x}{x^3}+\frac{2}{x^3}}{\frac{2x^3}{x^3}-\frac{1}{x^3}} = \lim_{x\to\infty}\frac{\frac{11}{x^2}+\frac{2}{x^3}}{2-\frac{1}{x^3}} = \frac{0+0}{2-0} = \frac{0}{2} = 0$$

$$(2)\ \lim_{x\to\infty}\frac{5x^2+8x-3}{3x^2+2} = \lim_{x\to\infty}\frac{\frac{5x^2}{x^2}+\frac{8x}{x^2}-\frac{3}{x^2}}{\frac{3x^2}{x^2}+\frac{2}{x^2}} = \lim_{x\to\infty}\frac{5+\frac{8}{x}-\frac{3}{x^2}}{3+\frac{2}{x^2}} = \frac{5+0-0}{3+0} = \frac{5}{3}$$

(3) $$\lim_{x\to\infty}\frac{x^3-4x^2+7}{2x^2-3}=\lim_{x\to\infty}\frac{\frac{x^3}{x^2}-\frac{4x^2}{x^2}+\frac{7}{x^2}}{\frac{2x^2}{x^2}-\frac{3}{x^2}}$$

$$=\lim_{x\to\infty}\frac{x-4+\frac{7}{x^2}}{2-\frac{3}{x^2}}=\frac{\infty-4+0}{2-0}=\frac{\infty}{2}=\infty$$

Example 1.17: Find

$$\lim_{x\to\infty}\frac{x^{\frac{3}{2}}+5}{\sqrt{x^3+4}}$$

Solution:

$$\lim_{x\to\infty}\frac{\frac{x^{\frac{3}{2}}}{x^{\frac{3}{2}}}+\frac{5}{x^{\frac{3}{2}}}}{\sqrt{\frac{x^3}{x^3}+\frac{4}{x^3}}}=\lim_{x\to\infty}\frac{1+\frac{5}{x^{\frac{3}{2}}}}{\sqrt{1+\frac{4}{x^3}}}=\frac{1+0}{\sqrt{1+0}}=1$$

Home works

Find the following limits

1) $\lim_{x\to 0}\frac{5x^3+8x^2}{3x^4-16x^2}$, 2) $\lim_{x\to a}\frac{x^3-a^2}{x^4-a^4}$, 3) $\lim_{x\to 0}\frac{\text{Sin}5x}{\text{Sin}3x}$, 4) $\lim_{y\to a}\frac{\tan 2y}{3y}$

5) $\lim_{x\to 0}\frac{\text{Sin}2x}{2x^2+x}$, 6) $\lim_{x\to\infty}\left(1+\text{Cos}\frac{1}{x}\right)$, 7) $\lim_{x\to\infty}\frac{3x^3+5x^2-7}{10x^3-11x+5}$, 8) $\lim_{x\to\infty}\frac{x^3-1}{2x^2-7x+5}$

9) $\lim_{x\to 0}\left(1-\frac{\text{Sinx}}{x}\right)$, 10) $\lim_{x\to 0}\text{Sin}\left(\frac{\pi}{2}\text{Cos}(\tan x)\right)$, 11) $\lim_{y\to\infty}\frac{3y+7}{y^2-2}$, 12) $\lim_{x\to -1^-}\frac{1}{x+1}$

2. Continuity

We noticed that the limit of a function as x approaches c can often be found simply by calculating the value of the function at c. Functions with this property are called ***continuous*** at c.

Definition 2.1 (**Continuity at a Point**): A function f is ***continous*** at c if

$$\lim_{x\to c} f(x) = f(c)$$

Continuity Test

A function $f(x)$ is continuous at a point $x = c$ if and only if it meets the following three conditions.

1. $f(c)$ exists (c lies in the domain of f).

2. $\lim_{x\to c} f(x)$ exists (f has a limit as $x \to c$).

3. $\lim_{x\to c} f(x) = f(c)$ (the limit equals the function value).

Remark 2.2: If a function f is not ***continuous*** at a point c, we say that f is ***discontinuous*** at c and c is a point of discontinuity of f.

Example 2.3: **1-** The identity function $f(x) = x$ and constant functions $f(x) = k$ are continuous everywhere

2- The function $f(x) = \frac{1}{x}$ is continuous at every point of f except $x = 0$. It has a point of discontinuity at $x = 0$.

Algebraic combinations of continuous functions are continuous wherever they are defined.

THEOREM 8—Properties of Continuous Functions

If the functions f and g are continuous at $x = c$, then the following algebraic combinations are continuous at $x = c$.

1. *Sums:* $f + g$

2. *Differences:* $f - g$

3. *Constant multiples:* $k \cdot f$, for any number k

4. *Products:* $f \cdot g$

5. *Quotients:* f/g, provided $g(c) \neq 0$

6. *Powers:* f^n, n a positive integer

7. *Roots:* $\sqrt[n]{f}$, provided it is defined on an interval containing c, where n is a positive integer

Example 2.4: Show that the function $f(x) = x^2 + 2x - 3$ is continuous at $x = 2$.

Solution:

1- $f(2) = 2^2 + 2(2) - 3 = 5$

2- $\lim_{x\to 2} f(x) = \lim_{x\to 2} x^2 + 2x - 3 = 2^2 + 2(2) - 3 = 5$

3- $\lim_{x\to 2} f(x) = f(2) = 5$

So, f is continuous at $x = 2$.

Remark 2.5: The polynomial functions are continuous everywhere.

Example 2.6: Where are each of the following functions discontinuous ?

$$(a)\ \ f(x) = \frac{x^2 - x - 2}{x - 2}$$

$$(b)\ \ f(x) = \begin{cases} \frac{1}{x^2} & if\ x \neq 0 \\ 1 & if\ x = 0 \end{cases}$$

$$(c)\ \ f(x) = \begin{cases} \frac{x^2 - x - 2}{x - 2} & if\ x \neq 2 \\ 1 & if\ x = 2 \end{cases}$$

Solution: (a) Notice that $f(2)$ does not exists, so f is discontinuous at 2. But $f(x)$ is continuous at all other numbers.

(b)

1- $f(0) = 1$ exists.

2- $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1}{x^2} = \infty$ does not exist.

So f is discontinuous at 0.

(c)

1- $f(2) = 1$ exists

2- $\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2-x-2}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+1)}{x-2} = \lim_{x\to 2}(x + 1) = 2 + 1 = 3$

3- $\lim_{x\to 2} f(x) \neq f(2)$

So f is discontinuous at 2.

Example 2.7: Is the function $f(x) = \begin{cases} x^2 + 1, & x \geq 1 \\ 2x, & x < 1 \end{cases}$ continuous at $x = 1$?

Solution:

1- $f(1) = 1^2 + 1 = 2$

2-

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x^2 + 1 = 1^2 + 1 = 2$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 2x = 2(1) = 2$$

So, $\lim_{x \to 1} f(x)$ exists

3- $\lim_{x \to 1} f(x) = f(1) = 2$

Therefore, f is continuous at $x = 1$.

Example 2.8: What value should be assigned to a to make the function

$$f(x) = \begin{cases} x^2 - 1, & x < 3 \\ 2ax, & x \geq 3 \end{cases}$$

continuous at $x = 3$?

Solution: To make $f(x)$ continuous at $x = 3$

$$\lim_{x \to 3} f(x) = f(3)$$

$$\lim_{x \to 3^-} x^2 - 1 = 2a(3)$$

$$3^2 - 1 = 6a$$

$8 = 6a \Rightarrow a = \frac{8}{6} \Rightarrow a = \frac{4}{3}$.

Home Works

1- Is the function $f(x) = \frac{x}{x+1}$ continuous at $x = 2$?

2- Is the function $f(x) = \begin{cases} 3x + 1, & x \geq -1 \\ x^2, & x < -1 \end{cases}$ continuous at $x = -1$?