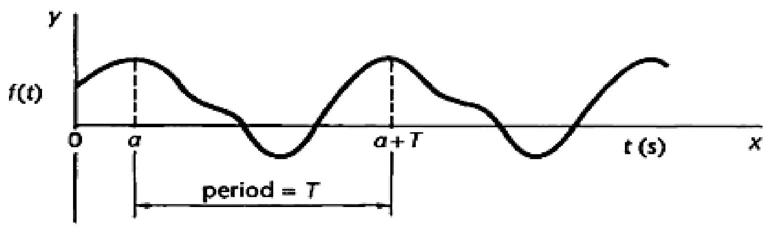
Functions with periods other than 2π

If $y = f(x)$ is defined in the range $-\frac{T}{2}$ to $\frac{T}{2}$, i.e. has a period T, we can convert this to an interval of 2π by changing the units of the independent variable.

In many practical cases involving physical oscillations, the independent variable is time (t) and the periodic interval is normally denoted by T, i.e.

$$f(t+T) = f(t)$$



Each cycle is therefore completed in T seconds and the frequency f **hertz** (oscillations per second) of the periodic function is therefore given by $f = \frac{1}{T}$. If the angular velocity, ω radians per second, is defined by $\omega = 2\pi f$, then

$$\omega = \frac{2\pi}{T} \quad \text{and} \quad T = \frac{2\pi}{\omega}.$$

The angle, x radians, at any time t is therefore $x = \omega t$ and the Fourier series to represent the function can be expressed as

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \{a_n \cos n\omega t + b_n \sin n\omega t\}$$
$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{a_n \cos \frac{2n\pi t}{T} + b_n \sin \frac{2n\pi t}{T}\right\}$$

Fourier coefficients

With the new variable, the Fourier coefficients become

$$f(t) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} \{a_n \cos n\omega t + b_n \sin n\omega t\}$$

$$a_0 = \frac{2}{T}\int_0^T f(t)\,dt = \frac{\omega}{\pi}\int_0^{2\pi/\omega} f(t)\,dt$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega t\,dt = \frac{\omega}{\pi}\int_0^{2\pi/\omega} f(t)\cos n\omega t\,dt$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega t\,dt = \frac{\omega}{\pi}\int_0^{2\pi/\omega} f(t)\sin n\omega t\,dt.$$

We can see that there is very little difference between these expressions and those that have gone before. The limits can, of course, be 0 to T, $-\frac{T}{2}$ to $\frac{T}{2}$, $-\frac{\pi}{\omega}$ to $\frac{\pi}{\omega}$, 0 to $\frac{2\pi}{\omega}$ etc. as is convenient, so long as they cover a complete period.

Let $f(x)$ be defined for $-L \le x \le L$. For the time being, we assume only that $\int_{-L}^{L} f(x)dx$ exists. We want to explore the possibility of choosing numbers $a_0, a_1, \dots, b_1, b_2, \dots$ such that

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

Let f be a Riemann integrable function on $[-L, L]$.

1. The numbers

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad \text{for} \quad n = 0, 1, 2, \dots$$

and

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad \text{for} \quad n = 1, 2, 3, \dots$$

are the Fourier coefficients of f on $[-L, L]$.

2. The series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

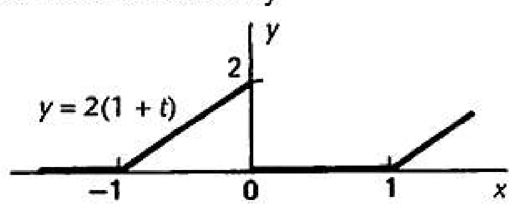
is the Fourier series of f on $[-L, L]$ when the constants are chosen to be the Fourier coefficients of f on $[-L, L]$.

Example

Determine the Fourier series for a periodic function defined by

$$f(t) = \begin{cases} 2(1+t) & -1 < t < 0 \\ 0 & 0 < t < 1 \end{cases}$$

$$f(t+2) = f(t)$$



$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n \cos\frac{2n\pi t}{T} + b_n \sin\frac{2n\pi t}{T}\right\}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\{a_n \cos n\pi t + b_n \sin n\pi t\} \quad \text{because } T = 2$$

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,\mathrm{d}t = \int_{-1}^{1} f(t)\,\mathrm{d}t = \int_{-1}^{0} 2(1+t)\,\mathrm{d}t + \int_{0}^{1} (0)\,\mathrm{d}t$$
$$= \Big[2t + t^2\Big]_{-1}^{0} = 1$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\pi t\,\mathrm{d}t = \int_{-1}^{1} f(t)\cos n\pi t\,\mathrm{d}t \quad = \int_{-1}^{0} 2(1+t)\,\cos n\pi t\,\mathrm{d}t$$

$$a_n = 0\ (n\text{ even});\quad a_n = \frac{4}{n^2\pi^2}\ (n\text{ odd})$$

$$a_n = \int_{-1}^{0} 2(1+t)\cos n\pi t\,\mathrm{d}t$$
$$= 2\left\{\left[(1+t)\frac{\sin\ n\pi t}{n\pi}\right]_{-1}^{0} - \frac{1}{n\pi}\int_{-1}^{0}\sin\ n\pi t\,\mathrm{d}t\right\}$$
$$= 2\left\{(0-0) - \frac{1}{n\pi}\left[-\frac{\cos n\pi t}{n\pi}\right]_{-1}^{0}\right\} = \frac{2}{n^2\pi^2}(1-\cos n\pi)$$
$$= \frac{2}{n^2\pi^2}(1-(-1)^n)$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\frac{2n\pi t}{T}\,\mathrm{d}t \qquad b_n = -\frac{2}{n\pi}$$

$$b_n = \int_{-1}^{0} 2(1+t)\sin n\pi t\,\mathrm{d}t$$
$$= 2\left\{\left[(1+t)\frac{-\cos n\pi t}{n\pi}\right]_{-1}^{0} + \frac{1}{n\pi}\int_{-1}^{0}\cos n\pi t\,\mathrm{d}t\right\}$$
$$= 2\left\{-\frac{1}{n\pi} + \left[\frac{\sin n\pi t}{n\pi}\right]_{-1}^{0}\right\} = -\frac{2}{n\pi} + \frac{2}{n^2\pi^2}(\sin n\pi) = -\frac{2}{n\pi}$$

$$f(t) = \frac{1}{2} + \frac{4}{\pi^2}\left\{\cos\pi t + \frac{1}{9}\cos 3\pi t + \frac{1}{25}\cos 5\pi t + \dots\right\}$$
$$-\frac{2}{\pi}\left\{\sin\pi t + \frac{1}{2}\sin 2\pi t + \frac{1}{3}\sin 3\pi t + \frac{1}{4}\sin 4\pi t + \dots\right\}$$

Example

Let

$$f(x) = \begin{cases} 0 & \text{for } -3 \le x < 0 \\ x & \text{for } 0 \le x \le 3 \end{cases}.$$

Here $L = 3$ and the Fourier coefficients are:

$$a_0 = \frac{1}{3}\int_{-3}^{3} f(x)dx = \frac{1}{3}\int_{0}^{3} xdx = \frac{3}{2},$$

$$\begin{aligned} a_n &= \frac{1}{3}\int_{-3}^{3} f(x)\cos\left(\frac{n\pi x}{3}\right)dx \\ &= \frac{1}{3}\int_{0}^{3} x\cos\left(\frac{n\pi x}{3}\right)dx \\ &= \frac{3}{n^2\pi^2}\cos\left(\frac{n\pi x}{3}\right) + \frac{x}{n\pi}\sin\left(\frac{n\pi x}{3}\right)\Big]_0^3 \\ &= \frac{3}{n^2\pi^2}[(-1)^n - 1], \end{aligned}$$

and

$$\begin{aligned} b_n &= \frac{1}{3}\int_{-3}^{3} f(x)\sin\left(\frac{n\pi x}{3}\right)dx = \frac{1}{3}\int_{0}^{3} x\sin\left(\frac{n\pi x}{3}\right)dx \\ &= \frac{3}{n^2\pi^2}\sin\left(\frac{n\pi x}{3}\right) - \frac{x}{n\pi}\cos\left(\frac{n\pi x}{3}\right)\Big]_0^3 \\ &= \frac{3}{n\pi}(-1)^{n+1}. \end{aligned}$$

The Fourier series of f on $[-3, 3]$ is

$$\frac{3}{4} + \sum_{n=1}^{\infty}\left(\frac{3}{n^2\pi^2}[(-1)^n - 1]\cos\left(\frac{n\pi x}{3}\right) + \frac{3}{n\pi}(-1)^{n+1}\sin\left(\frac{n\pi x}{3}\right)\right).$$

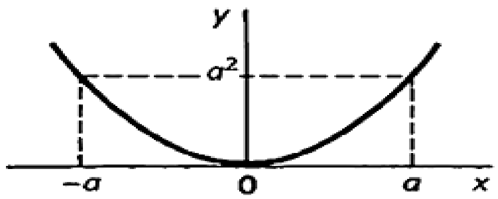
Odd and even functions

Even functions

A function $f(x)$ is said to be *even* if

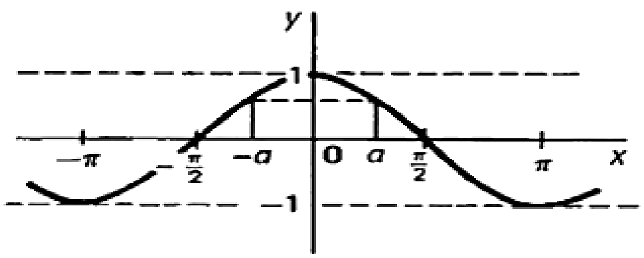
$$f(-x) = f(x)$$

i.e. the function value for a particular negative value of x is the same as that for the corresponding positive value of x. The graph of an even function is therefore *symmetrical about the y-axis*.



$y = f(x) = x^2$ is an even function because

$$f(-2) = 4 = f(2)$$
$$f(-3) = 9 = f(3) \quad \text{etc.}$$



$y = f(x) = \cos x$ is an even function because

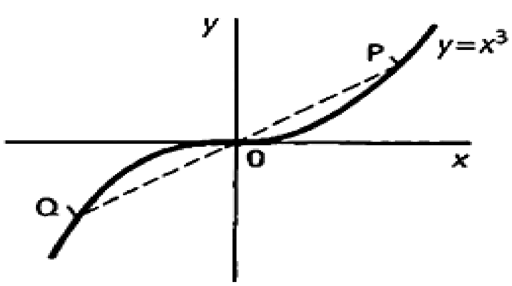
$$\cos(-x) = \cos x$$
$$f(-a) = \cos a = f(a).$$

Odd functions

A function $f(x)$ is said to be *odd* if

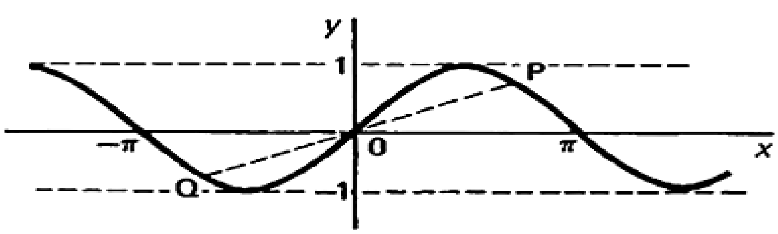
$$f(-x) = -f(x)$$

i.e. the function value for a particular negative value of x is numerically equal to that for the corresponding positive value of x but opposite in sign. The graph of an odd function is thus *symmetrical about the origin*.



$y = f(x) = x^3$ is an odd function because

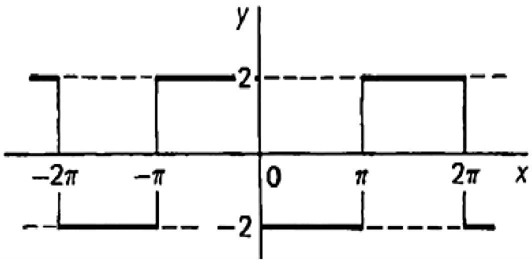
$$f(-2) = -8 = -f(2)$$
$$f(-5) = -125 = -f(5) \quad \text{etc.}$$



$y = f(x) = \sin x$ is an odd function because

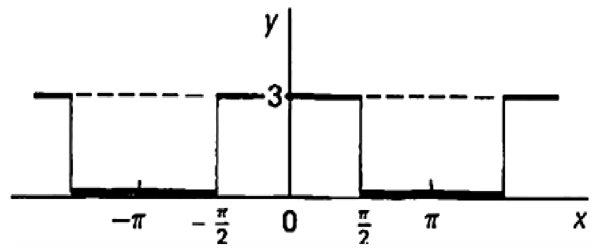
$$\sin(-x) = -\sin x$$
$$f(-a) = -f(a).$$

Example 1



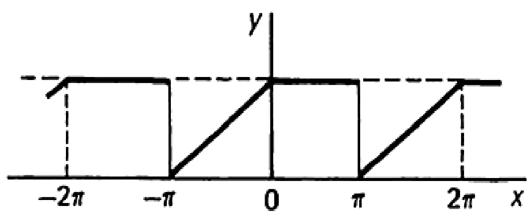
odd; symmetrical about the origin, i.e. $f(-x) = -f(x)$

Example 2



even: symmetrical about the y-axis. i.e. $f(-x) = f(x)$

Example 3



neither even nor odd; not symmetrical about either the y-axis or the origin

Products of odd and even functions

(even) × (even) = (even) like $(+) \times (+) = (+)$
(odd) × (odd) = (even) $(-) \times (-) = (+)$
(odd) × (even) = (odd) $(-) \times (+) = (-)$.

State whether each of the following products is odd, even, or neither

1 $x^2 \sin 2x$
2 $x^3 \cos x$
3 $\cos 2x \cos 3x$
4 $x \sin nx$
5 $3 \sin x \cos 4x$

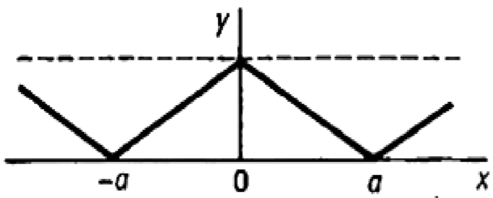
1 odd (E)(O) = (O)
2 odd (O)(E) = (O)
3 even (E)(E) = (E)
4 even (O)(O) = (E)
5 odd (O)(E) = (O)

6 $(2x+3) \sin 4x$
7 $\sin^2 x \cos 3x$
8 $x^3 e^x$
9 $(x^4+4) \sin 2x$
10 $\frac{1}{x+2} \cosh x$

6 neither (N)(O) = (N)
7 even (E)(E) = (E)
8 neither (O)(N) = (N)
9 odd (E)(O) = (O)
10 neither (N)(E) = (N)

Two useful facts emerge from odd and even functions

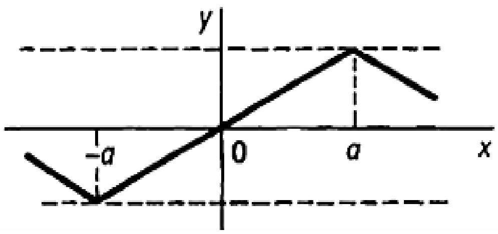
(a)



For an *even* function

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

(b)



For an *odd* function

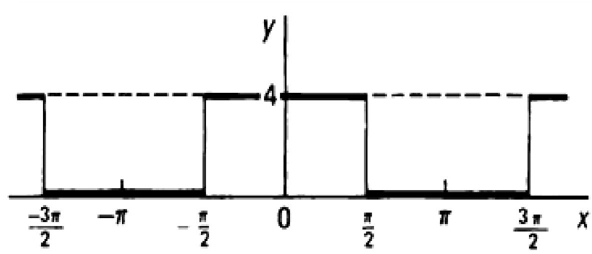
$$\int_{-a}^{a} f(x)dx = 0$$

Theorem 1

If $f(x)$ is defined over the interval $-\pi < x < \pi$ and $f(x)$ is *even*, then the Fourier series for $f(x)$ contains *cosine terms* only. Included in this is a_0 which may be regarded as $a_n \cos nx$ with $n = 0$.

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \qquad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \qquad b_n = 0$$

Example



$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{(a)}\ \ a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi/2} 4\,dx = \frac{2}{\pi}\Big[4x\Big]_0^{\pi/2} = 4$$

$$\text{(b)}\ \ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$a_n = 0 \quad (n\ \text{even}); \quad a_n = \frac{8}{\pi n} \quad (n = 1, 5, 9, \ldots);$$

Because

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi/2} 4\cos nx\,dx$$

$$= \frac{8}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi/2} = \frac{8}{\pi n}\sin\frac{n\pi}{2}$$

But $\sin\frac{n\pi}{2} = 0$ for n even

$= 1$ for $n = 1, 5, 9, \ldots$

$= -1$ for $n = 3, 7, 11, \ldots$

$$f(x) = 2 + \frac{8}{\pi}\left\{\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \frac{1}{7}\cos 7x + \ldots\right\}$$

Theorem 2

If $f(x)$ is an *odd* function defined over the interval $-\pi < x < \pi$, then the Fourier series for $f(x)$ contains *sine terms* only.

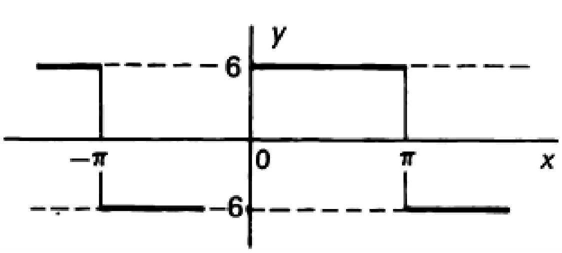
(a) $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx.$ But $f(x)$ is odd $\therefore\ a_0 = 0$

(b) $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

(c) $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

Example

Consider the function shown.



$$f(x) = -6 \qquad -\pi < x < 0$$
$$f(x) = 6 \qquad 0 < x < \pi$$
$$f(x + 2\pi) = f(x).$$

an odd function; sine terms only, i.e. $a_0 = 0$ and $a_n = 0$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

$$b_n = 0 \quad (n \text{ even}) \quad \text{or} \quad \frac{24}{\pi n} \quad (n \text{ odd})$$

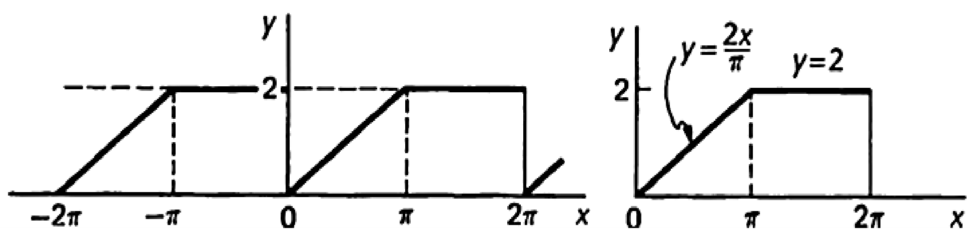
Because

$$b_n = \frac{2}{\pi}\int_0^{\pi} 6\sin nx\,dx = \frac{12}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi} = \frac{12}{\pi n}(1-\cos n\pi).$$

$$f(x) = \frac{24}{\pi}\left\{\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\right\}$$

Example

Determine the Fourier series for the function shown.



This is neither odd nor even. Therefore we must find a_0, a_n and b_n.

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n\cos nx + b_n\sin nx\}$$

$$f(x) = \frac{3}{2} - \frac{4}{\pi^2}\left\{\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \ldots\right\}$$
$$-\frac{2}{\pi}\left\{\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x \ldots\right\}$$

Sometimes we can save some work in computing Fourier coefficients by observing special properties of $f(x)$.

Even Function

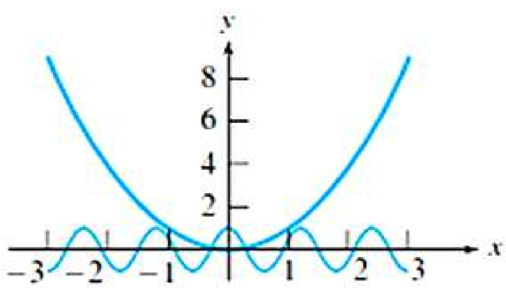
f is an even function on $[-L, L]$ if $f(-x) = f(x)$ for $-L \le x \le L$.

Odd Function

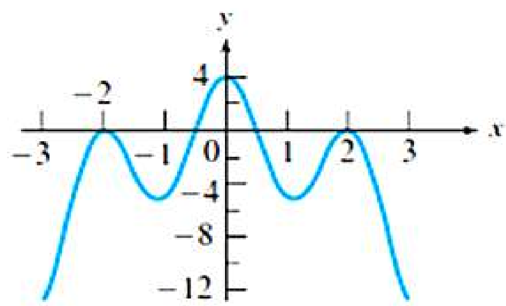
f is an odd function on $[-L, L]$ if $f(-x) = -f(x)$ for $-L \le x \le L$.

For example, x^2, x^4, $\cos(n\pi x/L)$ and $e^{-|x|}$ are even functions on any interval $[-L, L]$.

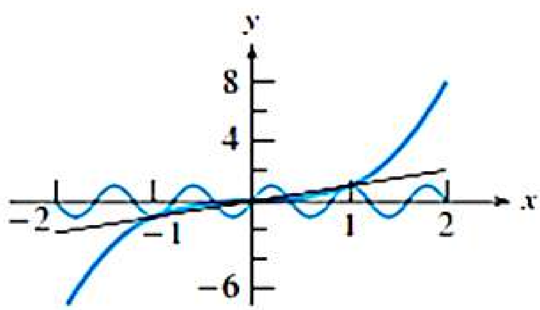
The functions x, x^3, x^5 and $\sin(n\pi x/L)$ are odd functions on any interval $[-L, L]$.



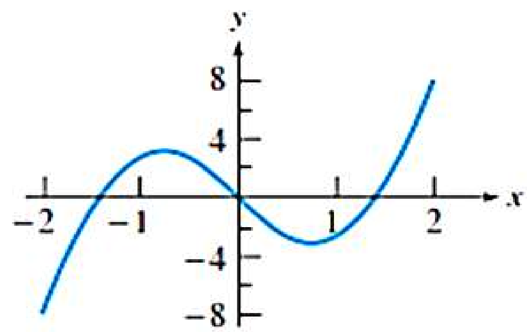
functions $y = x^2$ *and* $y = \cos(5\pi x/3)$.



even function, symmetric about the y axis.



odd functions $y = x$, $y = x^3$, *and* $y = \sin(5\pi x/2)$.



odd function, symmetric through the origin.

Even and odd functions behave as follows under multiplication:

$$\text{even} \cdot \text{even} = \text{even},$$

$$\text{odd} \cdot \text{odd} = \text{even},$$

and

$$\text{even} \cdot \text{odd} = \text{odd}.$$