Chapter Four

Applications of Derivatives

L'Hopital rule : **Suppose that $f\,(x_0) = g\,(x_0) = 0$ and that the functions f and g are both differentiable on an open interval (a, b) that contains the point x_0. Suppose also that $g'\,(x) \neq 0$ at every point in $(a, b\,)$ except possibly x_0. Then:**

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

provided the limit exists .

Differentiate f and g as long as you still get the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ at $x = x_0$. Stop differentiating as soon as you get something else.

L'Hopital's rule does not apply when either the numerator or denominator has a finite non-zero limit.

Example : **Evaluate the following limits:**

$$1)\ \lim_{x \to 0} \frac{\sin x}{x} \qquad 2)\ \lim_{x \to 2} \frac{\sqrt{x^2+5}-3}{x^2-4}$$

$$3)\ \lim_{x \to 0} \frac{x - \sin x}{x^3} \qquad 4)\ \lim_{x \to \frac{\pi}{2}} -(x - \frac{\pi}{2}).\tan x$$

Solution.

$$1)\ \lim_{x \to 0} \frac{\sin x}{x} \Rightarrow \frac{0}{0} \quad using\ L'Hoptal's\ rule \Rightarrow$$

$$= \lim_{x \to 0} \frac{\cos x}{1} = \cos 0 = 1$$

$$2)\ \lim_{x \to 2} \frac{\sqrt{x^2+5}-3}{x^2-4} \Rightarrow \frac{0}{0} \quad using\ L'Hoptal's\ rule \Rightarrow$$

$$= \lim_{x \to 2} \frac{\frac{x}{\sqrt{x^2+5}}}{2x} = \lim_{x \to 2} \frac{1}{2\sqrt{x^2+5}} = \frac{1}{2\sqrt{4+5}} = \frac{1}{6}$$

$$3)\ \lim_{x \to 0} \frac{x - \sin x}{x^3} \Rightarrow \frac{0}{0} \quad using\ L'Hoptal's\ rule \Rightarrow$$

$$= \lim_{x \to 0} \frac{1 - \cos x}{3x^2} \Rightarrow \frac{0}{0} \quad using\ L'Hopital's\ rule \Rightarrow$$

$$= \frac{1}{6} \lim_{x \to 0} \frac{\sin x}{x} = \frac{1}{6}$$

$$4)\quad \lim_{x\to\frac{\pi}{2}} -(x-\frac{\pi}{2})\tan x \Rightarrow 0.\infty \text{ we can't using L'Hoptal's rule} \Rightarrow$$

$$= \lim_{x\to\frac{\pi}{2}} -\frac{x-\frac{\pi}{2}}{\cos x}.\lim_{x\to\frac{\pi}{2}} \sin x \Rightarrow \frac{0}{0} \text{ using L'Hopital's rule} \Rightarrow$$

$$= \lim_{x\to\frac{\pi}{2}} -\frac{1}{-\sin x}.\lim_{x\to\frac{\pi}{2}} \sin x = \frac{1}{\sin\frac{\pi}{2}}.\sin\frac{\pi}{2} = 1$$

Velocity and Acceleration and Other Rates of Changes

The average velocity of a body moving along a line is:

$$v_{av} = \frac{\Delta s}{\Delta t} = \frac{f(t+\Delta t)-f(t)}{\Delta t} = \frac{displacement}{time\ travelled}$$

The instantaneous velocity of a body moving along a line is the derivative of its position $s = f(t)$ with respect to time t.

$$\text{i.e.}\quad v = \frac{ds}{dt} = \lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t}$$

The rate at which the particle's velocity increase is called its acceleration a. If a particle has an initial velocity v and a constant acceleration a, then its velocity after time t is $v + at$.

$$average\ acceleration = a_{av} = \frac{\Delta v}{\Delta t}$$

The acceleration at an instant is the limit of the average acceleration for an interval following that instant, as the interval tends to zero.

$$\text{i.e.}\quad a = \lim_{\Delta t\to 0}\frac{\Delta v}{\Delta t}$$

The average rate of a change in a function $y = f(x)$ over the interval from x to $x + \Delta x$ is :

$$average\ rate\ of\ change = \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

The instantaneous rate of change of f at x is the derivative.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

provided the limit exists.

Example: The position s (in meters) of a moving body as a function of time t (in second) is : $s(t) = 2t^2 + 5t - 3$; find : The body's velocity at $t = 2$ seconds.

Solution.

$$v(t) = s'(t)$$

$$v(t) = 4t + 5$$

$$\text{at } t = 2$$

$$v(2) = 4(2) + 5 = 13\ m/s$$

Example: A particle moves along a straight line so that after t (seconds) , its distance from O a fixed point on the line is s (meters) , where $s(t) = t^3 - 3t^2 + 2t$:

i) when is the particle at O ?

ii) what is its velocity and acceleration at these times ?

iii)what is its average velocity during the first second ?

iv) what is its average acceleration between t = 0 and t = 2 ?

Solution.

$$i)\quad at\ \ s = 0 \Rightarrow t^3 - 3t^2 + 2t = 0 \Rightarrow t(t-1)(t-2) = 0$$

$$either\ \ t = 0\ \ or\ \ t = 1\ \ or\ \ t = 2\ \ sec.$$

$$ii)\quad velocity = v(t) = 3t^2 - 6t + 2 \Rightarrow v(0) = 2m/s$$

$$\Rightarrow v(1) = -1m/s$$

$$\Rightarrow v(2) = 2m/s$$

$$acceleration = a(t) = 6t - 6 \Rightarrow a(0) = -6m/s^2$$

$$\Rightarrow a(1) = 0m/s^2$$

$$\Rightarrow a(2) = 6m/s^2$$

$$iii)\quad v_{av} = \frac{\Delta s}{\Delta t} = \frac{s(1) - s(0)}{1 - 0} = \frac{1 - 3 + 2 - 0}{1} = 0m/s$$

$$iv)\quad a_{av} = \frac{\Delta v}{\Delta t} = \frac{v(2) - v(0)}{2 - 0} = \frac{2 - 2}{2} = 0m/s^2$$

Determinants and their properties

The determinant of a Matrix: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix, then $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is called determinant of a Matrix A, and denoted by Δ or $det(A)$ where

$$\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example: Find the determinant of the following matrices

$$A = \begin{bmatrix} 4 & 1 \\ -2 & 3 \end{bmatrix}, B = \begin{bmatrix} \sqrt{2} & -3 \\ 2 & 3\sqrt{2} \end{bmatrix}$$

Solution.

$$det(A) = \begin{vmatrix} 4 & 1 \\ -2 & 3 \end{vmatrix} = 4 \times 3 - (1 \times -2) = 12 + 2 = 14$$

$$det(B) = \begin{vmatrix} \sqrt{2} & -3 \\ 2 & 3\sqrt{2} \end{vmatrix} = \sqrt{2} \times 3\sqrt{2} - (-3 \times 2) = 6 + 6 = 12$$

Example: Find the value of h if

$$\begin{vmatrix} 3h & -2 \\ 3 & h \end{vmatrix} = 9$$

Solution.

$$3h \times h - (-2 \times 3) = 9$$

$$3h^2 + 6 = 9$$

$$3h^2 = 3$$

$$h^2 = 1$$

$$h = \pm 1$$

Simultaneous Equations

Cramer's rule is a method for solving linear simultaneous equations. It makes use of determinants and so a knowledge of these is necessary before proceeding.

Cramer's Rule (two equations)

If we are given a pair of simultaneous equations

$$ax_1 + by_1 = c_1$$

$$ax_2 + by_2 = c_2$$

Determinants are used to solve two first-degree equations with two variables, where

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

,

$$\Delta \mathrm{x} = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$$

And

$$\Delta \mathrm{y} = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

So,

$$x = \frac{\Delta \mathrm{x}}{\Delta} \text{ and } y = \frac{\Delta \mathrm{y}}{\Delta}$$

Example: Using Cramer's rule to solve the following equations

$$5x - 2y = 11$$

$$2x + 3y = 12$$

Solution.

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} 5 & -2 \\ 2 & 3 \end{vmatrix} = 5 \times 3 - (-2) \times 2 = 15 + 4 = 19$$

$$\Delta \mathrm{x} = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = \begin{vmatrix} 11 & -2 \\ 12 & 3 \end{vmatrix} = 11 \times 3 - (-2) \times 12 = 33 + 24 = 57$$

$$\Delta \mathrm{y} = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = \begin{vmatrix} 5 & 11 \\ 2 & 12 \end{vmatrix} = 5 \times 12 - 11 \times 2 = 60 - 22 = 38$$

$$\therefore \ \mathrm{x} = \frac{\Delta \mathrm{x}}{\Delta} = \frac{57}{19} = 3$$

$$\mathrm{y} = \frac{\Delta \mathrm{y}}{\Delta} = \frac{38}{19} = 2$$