Chapter Three

Derivatives

Let $y = f(x)$ be a function of x. If the limit:

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

exists and is finite, we call this limit the derivative of f at x and say that f is differentiable at x.

Example 1: **Find the derivative of the function: $f(x) = x^2 + x + 1$ at $x = 2$.**

Solution.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 + (x+\Delta x) + 1 - [x^2 + x + 1]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 + x + \Delta x + 1 - x^2 - x - 1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + \Delta x + (\Delta x)^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x(2x + 1 + \Delta x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 2x + 1 + \Delta x = 2x + 1 + 0 = 2x + 1$$

$$f'(2) = 2(2) + 1 = 5$$

Example 2: **Find the derivative of the function: $f(x) = \sqrt{x+1}$**

Solution.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \Longrightarrow f'(x) = \lim_{\Delta x \to 0} \frac{\sqrt{(x+\Delta x)+1} - \sqrt{x+1}}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x+1} - \sqrt{x+1}}{\Delta x} \cdot \frac{\sqrt{x+\Delta x+1} + \sqrt{x+1}}{\sqrt{x+\Delta x+1} + \sqrt{x+1}}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{\left(\sqrt{x+\Delta x+1}\right)^2 - (\sqrt{x+1})^2}{\Delta x(\sqrt{x+\Delta x+1} + \sqrt{x+1})}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{(x + \Delta x + 1) - (x + 1)}{\Delta x(\sqrt{x + \Delta x + 1} + \sqrt{x + 1})}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{x + \Delta x + 1 - x - 1}{\Delta x\left(\sqrt{x + \Delta x + 1} + \sqrt{x + 1}\right)} = \frac{1}{\sqrt{x + 0 + 1} + \sqrt{x + 1}}$$

$$= \frac{1}{\sqrt{x + 1} + \sqrt{x + 1}} = \frac{1}{2\sqrt{x + 1}}$$

Rules of Derivatives

Let c and n are constants, u, v and w are differentiable functions of x:

1. $\frac{d}{dx} c = 0$
2. $\frac{d}{dx} u^n = nu^{n-1} \frac{du}{dx} \Rightarrow \frac{d}{dx}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{du}{dx}$
3. $\frac{d}{dx} cu = c\frac{du}{dx}$
4. $\frac{d}{dx}(u \mp v) = \frac{du}{dx} \mp \frac{dv}{dx}$; $\frac{d}{dx}(u \mp v \mp w) = \frac{du}{dx} \mp \frac{dv}{dx} \mp \frac{dw}{dx}$
5. $\frac{d}{dx}(u.v) = u.\frac{dv}{dx} + v\frac{du}{dx}$
6. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ *where* $v \neq 0$

Example 3: Find $\frac{dy}{dx}$ for the following functions:

$a)\ y = (x^2 + 1)^5$

$b)\ y = [(5 - x)(4 - 2x)]^2$

$c)\ y = (2x^3 - 3x^2 + 6x)^{-5}$

$d)\ y = \frac{12}{x} - \frac{4}{x^3} + \frac{3}{x^4}$

$e)\ y = \frac{(x^2 + x)(x^2 - x + 1)}{x^3}$

$f)\ y = \frac{x^2 - 1}{x^2 + x - 2}$

Solution.

$$a)\quad \frac{dy}{dx} = 5(x^2 + 1)^4.2x = 10x(x^2 + 1)^4$$

b) $\frac{dy}{dx}=2[(5-x)(4-2x)][-2(5-x)-(4-2x)]$

$=8(5-x)(2-x)(2x-7)$

c) $\frac{dy}{dx}=-5(2x^3-3x^2+6x)^{-6}(6x^2-6x+6)$

$=-30(2x^3-3x^2+6x)^{-6}(x^2-x+1)$

d) $y=12x^{-1}-4x^{-3}+3x^{-4}\Rightarrow\frac{dy}{dx}=-12x^{-2}+12x^{-4}-12x^{-5}$

$\Rightarrow\frac{dy}{dx}=-\frac{12}{x^2}+\frac{12}{x^4}-\frac{12}{x^5}$

e) $y=\frac{(x+1)(x^2-x+1)}{x^3}\Rightarrow$

$\frac{dy}{dx}=\frac{x^3[(x^2-x+1)+(x+1)(2x-1)]-3x^2(x+1)(x^2-x+1)}{x^6}=-\frac{3}{x^4}$

f) $\frac{dy}{dx}=\frac{2x(x^2+x-2)-(x^2-1)(2x+1)}{(x^2+x-2)^2}=\frac{x^2-2x+1}{(x^2+x-2)^2}$

The Chain Rule

1. Suppose that $h=go\,f$ is the composite of the differentiable functions $y=g(t)$ and $x=f(t)$, then h is a differentiable functions of x whose derivative at each value of x is:

$$\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}$$

2. If y is a differentiable function of t and t is differentiable function of x, then y is a differentiable function of x:

$$y=g(t)\ \text{ and }\ t=f(x)\Rightarrow\frac{dy}{dx}=\frac{dy}{dt}*\frac{dt}{dx}$$

Example 4: Use the chain rule to express $\frac{dy}{dx}$ in terms of x and y:

$$a)\quad y=\frac{t^2}{t^2+1} \qquad and \qquad t=\sqrt{2x+1}$$

$$b)\quad y=\frac{1}{t^2+1} \qquad and \qquad x=\sqrt{4t+1}$$

Solution.

$$a)\quad y=\frac{t^2}{t^2+1}\Rightarrow\frac{dy}{dt}=\frac{2t(t^2+1)-2t.t^2}{(t^2+1)^2}=\frac{2t}{(t^2+1)^2}$$

$$t=(2x+1)^{\frac{1}{2}}\Rightarrow\frac{dt}{dx}=\frac{1}{2}.(2x+1)^{-\frac{1}{2}}.2=\frac{1}{\sqrt{2x+1}}$$

$$\frac{dy}{dx}=\frac{dy}{dt}.\frac{dt}{dx}=\frac{2t}{(t^2+1)^2}.\frac{1}{\sqrt{2x+1}}=\frac{2\sqrt{2x+1}}{((2x+1)+1)^2}.\frac{1}{\sqrt{2x+1}}=\frac{1}{2(x+1)^2}$$

$$b)\quad y=(t^2+1)^{-1}\Rightarrow\frac{dy}{dx}=-2t(t^2+1)^{-2}=-\frac{2t}{(t^2+1)^2}$$

$$x=(4t+1)^{\frac{1}{2}}\Rightarrow\frac{dx}{dt}=\frac{1}{2}(4t+1)^{-\frac{1}{2}}.4=\frac{2}{\sqrt{4t+1}}$$

$$\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}=-\frac{2t}{(t^2+1)^2}\div\frac{2}{\sqrt{4t+1}}=-\frac{t\sqrt{4t+1}}{(t^2+1)^2}$$

$$=-\frac{x^2-1}{4}.x\div\frac{1}{y^2}=-\frac{xy^2(x^2-1)}{4}$$

$$where \quad x=\sqrt{4t+1}\Rightarrow t=\frac{x^2-1}{4}$$

$$where \quad y=\frac{1}{t^2+1}\Rightarrow t^2+1=\frac{1}{y}$$

Higher Derivatives

If a function $y = f(x)$ possesses a derivative at every point of some interval, we may form the function $f'(x)$ and talk about its derivate, if it has one. The procedure is formally identical with that used before, that is:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx} f'(x) = \lim_{\Delta x \to 0} \frac{f'(x+\Delta x) - f'(x)}{\Delta x}$$

if the limit exists. This derivative is called the second derivative of y with respect to x. It is written in a number of ways, for example,

$$y^{(n)}, f^{(n)}(x), \frac{d^n y}{dx^n}$$

Example 5: Find all derivatives of the following function:

$$y = 3x^3 - 4x^2 + 7x + 10$$

Solution.

$$\frac{dy}{dx} = 9x^2 - 8x + 7 \quad , \quad \frac{d^2y}{dx^2} = 18x - 8$$

$$\frac{d^3y}{dx^3} = 18 \quad , \quad \frac{d^4y}{dx^4} = 0 = \frac{d^5y}{dx^5} = \ldots.$$

Example 6: Find the third derivative of the following function:

$$y = \frac{1}{x} + \sqrt{x^3}$$

Solution.

$$y = x^{-1} + x^{\frac{3}{2}} \Rightarrow \frac{dy}{dx} = -x^{-2} + \frac{3}{2}x^{\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = 2x^{-3} + \frac{3}{4}x^{-\frac{1}{2}} \Rightarrow \frac{d^3y}{dx^3} = -6x^{-4} - \frac{3}{8}x^{-\frac{3}{2}} \Rightarrow \frac{d^3y}{dx^3} = -\frac{6}{x^4} - \frac{3}{8\sqrt{x^3}}$$

Implicit Differentiation

If the formula for f is an algebraic combination of powers of x and y. To calculate the derivatives of these implicitly defined functions , we simply differentiate both sides of the defining equation with respect to x.

Example 7: Find $\frac{dy}{dx}$ for the following functions:

$a)\ x^2.y^2 = x^2 + y^2$ $\qquad$ $b)\ (x+y)^3 + (x-y)^3 = x^4 + y^4$

$c)\ \frac{x-y}{x-2y} = 2 \ \ at \ \ P(3,1)$ $\qquad$ $d)\ xy + 2x - 5y = 2 \ \ at \ \ P(3,2)$

Solution.

$$a)\ \ x^2.2y\frac{dy}{dx} + y^2.2x = 2x + 2y\frac{dy}{dx}$$

$$2x^2y\frac{dy}{dx} - 2y\frac{dy}{dx} = 2x - 2xy^2 \ \Rightarrow 2\frac{dy}{dx}(x^2y - y) = 2(x - xy^2\ )$$

$$\Rightarrow \frac{dy}{dx} = \frac{x - xy^2}{x^2y - y}$$

$$b)\ \ (x+y)^3 + (x-y)^3 = x^4 + y^4$$

$$3(x+y)^2\left(1 + \frac{dy}{dx}\right) + 3(x-y)^2\left(1 - \frac{dy}{dx}\right) = 4x^3 + 4y^3\frac{dy}{dx}$$

$$3(x+y)^2 + 3(x+y)^2\frac{dy}{dx} + 3(x-y)^2 - 3(x-y)^2\frac{dy}{dx} = 4x^3 + 4y^3\frac{dy}{dx}$$

$$3(x+y)^2\frac{dy}{dx} - 3(x-y)^2\frac{dy}{dx} - 4y^3\frac{dy}{dx} = 4x^3 - 3(x+y)^2 - 3(x-y)^2$$

$$\frac{dy}{dx}(3(x+y)^2 - 3(x-y)^2 - 4y^3) = 4x^3 - 3(x+y)^2 - 3(x-y)^2$$

$$\frac{dy}{dx} = \frac{4x^3 - 3(x+y)^2 - 3(x-y)^2}{3(x+y)^2 - 3(x-y)^2 - 4y^3}$$

$c)$ $\frac{x-y}{x-2y} = 2$ at $P(3,1)$

$$\frac{(x-2y)\left(1-\frac{dy}{dx}\right)-(x-y)\left(1-2\frac{dy}{dx}\right)}{(x-2y)^2} = 0$$

$$\Rightarrow \frac{(x-2y)-(x-2y)\frac{dy}{dx}-(x-y)+2(x-y)\frac{dy}{dx}}{(x-2y)^2} = 0$$

$$(x-2y)-(x-2y)\frac{dy}{dx}-(x-y)+2(x-y)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\big(2(x-y)-(x-2y)\big) = (x-y)-(x-2y)$$

$$\frac{dy}{dx} = \frac{(x-y)-(x-2y)}{2(x-y)-(x-2y)}$$

$$\left[\frac{dy}{dx}\right]_{(3,1)} = \frac{(3-1)-(3-2(1))}{2(3-1)-(3-2(1))} = \frac{1}{3}$$

$d)$ $xy + 2x - 5y = 2$ at $P(3,2)$

$$x\frac{dy}{dx}+y+2-5\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx}(x-5) = -y-2 \Rightarrow \frac{dy}{dx} = \frac{-y-2}{x-5}$$

$$\left[\frac{dy}{dx}\right]_{(3,2)} = \frac{-2-2}{3-5} = \frac{-4}{-2} = 2$$

Exponential functions : **If u is any differentiable function of x, then:**

$$7)\quad \frac{d}{dx}a^u = a^u.\ln a.\frac{du}{dx} \qquad and \qquad \frac{d}{dx}e^u = e^u.\frac{du}{dx}$$

Example 8: **Find $\frac{dy}{dx}$ for the following functions:**

$a)\ y = 2^{3x}$ $\qquad$ $b)\ y = 2^x.3^x$

$c)\ y = (2^x)^2$ $\qquad$ $d)\ y = x.2^{x^2}$

$e)\ y = e^{(x+e^{5x})}$ $\qquad$ $f)\ y = e^{\sqrt{1+5x^2}}$

Solution.

$$a)\ y = 2^{3x} \Rightarrow \frac{dy}{dx} = 2^{3x} * 3\ln 2$$

$$b)\ y = 2^x.3^x \Rightarrow y = 6^x \Rightarrow \frac{dy}{dx} = 6^x.\ln 6$$

$$c)\ y = (2^x)^2 \Rightarrow y = 2^{2x} \Rightarrow \frac{dy}{dx} = 2^{2x}\ln 2.2 = 2^{2x+1}\ln 2$$

$$d)\ y = x.2^{x^2} \Rightarrow \frac{dy}{dx} = x.2^{x^2}\ln 2.2x + 2^{x^2} = 2^{x^2}(2x^2\ln 2 + 1)$$

$$e)\ y = e^{(x+e^{5x})} \Rightarrow \frac{dy}{dx} = e^{(x+e^{5x})}(1+5e^{5x})$$

$$f)\ y = e^{(1+5x^2)^{\frac{1}{2}}} \Rightarrow \frac{dy}{dx} = e^{(1+5x^2)^{\frac{1}{2}}}\frac{1}{2}(1+5x^2)^{-\frac{1}{2}}.10x = e^{\sqrt{1+5x^2}}\frac{5x}{\sqrt{1+5x^2}}$$

Logarithm functions: If u is any differentiable function of x, then:

$$8)\ \frac{d}{dx}\log_a u=\frac{1}{u.\ln a}.\frac{du}{dx} \qquad and \qquad \frac{d}{dx}\ln u=\frac{1}{u}.\frac{du}{dx}$$

Example 9: Find $\frac{dy}{dx}$ for the following functions:

$a)\ y=\log_{10}e^x$

$b)\ y=\log_5(x+1)^2$

$c)\ y=\log_2(3x^2+1)^3$

$d)\ y=\left[\ln(x^2+2)^2\right]^3$

$e)\ y+\ln(xy)=1$

$f)\ y=\dfrac{(2x^3-4)^{\frac{2}{3}}.(2x^2+3)^{\frac{5}{2}}}{(7x^3+4x-3)^2}$

Solution.

$$a)\ y=\log_{10}e^x \Rightarrow y=x\log_{10}e \Rightarrow \frac{dy}{dx}=\log_{10}e=\frac{\ln e}{\ln 10}=\frac{1}{\ln 10}$$

$$b)\ y=\log_5(x+1)^2=2\log_5(x+1) \Rightarrow \frac{dy}{dx}=\frac{2}{(x+1)\ln 5}$$

$$c)\ y=3\log_2(3x^2+1) \Rightarrow \frac{dy}{dx}=\frac{3}{3x^2+1}.\frac{6x}{\ln 2}=\frac{18x}{(3x^2+1)\ln 2}$$

$$d)\ \frac{dy}{dx}=3\left[2\ln(x^2+2)\right]^2\frac{2}{x^2+2}.2x=\frac{48x\left[\ln(x^2+2)\right]^2}{x^2+2}$$

$$e)\ y+\ln x+\ln y=1 \Rightarrow \frac{dy}{dx}+\frac{1}{x}+\frac{1}{y}.\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\frac{y}{x(y+1)}$$

$$f)\ \ln y=\frac{2}{3}\ln(2x^3-4)+\frac{5}{2}\ln(2x^2+3)-2\ln(7x^3+4x-3)$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx}=\frac{2}{3}.\frac{6x^2}{2x^3-4}+\frac{5}{2}.\frac{4x}{2x^2+3}-2.\frac{21x^2+4}{7x^3+4x-3}$$

$$\Rightarrow \frac{dy}{dx}=2y\left[\frac{2x^2}{2x^3-4}+\frac{5x}{2x^2+3}-\frac{21x^2+4}{7x^3+4x-3}\right]$$

Trigonometric functions : If u is any differentiable function of x , then:

$$9)\ \frac{d}{dx}\sin u = \cos u.\frac{du}{dx}$$
$$10)\ \frac{d}{dx}\cos u = -\sin u.\frac{du}{dx}$$
$$11)\ \frac{d}{dx}\tan u = \sec^2 u.\frac{du}{dx}$$
$$12)\ \frac{d}{dx}\cot u = -\csc^2 u.\frac{du}{dx}$$
$$13)\ \frac{d}{dx}\sec u = \sec u.\tan u.\frac{du}{dx}$$
$$14)\ \frac{d}{dx}\csc u = -\csc u.\cot u.\frac{du}{dx}$$

Example 10: Find $\frac{dy}{dx}$ for the following functions:

a) $y = \tan(3x^2)$ b) $y = (\csc x + \cot x)^2$

c) $y = 2\sin\frac{x}{2} - x\mathrm{Cos}\frac{x}{2}$ d) $y = \tan^2(\cos x)$

e) $x + \tan(xy) = 0$ f) $y = \sec^4 x - \tan^4 x$

Solution.

$$a)\ \frac{dy}{dx} = \sec^2(3x^2).6x = 6x.\sec^2(3x^2)$$
$$b)\ \frac{dy}{dx} = 2(\csc x + \cot x)(-\csc x.\cot x - \csc^2 x) = -2\csc x.(\csc x + \cot x)^2$$
$$c)\ \frac{dy}{dx} = 2\cos\frac{x}{2}.\frac{1}{2} - \left[x(-\sin\frac{x}{2}).\frac{1}{2} + \cos\frac{x}{2}\right] = \frac{x}{2}.\sin\frac{x}{2}$$
$$d)\ \frac{dy}{dx} = 2.\tan(\cos x).\sec^2(\cos x).(-\sin x) = -2.\sin x.\tan(\cos x).\sec^2(\cos x)$$
$$e)\ 1 + \sec^2(xy).(x\frac{dy}{dx} + y) = 0 \Rightarrow \frac{dy}{dx} = -\frac{1 + y.\sec^2(xy)}{x.\sec^2(xy)} = -\frac{\cos^2(xy) + y}{x}$$
$$f)\ \frac{dy}{dx} = 4\sec^3 x.\sec x.\tan x - 4.\tan^3 x.\sec^2 x = 4\tan x.\sec^2 x$$