Bernoulli's equation

A Bernoulli equation is a differential equation of the form:

$$\frac{dy}{dx} + Py = Qy^n$$

This is solved by:

(a) Divide both sides by y^n to give:

$$y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$$

(b) Let $z = y^{1-n}$ so that:

$$\frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

Substitution yields:

$$\frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

then:

$$(1-n)\left[y^{-n}\frac{dy}{dx} + Py^{1-n}\right] = (1-n)Q$$

becomes:

$$\frac{dz}{dx} + P_1 z = Q_1$$

Which can be solved using the integrating factor method.

A Bernoulli equation is a first order equation

$$y' + p(x)\, y = R(x)\, y^{\alpha}$$

in which α is a real number.

Consider the equation

$$y' + \frac{1}{x}y = 3x^2y^3,$$

which is Bernoulli with $P(x) = 1/x$, $R(x) = 3x^2$, and $\alpha = 3$. Make the change of variables

$$v = y^{-2}.$$

Then $y = v^{-1/2}$ and

$$y'(x) = -\frac{1}{2}v^{-3/2}v'(x),$$

so the differential equation becomes

$$-\frac{1}{2}v^{-3/2}v'(x) + \frac{1}{x}v^{-1/2} = 3x^2v^{-3/2},$$

or, upon multiplying by $-2v^{3/2}$,

$$v' - \frac{2}{x}v = -6x^2,$$

a linear equation. An integrating factor is $e^{-\int(2/x)\,dx} = x^{-2}$. Multiply the last equation by this factor to get

$$x^{-2}v' - 2x^{-3}v = -6,$$

which is

$$(x^{-2}v)' = -6.$$

Integrate to get

$$x^{-2}v = -6x + C,$$

so

$$v = -6x^3 + Cx^2.$$

The general solution of the Bernoulli equation is

$$y(x) = \frac{1}{\sqrt{v(x)}} = \frac{1}{\sqrt{Cx^2 - 6x^3}}.$$

Second-order differential equations

Homogeneous equations

The differential equation:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

Is a *second-order, constant coefficient, linear, homogeneous differential equation*. Its solution is found from the solutions to the **auxiliary equation**:

$$am^2 + bm + c = 0$$

These are:

$$m_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad m_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The auxiliary equation

- Real and different roots

If the auxiliary equation: $am^2 + bm + c = 0$

With solution :

$$m_1 = \frac{\sqrt{b^2 - 4ac}}{2a} \quad and \;\; m_2 = \frac{\sqrt{b^2 - 4ac}}{2a}$$

Where m_1 and m_2 are real and $m_1 \neq m_2$

Then the solution to :

$$a\frac{dy^2}{dx^2} + b\frac{dy}{dx}\,cy = 0 \quad is \qquad |y = Ae^{m_1 x} + Be^{m_2 x}|$$

- Real and equal roots

  If the auxiliary equation: $am^2 + bm + c = 0$

  With solution :

$$m_1 = \frac{\sqrt{b^2 - 4ac}}{2a} \quad and \;\; m_2 = \frac{\sqrt{b^2 - 4ac}}{2a}$$

  Where m_1 and m_2 are real and $m_1 = m_2$

  Then the solution to :

$$a\frac{dy^2}{dx^2} + b\frac{dy}{dx}\,cy = 0 \quad is \qquad |y = (A + Bx)e^{m_1 x}|$$

Complex roots

If the auxiliary equation:

$$am^2 + bm + c = 0$$

with solution:

$$m_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad m_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

where:

$$m_1 \text{ and } m_2 \text{ are } complex$$

Then the solutions to the auxiliary equation are complex conjugates. That is:

$$m_1 = \alpha + j\beta \quad \text{and} \quad m_2 = \alpha - j\beta$$

Complex roots

Complex roots to the auxiliary equation:

$$am^2 + bm + c = 0$$

means that the solution of the differential equation:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

is of the form:

$$y = Ae^{(\alpha + j\beta)x} + Be^{(\alpha - j\beta)x}$$
$$= e^{\alpha x}\left(Ae^{j\beta x} + Be^{-j\beta x}\right)$$

Since:

$$e^{j\beta x} = \cos\beta x + j\sin\beta x \quad \text{and} \quad e^{-j\beta x} = \cos\beta x - j\sin\beta x$$

then:

$$Ae^{j\beta x} + Be^{-j\beta x} = (A+B)\cos\beta x + j(A-B)\sin\beta x$$
$$= C\cos\beta x + D\sin\beta x$$

The solution to the differential equation whose auxiliary equation has complex roots can be written as::

$$y = e^{\alpha x}\left(C\cos\beta x + D\sin\beta x\right)$$

Differential equations of the form:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \quad \text{where } a, b \text{ and } c \text{ are contants}$$

Auxiliary equation:

$$am^2 + bm + c = 0 \quad \text{with roots } m_1 \text{ and } m_2$$

Roots real and different: Solution $y = Ae^{m_1x} + Be^{m_2x}$

Roots real and the same: Solution $y = (A + Bx)e^{m_1x}$

Roots complex $(\alpha \pm j\beta)$: Solution $y = e^{\alpha x}\left(C\cos\beta x + D\sin\beta x\right)$

The characteristic equation of $y'' - y' - 6y = 0$ is

$$\lambda^2 - \lambda - 6 = 0,$$

with roots $a = -2$ and $b = 3$. The general solution is

$$y = c_1 e^{-2x} + c_2 e^{3x}. \blacksquare$$

The characteristic equation of $y'' - 6y' + 9y = 0$ is $\lambda^2 - 6\lambda + 9 = 0$, with repeated root $\lambda = 3$. The general solution is

$$y(x) = e^{3x}(c_1 + c_2 x). \blacksquare$$

The characteristic equation of $y'' + 2y' + 6y = 0$ is $\lambda^2 + 2\lambda + 6 = 0$, with roots $-1 \pm \sqrt{5}i$. The general solution is

$$y(x) = c_1 e^{(-1+\sqrt{5}i)x} + c_2 e^{(-1-\sqrt{5}i)x}. \blacksquare$$

Solve the initial value problem

$$y'' - 4y' + 53y = 0; \qquad y(\pi) = -3,\ y'(\pi) = 2.$$

First solve the differential equation. The characteristic equation is

$$\lambda^2 - 4\lambda + 53 = 0,$$

with complex roots $2 \pm 7i$. The general solution is

$$y(x) = c_1 e^{2x}\cos(7x) + c_2 e^{2x}\sin(7x).$$

Now

$$y(\pi) = c_1 e^{2\pi}\cos(7\pi) + c_2 e^{2\pi}\sin(7\pi) = -c_1 e^{2\pi} = -3,$$

so

$$c_1 = 3e^{-2\pi}.$$

Thus far

$$y(x) = 3e^{-2\pi}e^{2x}\cos(7x) + c_2 e^{2x}\sin(7x).$$

Compute

$$y'(x) = 3e^{-2\pi}[2e^{2x}\cos(7x) - 7e^{2x}\sin(7x)] + 2c_2 e^{2x}\sin(7x) + 7c_2 e^{2x}\cos(7x).$$

Then

$$y'(\pi) = 3e^{-2\pi}2e^{2\pi}(-1) + 7c_2 e^{2\pi}(-1) = 2,$$

so

$$c_2 = -\frac{8}{7}e^{-2\pi}.$$

The solution of the initial value problem is

$$y(x) = 3e^{-2\pi}e^{2x}\cos(7x) - \frac{8}{7}e^{-2\pi}e^{2x}\sin(7x)$$

$$= e^{2(x-\pi)}\left[3\cos(7x) - \frac{8}{7}\sin(7x)\right]. \blacksquare$$