Transform near Homogeneous equation to Homogeneous

$$\frac{dy}{dx} = \frac{y+1}{x+2y} \quad \text{(Near Homogeneous equation)}$$

1- *Let* $y+1=0 \rightarrow y=-1$

2- *Let* $x+2y=0 \rightarrow x=-2y \rightarrow x=2$

3- *Replace* y *with* $y-1$

4- *Replace* x *with* $x+2$

5- *Substitute x and y in the equation :* $\dfrac{dy}{dx} = \dfrac{(y-1)+1}{(x+2)+2(y-1)}$

$$\frac{dy}{dx} = \frac{y}{x+2y} \quad \textit{(Homogeneous equation)}$$

6- ***Let*** $y=u\,x \rightarrow dy = u\,dx + x\,du$

7- ***Continue to solve the equation to*** *:*

$x = 2y \ln(c.\,y)$

8- ***Let*** $x=x+2 \rightarrow x=x-2$ *, and* $y=y-1 \rightarrow y=y+1$

$x-2 = 2(y+1)\ln(c(y+1))$

or $x = 2+2(y+1)\ln(c(y+1))$

at $x=2,\ y=0 \rightarrow c = e0 = 1$

i.e., $x = 2+2(y+1)\ln(y+1)$ *(ans.)*

Linear equations – use of integrating factor

Consider the equation:

$$\frac{dy}{dx} + 5y = e^{2x}$$

Multiply both sides by e^{5x} to give:

$$e^{5x}\frac{dy}{dx} + e^{5x}5y = e^{5x}e^{2x} \quad \text{that is} \quad \frac{d}{dx}\left(ye^{5x}\right) = e^{7x}$$

then:

$$\int d\left(ye^{5x}\right) = \int e^{7x}dx \quad \text{so that} \quad ye^{5x} = e^{7x} + C$$

That is:

$$\boxed{y = e^{2x} + Ce^{-5x}}$$

A first-order differential equation is linear if it has the form

$$y'(x) + p(x)y = q(x).$$

Linear equations – use of integrating factor

The multiplicative factor e^{5x} that permits the equation to be solved is called the *integrating factor* and the method of solution applies to equations of the form:

$$\frac{dy}{dx} + Py = Q \quad \text{where} \quad e^{\int Pdx} \quad \text{is the integrating factor}$$

The solution is then given as:

$$\boxed{y.\mathrm{IF} = \int Q.\mathrm{IF}dx \quad \text{where IF} = e^{\int Pdx}}$$

The equation $y' + y = \sin(x)$ is linear. Here $p(x) = 1$ and $q(x) = \sin(x)$, both continuous for all x. An integrating factor is

$$e^{\int dx},$$

or e^x. Multiply the differential equation by e^x to get

$$y'e^x + ye^x = e^x \sin(x),$$

or

$$(ye^x)' = e^x \sin(x).$$

Integrate to get

$$ye^x = \int e^x \sin(x)\, dx = \frac{1}{2} e^x [\sin(x) - \cos(x)] + C.$$

The general solution is

$$y(x) = \frac{1}{2}[\sin(x) - \cos(x)] + Ce^{-x}.$$

Solve the initial value problem

$$y' = 3x^2 - \frac{y}{x}; \quad y(1) = 5.$$

First recognize that the differential equation can be written in linear form:

$$y' + \frac{1}{x}y = 3x^2.$$

An integrating factor is $e^{\int (1/x)\, dx} = e^{\ln(x)} = x$, for $x > 0$. Multiply the differential equation by x to get

$$xy' + y = 3x^3,$$

or

$$(xy)' = 3x^3.$$

Integrate to get

$$xy = \frac{3}{4}x^4 + C.$$

Then

$$y(x) = \frac{3}{4}x^3 + \frac{C}{x}$$

for $x > 0$. For the initial condition, we need

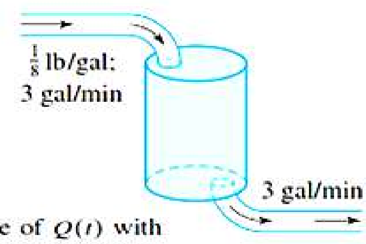
$$y(1) = 5 = \frac{3}{4} + C$$

so $C = 17/4$ and the solution of the initial value problem is

$$y(x) = \frac{3}{4}x^3 + \frac{17}{4x}$$

for $x > 0$.

As an example, suppose a tank contains 200 gallons of brine (salt mixed with water), in which 100 pounds of salt are dissolved. A mixture consisting of $\frac{1}{8}$ pound of salt per gallon is flowing into the tank at a rate of 3 gallons per minute, and the mixture is continuously stirred. Meanwhile, brine is allowed to empty out of the tank at the same rate of 3 gallons per minute



Now let $Q(t)$ be the amount of salt in the tank at time t. The rate of change of $Q(t)$ with time must equal the rate at which salt is pumped in, minus the rate at which it is pumped out. Thus

$$\begin{aligned}\frac{dQ}{dt} &= (\text{rate in}) - (\text{rate out}) \\ &= \left(\frac{1}{8}\,\frac{\text{pounds}}{\text{gallon}}\right)\left(3\,\frac{\text{gallons}}{\text{minute}}\right) - \left(\frac{Q(t)}{200}\,\frac{\text{pounds}}{\text{gallon}}\right)\left(3\,\frac{\text{gallons}}{\text{minute}}\right) \\ &= \frac{3}{8} - \frac{3}{200}Q(t).\end{aligned}$$

This is the linear equation

$$Q'(t) + \frac{3}{200}Q = \frac{3}{8}.$$

An integrating factor is $e^{\int (3/200)\,dt} = e^{3t/200}$. Multiply the differential equation by this factor to obtain

$$Q'e^{3t/200} + \frac{3}{200}e^{3t/200}Q = \frac{3}{8}e^{3t/200},$$

or

$$(Qe^{3t/200})' = \frac{3}{8}e^{3t/200}.$$

Then

$$Qe^{3t/200} = \frac{3}{8}\,\frac{200}{3}e^{3t/200} + C,$$

so

$$Q(t) = 25 + Ce^{-3t/200}.$$

Now

$$Q(0) = 100 = 25 + C$$

so $C = 75$ and

$$Q(t) = 25 + 75e^{-3t/200}.$$

Q/ For the system shown in Fig (1):

i. Find the amount of salt in tank (3) after 3 hour.
ii. Derive the simultaneous differential equation that describe the change of salt amount in tanks (2) and (3) (Do not solve it).

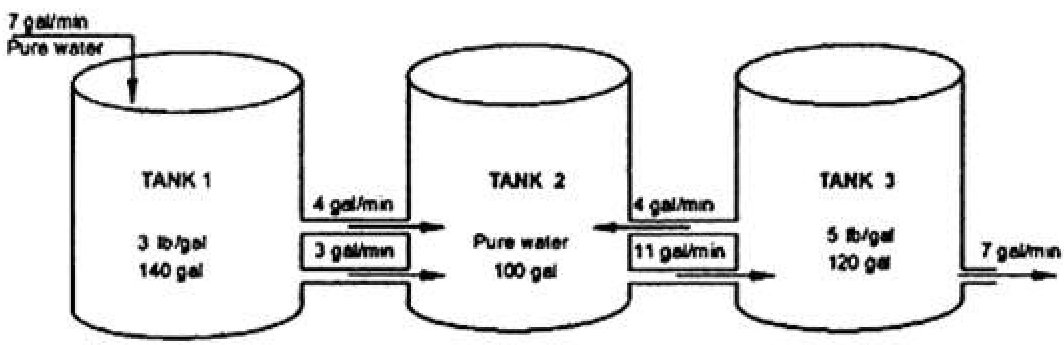


Figure 1

Solution:

Let the amount of salt in tank1 at any time = Q (Ib)
Let the amount of salt in tank2 at any time = S (Ib)
Let the amount of salt in tank3 at any time = Z (Ib)

i- Tank1 will be solved separately since there is no feedback from another tank (not cycled)

$$Q'(t) = \text{In salt proportion} * \text{Inflow rate} - \textit{out} \text{ salt proportion} * \text{Outflow rate}$$

$$Q'(t) = \text{In salt proportion} * \text{Inflow rate} - \frac{\text{Q(t)}}{\text{Amount of solution}} * \text{Outflow rate}$$

or

$$Q'(t) = S_{in} - \frac{\text{Q(t)}}{\text{V}} F_{out}$$

Where

V= 140 gal , $S_{in} = P_{in} * F_{in} = 0 * 7$ (gal/min) = 0 , F_{out} = 7 gal/min

Then

$$Q'(t) = 0 - \frac{Q(t)}{140} * 7 \rightarrow Q'(t) = -5x10^{-2} Q(t)$$

Using linear equation

$$Q'(t) + 5x10^{-2} Q(t) = 0$$

The integrating factor is :

$$e^{\int p(x)dx} = e^{\int 5x10^{-2} dt} = e^{5x10^{-2} t}$$

Multiply the differential equation by this factor to obtain

$$Q'(t)e^{5x10^{-2} t} + 5x10^{-2} Q(t)e^{5x10^{-2}} = 0$$

Or

$$\left(Q(t)e^{5x10^{-2} t}\right)' = 0$$

$$Q(t)e^{5x10^{-2}t} = c \rightarrow Q(t) = c\, e^{-5x10^{-2} t}$$

But we have from the initial boundary condition Q(0) = 3*140 = 420 lb, then:

$c = 420$

and

$$Q(t) = 420\, e^{-5x10^{-2} t}$$

At t= 3 hr (180 min) the amount of salt is:

$$Q(180) = 420 * e^{-5x10^{-2}*180} = 0.0752\ lb$$

The system of differential equations for tank2 and tank3 are:

$$\frac{dS}{dt} = \frac{7\,Q}{140} - \frac{11\,S}{100} + \frac{4\,Z}{120} \rightarrow \frac{dS}{dt} + \frac{11\,S}{100} - \frac{4\,Z}{120} = 21e^{-5x10^{-2} t} \quad \ldots\ldots\ldots.1$$

$$\frac{dZ}{dt} = \frac{11\,S}{100} - \frac{11\,Z}{120} \rightarrow \frac{dZ}{dt} + \frac{11\,Z}{120} - \frac{11\,S}{100} = 0 \quad \ldots\ldots\ldots\ldots 2$$

Using operating notation the system of equations can be write as:

$$\left(D+\frac{11}{100}\right)S+\left(-\frac{1}{30}\right)Z=21\,e^{-5x10^{-2}\,t}$$

$$\left(-\frac{11}{100}\right)S+\left(D+\frac{11}{120}\right)Z=0$$

The matrix of coefficients is

$$A=\begin{bmatrix}\left(D+\frac{11}{100}\right) & -\frac{1}{30}\\ \left(-\frac{11}{100}\right) & \left(D+\frac{11}{120}\right)\end{bmatrix}$$

By Cramer's rule:

$$AS=\begin{bmatrix}\left(D+\frac{11}{100}\right) & 21e^{-5x10^{-2}\,t}\\ \left(-\frac{11}{100}\right) & 0\end{bmatrix}$$

$$Az=\begin{bmatrix}21e^{-5x10^{-2}\,t} & -\frac{1}{30}\\ 0 & \left(D+\frac{11}{120}\right)\end{bmatrix}$$