The inverse Fourier transform

if $F(\omega)$ is the Fourier transform of $f(t)$, *i.e.*,

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\, dt$$

then

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\, d\omega$$

let's check

$$\begin{aligned}
\frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} F(\omega)e^{j\omega t}\, d\omega &= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty}\left(\int_{\tau=-\infty}^{\infty} f(\tau)e^{-j\omega\tau}\right)e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{\tau=-\infty}^{\infty} f(\tau)\left(\int_{\omega=-\infty}^{\infty} e^{-j\omega(\tau-t)}d\omega\right)d\tau \\
&= \int_{-\infty}^{\infty} f(\tau)\delta(\tau-t)d\tau \\
&= f(t)
\end{aligned}$$

Example

Determine the inverse transform, if the Fourier transform F (ω) are given as;

$$F(\omega) = \begin{cases} 1 & |\omega| \le \omega_c \\ 0 & \omega_c < |\omega| \le \pi \end{cases}$$

Sol:

$$f(n) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega n}d\omega$$

$$f(n) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega = \frac{\sin \omega_c n}{\pi n} \qquad n \ne 0$$

For n=0, we have

$$f(0) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} d\omega = \frac{\omega_c}{\pi}$$

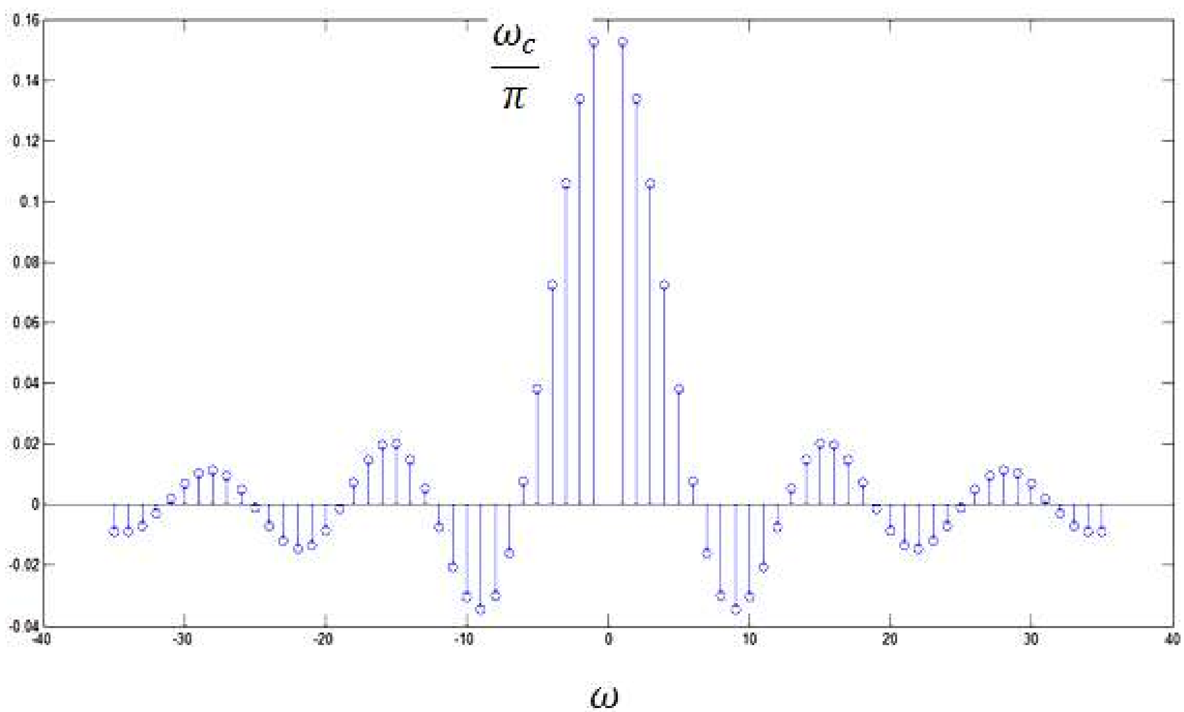
Hence

$$f(n) = \begin{cases} \dfrac{\omega_c}{\pi} & n = 0 \\ \dfrac{\omega_c}{\pi}\dfrac{\sin\omega_c n}{\omega_c n} & n \neq 0 \end{cases}$$

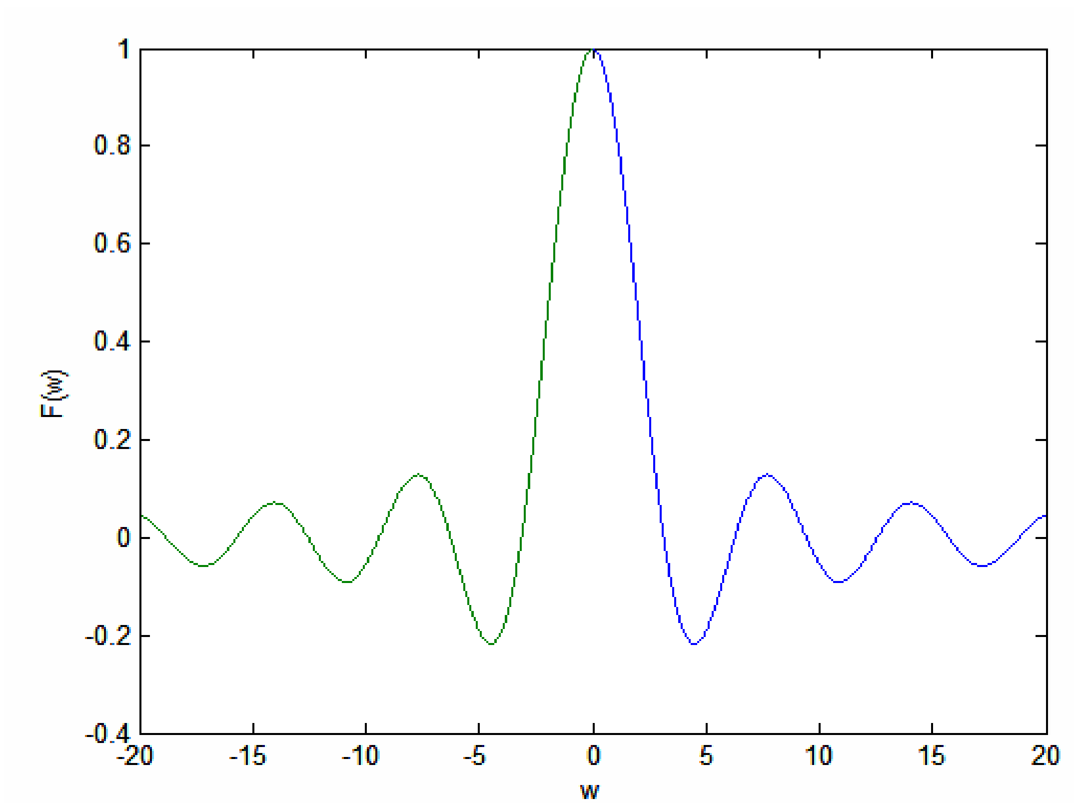
Matlab

```
clc
clear
wc=0.5;
n=[0:35];
m=[0:-1:-35];
a=1./(pi*n);
f1=a.*sin(wc*n);
f2=f1;
d=[1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0];
[x1 y1]=dimpulse(f1,d,n)
nm=[m n];
f=[f2 f1];
stem(nm,f)

```



```
clear
clc
w=[0:.01:20];
A=1;
b=A*sin(w);
f=b./w;
plot(w,f,-w,f)
xlabel('w')
ylabel('F(w)')
title( 'the sinc(w) with amplit
```



The unit sample and unit step

Let’s examine some special signals, first in discrete time, then in continuous time.

Definition 1 The discrete time unit step is given by

$$u[n] = \begin{cases} 1, n \geq 0 \\ 0, n < 0 \end{cases}$$

The unit sample or impulse is defined as

$$\delta[n] = u[n] - u[n-1]$$

We notice that they are related via the sum relation

$$u[n] = \sum_{k=-\infty}^{n} \delta[k]$$

Notice the unit sample sifts signals

Proposition 1 The unit sample has the "sampling property," picking off values of signals that it sums against:

$$x[n] = \sum_{k} x[n-k]\delta[k]$$

This is true for all signals, implying we can derive various properties, the "summed" and "differenced" versions. Defining x[n] = u[n], then

$$u[n] = \sum_{k=-\infty}^{n} \delta[k]$$

Odd and Even Functions

Even	Odd
$f(-t) = f(t)$ Symmetric Cosines Transform is real*	$f(-t) = -f(t)$ Anti-symmetric Sines Transform is imaginary*

* for real-valued signals

Sinusoids

Spatial Domain $f(t)$	Frequency Domain $F(u)$
$\cos(2\pi st)$	$\frac{1}{2}[\delta(u+s)+\delta(u-s)]$
$\sin(2\pi st)$	$\frac{1}{2}i[\delta(u+s)-\delta(u-s)]$

Constant Functions

Spatial Domain $f(t)$	Frequency Domain $F(u)$
1	$\delta(u)$
a	$a\ \delta(u)$

Matlab

Syms t w wc u

Fourier (dirac (t), w)   return   1

Fourier (dirac (1), w)   return   0

iFourier (dirac (w), t)   return   1/(2*pi)

iFourier (a*dirac (w), t)   return   a/(2*pi)

iFourier (dirac (1), t)   return 0

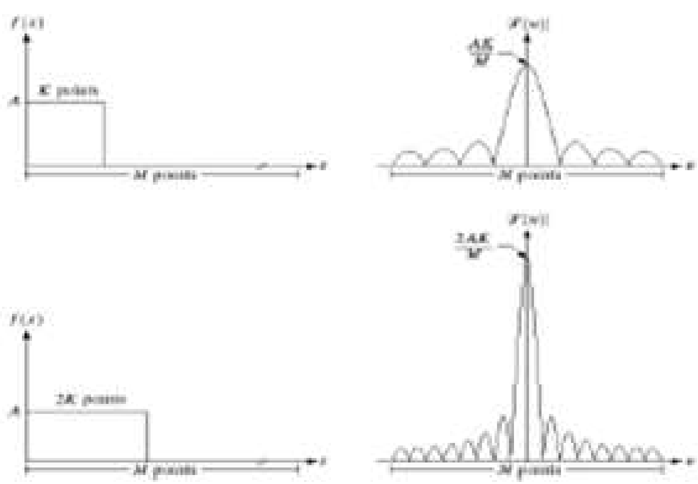
Delta Functions

Spatial Domain $f(t)$	Frequency Domain $F(u)$
$\delta(t)$	1

Square Pulse

Spatial Domain $f(t)$	Frequency Domain $F(u)$
$\begin{cases} 1 & \text{if } -a/2 \leq t \leq a/2 \\ 0 & \text{otherwise} \end{cases}$	$\text{sinc}(a\pi u) = \frac{\sin(a\pi u)}{a\pi u}$

Square Pulse



Differentiation

Spatial Domain $f(t)$	Frequency Domain $F(u)$
$\frac{d}{dt}$	$2\pi iu$

Gaussian

Spatial Domain $f(t)$	Frequency Domain $F(u)$
$e^{-\pi t^2}$	$e^{-\pi u^2}$

Some Common Fourier Transform Pairs

Spatial Domain $f(t)$		Frequency Domain $F(u)$	
Cosine	$\cos(2\pi st)$	Deltas	$\frac{1}{2}[\delta(u+s)+\delta(u-s)]$
Sine	$\sin(2\pi st)$	Deltas	$\frac{1}{2}i[\delta(u+s)-\delta(u-s)]$
Unit	1	Delta	$\delta(u)$
Constant	a	Delta	$a\delta(u)$
Delta	$\delta(t)$	Unit	1
Comb	$\delta(t \bmod k)$	Comb	$\delta(u \bmod 1/k)$

Properties: Linearity

Adding two functions together adds their Fourier Transforms together:

$$\mathcal{F}(f+g)=\mathcal{F}(f)+\mathcal{F}(g)$$

Multiplying a function by a scalar constant multiplies its Fourier Transform by the same constant:

$$\mathcal{F}(af)=a\ \mathcal{F}(f)$$

Properties: Notation

Let $\mathcal{F}$ denote the Fourier Transform:

$$F=\mathcal{F}(f)$$

Let $\mathcal{F}^{-1}$ denote the Inverse Fourier Transform:

$$f=\mathcal{F}^{-1}(F)$$

More Common Fourier Transform Pairs

Spatial Domain $f(t)$		Frequency Domain $F(u)$	
Square	1 if $-a/2 \le t \le a/2$ 0 otherwise	Sinc	$\text{sinc}(a\pi u)$
Triangle	$1-\lvert t\rvert$ if $-a \le t \le a$ 0 otherwise	Sinc^2	$\text{sinc}^2(a\pi u)$
Gaussian	$e^{-\pi t^2}$	Gaussian	$e^{-\pi u^2}$
Differentiation	$\frac{d}{dt}$	Ramp	$2\pi i u$

Rayleigh's Theorem

Total "energy" (sum of squares) is the same in either domain:

$$\int_{-\infty}^{\infty} |f(t)|^2\ dt = \int_{-\infty}^{\infty} |F(u)|^2\ du$$

Fourier Transform	MATLAB Commands

$f(x) = e^{-x^2}$
$F[f](w) = \int_{-\infty}^{\infty} f(x)e^{-ixw}dx$
$= \sqrt{\pi}e^{-w^2/4}$
	`syms x;`
	`f = exp(-x^2);`
	`fourier(f)`
	returns
	`ans =`
	`pi^(1/2)/exp(w^2/4)`

$g(w) = e^{-\|w\|}$
$F[g](t) = \int_{-\infty}^{\infty} g(w)e^{-itw}dw$
$= \frac{2}{1+t^2}$
	`syms w;`
	`g = exp(-abs(w));`
	`fourier(g)`
	returns
	`ans =`
	`2/(v^2 + 1)`

$f(x) = xe^{-\|x\|}$
$F[f](u) = \int_{-\infty}^{\infty} f(x)e^{-ixu}dx$
$= -\frac{4iu}{(1+u^2)^2}$
	`syms x u;`
	`f = x*exp(-abs(x));`
	`fourier(f,u)`
	returns
	`ans =`
	`-(u*4*i)/(u^2 + 1)^2`

$f(x,v) = e^{-x^2 \|v\| \frac{\sin v}{v}}$, x real
$F[f(v)](u) = \int_{-\infty}^{\infty} f(x,v)e^{-ivu}dv$
$= -\arctan\frac{u-1}{x^2} + \arctan\frac{u+1}{x^2}$
	`syms v u;`
	`syms x real;`
	`f = exp(-x^2*abs(v))*sin(v)/v;`
	`fourier(f,v,u)`
	returns
	`ans =`
	`piecewise([x <> 0,...`
	`atan((u + 1)/x^2)...`
	`- atan(1/x^2*(u - 1))])`