Laplace transforms

Laplace transforms are invaluable for any engineer's mathematical toolbox as they make solving linear ODEs and related initial value problems, as well as systems of linear ODEs, much easier. Applications abound: electrical networks, springs, mixing problems, signal processing, and other areas of engineering and physics.

The Laplace transform of an expression $f(t)$ is denoted by $L\{f(t)\}$ and is defined as the semi-infinite integral

$$L\{f(t)\} = \int_{t=0}^{\infty} f(t)e^{-st}\,dt = F(s)$$

Example 1

To find the Laplace transform of $f(t) = a$ (constant).

$$L\{a\} = \int_0^{\infty} ae^{-st}dt = a\left[\frac{e^{-st}}{-s}\right]_0^{\infty} = -\frac{a}{s}\left[e^{-st}\right]_0^{\infty}$$
$$= -\frac{a}{s}\{0-1\} = \frac{a}{s}$$
$$\therefore\ L\{a\} = \frac{a}{s} \qquad (s > 0)$$

Example 2

To find the Laplace transform of $f(t) = e^{at}$ (a constant). As with all cases, we multiply $f(t)$ by e^{-st} and integrate between $t = 0$ and $t = \infty$.

$$\therefore\ L\{e^{at}\} = \int_0^{\infty} e^{at}e^{-st}\,dt = \int_0^{\infty} e^{-(s-a)t}\,dt$$

Because

$$L\{e^{at}\} = \int_0^{\infty} e^{at}e^{-st}\,dt = \int_0^{\infty} e^{-(s-a)t}\,dt = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^{\infty}$$
$$= -\frac{1}{s-a}\{0-1\} = \frac{1}{s-a}$$
$$\therefore\ L\{e^{at}\} = \frac{1}{s-a} \qquad (s > a)$$

So we already have two standard transforms

$$L\{a\} = \frac{a}{s} \quad \text{and} \quad L\{e^{at}\} = \frac{1}{s-a}$$

$\therefore$ $L\{4\} = \ldots\ldots\ldots\ldots;$ $\quad L\{e^{4t}\} = \ldots\ldots\ldots\ldots$

$L\{-5\} = \ldots\ldots\ldots\ldots;$ $\quad L\{e^{-2t}\} = \ldots\ldots\ldots\ldots$

$$L\{4\} = \frac{4}{s}; \qquad L\{e^{4t}\} = \frac{1}{s-4}$$

$$L\{-5\} = -\frac{5}{s}; \qquad L\{e^{-2t}\} = \frac{1}{s+2}$$

Note that, as we said earlier, the Laplace transform is always an expression in s.

Example 3

To find the Laplace transform of $f(t) = \sin at$. We could, of course, apply the definition and evaluate

$$L\{\sin at\} = \int_0^\infty \sin at \cdot e^{-st}\,dt$$

using integration by parts.

However, it is much shorter if we use the fact that

$$e^{j\theta} = \cos\theta + j\sin\theta$$

so that $\sin\theta$ is the imaginary part of $e^{j\theta}$, written $\mathscr{I}(e^{j\theta})$.

The function $\sin at$ can therefore be written $\mathscr{I}(e^{jat})$ so that

$$L\{\sin at\} = L\{\mathscr{I}(e^{jat})\} = \mathscr{I}\int_0^\infty e^{jat}e^{-st}\,dt = \mathscr{I}\int_0^\infty e^{-(s-ja)t}\,dt$$

$$= \mathscr{I}\left\{\left[\frac{e^{-(s-ja)t}}{-(s-ja)}\right]_0^\infty\right\} = \mathscr{I}\left\{-\frac{1}{(s-ja)}[0-1]\right\}$$

$$= \mathscr{I}\left\{\frac{1}{s-ja}\right\}$$

We can rationalise the denominator by multiplying top and bottom by $\ldots\ldots\ldots\ldots$

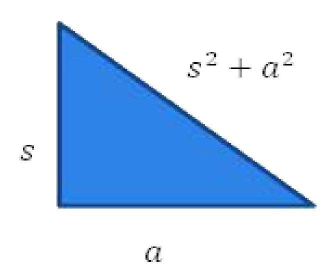
$$s + ja$$

$$\therefore \; L\{\sin at\} = \mathscr{I}\left\{\frac{s+ja}{s^2+a^2}\right\} = \frac{a}{s^2+a^2}$$

$$\therefore \; L\{\sin at\} = \frac{a}{s^2+a^2}$$

We can use the same method to determine $L\{\cos at\}$ since $\cos at$ is the real part of e^{jat}, written $\Re(e^{jat})$.

Then $L\{\cos at\} = \ldots\ldots\ldots\ldots$



$$L\{\cos at\} = \frac{s}{s^2 + a^2}$$

Because

$$L\{\cos at\} = \Re\left\{\frac{s + ja}{s^2 + a^2}\right\} = \frac{s}{s^2 + a^2}$$

Recapping then: $L\{1\} = \ldots\ldots\ldots\ldots;$ $L\{e^{3t}\} = \ldots\ldots\ldots\ldots$
$L\{\sin 2t\} = \ldots\ldots\ldots\ldots;$ $L\{\cos 4t\} = \ldots\ldots\ldots$

$$L\{1\} = \frac{1}{s}; \qquad L\{e^{3t}\} = \frac{1}{s-3}$$
$$L\{\sin 2t\} = \frac{2}{s^2+4}; \qquad L\{\cos 4t\} = \frac{s}{s^2+16}$$

Example 4

To find the transform of $f(t) = t^n$ where n is a positive integer.

By the definition $L\{t^n\} = \int_0^\infty t^n e^{-st}\, dt$.

$$\therefore\ L\{t^n\} = \frac{n!}{s^{n+1}}$$

$$\therefore\ L\{t\} = \frac{1}{s^2};\quad L\{t^2\} = \frac{2}{s^3};\quad L\{t^3\} = \frac{6}{s^4}$$

Example 5

Laplace transforms of $f(t) = \sinh at$ and $f(t) = \cosh at$.

Starting from the exponential definitions of $\sinh at$ and $\cosh at$, i.e.

$\sinh at = \frac{1}{2}(e^{at} - e^{-at})$ and $\cosh at = \frac{1}{2}(e^{at} + e^{-at})$

we proceed as follows.

(a) $f(t) = \sinh at$. $L\{\sinh at\} = \int_0^\infty \sinh at\, e^{-st}\, dt$

$$= \frac{1}{2}\int_0^\infty (e^{at} - e^{-at})e^{-st}\, dt$$
$$= \frac{1}{2}\int_0^\infty \left\{e^{-(s-a)t} - e^{-(s+a)t}\right\} dt$$

$$L\{\sinh at\} = \frac{a}{s^2 - a^2}$$

Because

$$\frac{1}{2}\int_0^\infty \left\{e^{-(s-a)t} - e^{-(s+a)t}\right\} dt = \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)} - \frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty$$

$$= \frac{1}{2}\left\{\frac{1}{s-a} - \frac{1}{s+a}\right\} = \frac{a}{s^2 - a^2}$$

$$\therefore\ L\{\sinh at\} = \frac{a}{s^2 - a^2}$$

(b) $f(t) = \cosh at$. **Proceeding in the same way**

$$L\{\cosh at\} = \ldots\ldots\ldots\ldots$$

$$L\{\cosh at\} = \frac{1}{2}\int_0^\infty (e^{at} + e^{-at})e^{-st}\, dt = \frac{1}{2}\int_0^\infty \left\{e^{-(s-a)t} + e^{-(s+a)t}\right\} dt$$

$$= \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)} + \frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty = \frac{1}{2}\left\{\frac{1}{s-a} + \frac{1}{s+a}\right\}$$

$$= \frac{1}{2}\left\{\frac{2s}{s^2 - a^2}\right\} = \frac{s}{s^2 - a^2}$$

$$\therefore\ L\{\cosh at\} = \frac{s}{s^2 - a^2}$$

So we have accumulated several standard results:

$$L\{a\} = \frac{a}{s}; \qquad L\{e^{at}\} = \frac{1}{s-a}; \qquad L\{t^n\} = \frac{n!}{s^{n+1}}$$

$$L\{\sin at\} = \frac{a}{s^2 + a^2}; \qquad L\{\cos at\} = \frac{s}{s^2 + a^2}$$

$$L\{\sinh at\} = \frac{a}{s^2 - a^2}; \qquad L\{\cosh at\} = \frac{s}{s^2 - a^2}$$

The Laplace transform is a linear transform, by which is meant that:

(1) *The transform of a sum (or difference) of expressions is the sum (or difference) of the individual transforms. That is*

$$L\{f(t) \pm g(t)\} = L\{f(t)\} \pm L\{g(t)\}$$

(2) *The transform of an expression that is multiplied by a constant is the constant multiplied by the transform of the expression. That is*

$$L\{kf(t)\} = kL\{f(t)\}$$

Example 6

(a) $L\{2e^{-t} + t\} = L\{2e^{-t}\} + L\{t\}$
$= 2L\{e^{-t}\} + L\{t\} = \dfrac{2}{s+1} + \dfrac{1}{s^2} = \dfrac{2s^2+s+1}{s^2(s+1)}$

(b) $L\{2\sin 3t + \cos 3t\} = 2L\{\sin 3t\} + L\{\cos 3t\} = 2.\dfrac{3}{s^2+9} + \dfrac{s}{s^2+9} = \dfrac{s+6}{s^2+9}$

(c) $L\{4e^{2t} + 3\cosh 4t\} = 4L\{e^{2t}\} + 3L\{\cosh 4t\} = 4.\dfrac{1}{s-2} + 3.\dfrac{s}{s^2-16} = \dfrac{4}{s-2} + \dfrac{3s}{s^2-16}$
$= \dfrac{7s^2-6s-64}{(s-2)(s^2-16)}$

1. $L\{2\sin 3t + 4\sinh 3t\} = 2.\dfrac{3}{s^2+9} + 4.\dfrac{3}{s^2-9} = \dfrac{6}{s^2+9} + \dfrac{12}{s^2-9} = \dfrac{18(s^2+3)}{s^4-81}$

2. $L\{5e^{4t} + \cosh 2t\} = \dfrac{5}{s-4} + \dfrac{s}{s^2-4} = \dfrac{6s^2-4s-20}{(s-4)(s^2-4)}$

3. $L\{t^3 + 2t^2 - 4t + 1\} = \dfrac{3!}{s^4} + 2.\dfrac{2!}{s^3} - 4.\dfrac{1!}{s^2} + \dfrac{1}{s} = \dfrac{1}{s^4}\{s^3 - 4s^2 + 4s + 6\}$

Theorem 1 The first shift theorem

The first shift theorem states that if $L\{f(t)\} = F(s)$ then
$L\{e^{-at}f(t)\} = F(s+a)$

Because $L\{e^{-at}f(t)\} = \int_{t=0}^{\infty} e^{-at}f(t)e^{-st}\,dt = \int_{t=0}^{\infty} f(t)e^{-(s+a)t}\,dt = F(s+a)$

That is

$$L\{e^{-at}f(t)\} = F(s+a)$$

For example $L\{\sin 2t\} = \dfrac{2}{s^2+4}$

then $L\{e^{-3t}\sin 2t\} = \dfrac{2}{(s+3)^2+4} = \dfrac{2}{s^2+6s+13}$

Similarly, $L\{t^2\} = \dfrac{2}{s^3}$ $\therefore$ $L\{t^2e^{4t}\} =$. $\boxed{\dfrac{2}{(s-4)^3}}$

Because $L\{t^2\} = \dfrac{2}{s^3}$. $\therefore$ $L\{t^2e^{4t}\}$ is the same with s replaced by $(s-4)$.

$\therefore$ $L\{t^2e^{4t}\} = \dfrac{2}{(s-4)^3}$

Exercise

Determine the following.

1. $L\{e^{-2t}\cosh 3t\}$ $\quad L\{\cosh 3t\} = \frac{s}{s^2-9}$ $\quad \therefore L\{e^{-2t}\cosh 3t\} = \frac{s+2}{(s+2)^2-9} = \frac{s+2}{s^2+4s-5}$

2. $L\{2e^{3t}\sin 3t\}$ $\quad L\{\sin 3t\} = \frac{3}{s^2+9}$ $\quad \therefore L\{2e^{3t}\sin 3t\} = \frac{6}{(s-3)^2+9} = \frac{6}{s^2-6s+18}$

3. $L\{4te^{-t}\}$ $\quad L\{4t\} = 4.\frac{1}{s^2}$ $\quad \therefore L\{4te^{-t}\} = \frac{4}{(s+1)^2}$

4. $L\{e^{2t}\cos t\}$ $\quad L\{\cos t\} = \frac{s}{s^2+1}$ $\quad \therefore L\{e^{2t}\cos t\} = \frac{s-2}{(s-2)^2+1} = \frac{s-2}{s^2-4s+5}$

5. $L\{e^{3t}\sinh 2t\}$ $\quad L\{\sinh 2t\} = \frac{2}{s^2-4}$ $\quad \therefore L\{e^{3t}\sinh 2t\} = \frac{2}{(s-3)^2-4} = \frac{2}{s^2-6s+5}$

6. $L\{t^3e^{-4t}\}$ $\quad L\{t^3\} = \frac{3!}{s^4}$ $\quad \therefore L\{t^3e^{-4t}\} = \frac{6}{(s+4)^4}$

Theorem 2

Theorem 2 **Multiplying by t**

If $L\{f(t)\} = F(s)$, then $L\{tf(t)\} = -\frac{d}{ds}\{F(s)\}$

For example, $L\{\sin 2t\} = \dfrac{2}{s^2+4}$

$$\therefore\ L\{t\sin 2t\} = -\frac{d}{ds}\left(\frac{2}{s^2+4}\right) = \frac{4s}{(s^2+4)^2}$$

and similarly, $L\{t\cosh 3t\} = \ldots\ldots\ldots\ldots$

Because $L\{t\cosh 3t\} = -\dfrac{d}{ds}\left(\dfrac{s}{s^2-9}\right) = -\dfrac{(s^2-9)-s(2s)}{(s^2-9)^2} = \dfrac{s^2+9}{(s^2-9)^2}$

$$L\{t^2\cosh 3t\} = L\{t(t\cosh 3t)\} = -\frac{d}{ds}\left\{\frac{s^2+9}{(s^2-9)^2}\right\} = \frac{2s(s^2+27)}{(s^2-9)^3}$$

Likewise, starting with $L\{\sin 4t\} = \dfrac{4}{s^2+16}$

$L\{t\sin 4t\} = \ldots\ldots\ldots\ldots$ and $L\{t^2\sin 4t\} = \ldots\ldots$ $\dfrac{8s}{(s^2+16)^2};\quad \dfrac{8(3s^2-16)}{(s^2+16)^3}$

applying $L\{tf(t)\} = -\dfrac{d}{ds}\{F(s)\}$ in each case.

Theorem 2 obviously extends the range of functions that we can deal with.

So, in general, if $L\{f(t)\} = F(s)$, then

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\{F(s)\}$$

Theorem 3 Dividing by t

If $L\{f(t)\} = F(s)$ then $L\left\{\dfrac{f(t)}{t}\right\} = \displaystyle\int_{\sigma=s}^{\infty} F(\sigma)\,d\sigma$

provided $\displaystyle\lim_{t\to 0}\left(\frac{f(t)}{t}\right)$ exists. To demonstrate this we start from the right-hand side of the result

$$\begin{aligned}\int_{\sigma=s}^{\infty} F(\sigma)\,d\sigma &= \int_{\sigma=s}^{\infty}\left\{\int_{t=0}^{\infty} f(t)e^{-\sigma t}\,dt\right\}d\sigma \\ &= \int_{t=0}^{\infty}\int_{\sigma=s}^{\infty} f(t)e^{-\sigma t}\,d\sigma\,dt \\ &= \int_{t=0}^{\infty} f(t)\left\{\int_{\sigma=s}^{\infty} e^{-\sigma t}\,d\sigma\right\}dt \\ &= \int_{t=0}^{\infty} f(t)\frac{e^{-st}}{t}\,dt \\ &= L\left\{\frac{f(t)}{t}\right\}\end{aligned}$$

Notice the dummy variable σ. The end result is an expression in s which comes from the lower limit of the integral so the variable of integration, which is absorbed during the process of integration, is changed to σ. Notice also that we interchange the order of integration.

This rule is somewhat restricted in use, since it is applicable only if $\lim_{t\to 0}\left(\frac{f(t)}{t}\right)$ exists. In indeterminate cases, we use L'Hôpital's rule to find out. Let's try a couple of examples.

Example 1

Determine $L\left\{\frac{\sin at}{t}\right\}$

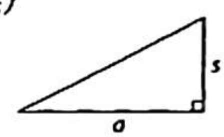
First we test $\lim_{t\to 0}\left\{\frac{\sin at}{t}\right\} = \left\{\frac{0}{0}\right\} = ?$ By L'Hôpital's rule, we differentiate top and bottom separately and substitute $t = 0$ in the result to ascertain the limit of the new expression.

$\lim_{t\to 0}\left\{\frac{\sin at}{t}\right\} = \lim_{t\to 0}\left\{\frac{a\cos at}{1}\right\} = a$, that is, the limit exists and the theorem can therefore be applied.

So $L\{\sin at\} = \frac{a}{s^2 + a^2}$, therefore

$$L\left\{\frac{\sin at}{t}\right\} = \int_s^\infty \frac{a}{\sigma^2 + a^2}\,d\sigma = \left[\arctan\left(\frac{\sigma}{a}\right)\right]_s^\infty = \frac{\pi}{2} - \arctan\left(\frac{s}{a}\right) = \arctan\left(\frac{a}{s}\right)$$

Notice that $\arctan\left(\frac{a}{s}\right) + \arctan\left(\frac{s}{a}\right) = \frac{\pi}{2}$, as can be



Example 2

Determine $L\left\{\frac{1 - \cos 2t}{t}\right\}$

First we test whether $\lim_{t\to 0}\left\{\frac{1-\cos 2t}{t}\right\}$ exists.

$\lim_{t\to 0}\left\{\frac{1-\cos 2t}{t}\right\} = \frac{1-1}{0} = \frac{0}{0} = ?\quad \therefore$ Apply l'Hôpital's rule.

$\lim_{t\to 0}\left\{\frac{1-\cos 2t}{t}\right\} = \lim_{t\to 0}\left\{\frac{2\sin 2t}{1}\right\} = \frac{0}{1} = 0\quad \therefore$ limit exists.

$$L\{1 - \cos 2t\} = \frac{1}{s} - \frac{s}{s^2+4}$$

Then, by Theorem 3

$$L\left\{\frac{1-\cos 2t}{t}\right\} = \int_{\sigma=s}^\infty \left\{\frac{1}{\sigma} - \frac{\sigma}{\sigma^2+4}\right\}d\sigma = \left[\ln\sigma - \frac{1}{2}\ln(\sigma^2+4)\right]_{\sigma=s}^\infty = \frac{1}{2}\left[\ln\left(\frac{\sigma^2}{\sigma^2+4}\right)\right]_{\sigma=s}^\infty$$

$$\boxed{\ln\sqrt{\frac{s^2+4}{s^2}}}$$

When $\sigma \to \infty$, $\ln\left(\frac{\sigma^2}{\sigma^2+4}\right) \to \ln 1 = 0$

Therefore, $L\left\{\frac{1-\cos 2t}{t}\right\} = \ldots\ldots\ldots\ldots$

Because

$$L\left\{\frac{1-\cos 2t}{t}\right\} = -\frac{1}{2}\ln\left(\frac{s^2}{s^2+4}\right) = \ln\left(\frac{s^2}{s^2+4}\right)^{-1/2} = \ln\sqrt{\frac{s^2+4}{s^2}}$$

1 *Standard transforms*

$f(t)$	$L\{f(t)\} = F(s)$
a	$\frac{a}{s}$
e^{at}	$\frac{1}{s-a}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$
t^n	$\frac{n!}{s^{n+1}}$

(n a positive integer)

2 *Theorem 1* The first shift theorem

If $L\{f(t)\} = F(s)$, then $L\{e^{-at}f(t)\} = F(s+a)$

3 *Theorem 2* Multiplying by t

If $L\{f(t)\} = F(s)$, then $L\{tf(t)\} = -\frac{d}{ds}\{F(s)\}$

4 *Theorem 3* Dividing by t

If $L\{f(t)\} = F(s)$, then $L\left\{\frac{f(t)}{t}\right\} = \int_{\sigma=s}^{\infty} F(\sigma)\, d\sigma$

provided $\lim_{t\to 0}\left\{\frac{f(t)}{t}\right\}$ exists.