Engineering Analysis   التحليل الهندسي

يختص **التحليل الهندسي** بتطبيق أسس وعمليات التحليل العلمي على أي نظام أو جهاز أو تقنية محل دراسة، بهدف معرفة خصائصه وحالته. ويعتمد التحليل الهندسي على التفكيك؛ حيث يقوم بفصل التصميم الهندسي إلى آليات تشغيل أو آليات إخفاق، ثم يبدأ في تحليل أو تقييم مكوّنات كل آلية منها على حدة، وبعد ذلك يُعيد تجميع المكوّنات وفقًا للأسس الفيزيائية والقوانين الطبيعية الأساسية.

1- First-order differential equations
2- Second -order differential equations
3- Fourier Series Transform
4- Laplace Transform
5- z-transform
6- Wave Equation

First-order differential equations

A differential equation is a relationship between an independent variable x, a dependent variable y and one or more derivatives of y with respect to x.

The order of a differential equation is given by the highest derivative involved.

$x\frac{dy}{dx} - y^2 = 0$ is an equation of the 1st order

$xy\frac{d^2y}{dx^2} - y^2 \sin x = 0$ is an equation of the 2nd order

$\frac{d^3y}{dx^3} - y\frac{dy}{dx} + e^{4x} = 0$ is an equation of the 3rd order

Formation of differential equations

Differential equations may be formed from a consideration of the physical problems to which they refer. Mathematically, they can occur when arbitrary constants are eliminated from a given function. For example, let:

$y = A\sin x + B\cos x$ so that $\frac{dy}{dx} = A\cos x - B\sin x$ therefore

$$\frac{d^2y}{dx^2} = -A\sin x - B\cos x = -y$$

That is
$$\boxed{\frac{d^2y}{dx^2} + y = 0}$$

Here the given function had *two* arbitrary constants:

$$y = A\sin x + B\cos x$$

and the end result was a *second order* differential equation:

$$\frac{d^2y}{dx^2} + y = 0$$

In general an nth order differential equation will result from consideration of a function with n arbitrary constants.

Solving a differential equation is the reverse process to the one just considered. To solve a differential equation a function has to be found for which the equation holds true.

The solution will contain a number of arbitrary constants – the number equalling the order of the differential equation.

Direct integration

If the differential equation to be solved can be arranged in the form:

$$\frac{dy}{dx} = f(x)$$

the solution can be found by direct integration. That is:

$$y = \int f(x)dx$$

Direct integration

For example:

$$\frac{dy}{dx} = 3x^2 - 6x + 5$$

so that:

$$y = \int (3x^2 - 6x + 5)dx$$
$$= x^3 - 3x^2 + 5x + C$$

This is the *general solution* (or *primitive*) of the differential equation. If a value of y is given for a specific value of x then a value for C can be found. This would then be a *particular solution* of the differential equation.

Separating the variables

If a differential equation is of the form:

$$\frac{dy}{dx} = \frac{f(x)}{F(y)}$$

Then, after some manipulation, the solution can be found by direct integration.

$$F(y)dy = f(x)dx \quad \text{so} \quad \int F(y)dy = \int f(x)dx$$

Separating the variables

For example:

$$\frac{dy}{dx}=\frac{2x}{y+1}$$

so that:

$$(y+1)dy=2xdx \quad \text{so} \quad \int(y+1)dy=\int 2xdx$$

That is:

$$y^2+y+C_1=x^2+C_2$$

Finally:

$$y^2+y=x^2+C$$

$y'=y^2e^{-x}$ is separable. Write

$$\frac{dy}{dx}=y^2e^{-x}$$

as

$$\frac{1}{y^2}\,dy=e^{-x}\,dx$$

for $y\neq 0$. Integrate this equation to obtain

$$-\frac{1}{y}=-e^{-x}+k,$$

an equation that implicitly defines the general solution. In this example we can explicitly solve for y, obtaining the general solution

$$y=\frac{1}{e^{-x}-k}.$$

$x^2y' = 1 + y$ is separable, and we can write

$$\frac{1}{1+y}\ dy = \frac{1}{x^2}\ dx.$$

The algebra of separation has required that $x \neq 0$ and $y \neq -1$, even though we can put $x = 0$ and $y = -1$ into the differential equation to obtain the correct equation $0 = 0$.

Now integrate the separated equation to obtain

$$\ln|1+y| = -\frac{1}{x} + k.$$

This implicitly defines the general solution. In this case, we can solve for $y(x)$ explicitly. Begin by taking the exponential of both sides to obtain

$$|1+y| = e^k e^{-1/x} = Ae^{-1/x},$$

in which we have written $A = e^k$. Since k could be any number, A can be any positive number. Then

$$1 + y = \pm Ae^{-1/x} = Be^{-1/x},$$

in which $B = \pm A$ can be any nonzero number. The general solution is

$$y = -1 + Be^{-1/x},$$

in which B is any nonzero number.

Solve the initial value problem

$$y' = y^2e^{-x};\ y(1) = 4.$$

We know from Example that the general solution of $y' = y^2e^{-x}$ is

$$y(x) = \frac{1}{e^{-x} - k}.$$

Now we need to choose k so that

$$y(1) = \frac{1}{e^{-1} - k} = 4,$$

from which we get

$$k = e^{-1} - \frac{1}{4}.$$

The solution of the initial value problem is

$$y(x) = \frac{1}{e^{-x} + \frac{1}{4} - e^{-1}}.$$

Some Applications of Separable Differential Equations

(Radioactive Decay and Carbon Dating) In radioactive decay, mass is converted to energy by radiation. It has been observed that the rate of change of the mass of a radioactive substance is proportional to the mass itself. This means that, if $m(t)$ is the mass at time t, then for some constant of proportionality k that depends on the substance,

$$\frac{dm}{dt} = km.$$

This is a separable differential equation. Write it as

$$\frac{1}{m}\,dm = k\,dt$$

and integrate to obtain

$$\ln|m| = kt + c.$$

Since mass is positive, $|m| = m$ and

$$\ln(m) = kt + c.$$

Then

$$m(t) = e^{kt+c} = Ae^{kt},$$

in which A can be any positive number.

Determination of A and k for a given element requires two measurements. Suppose at some time, designated as time zero, there are M grams present. This is called the initial mass. Then

$$m(0) = A = M,$$

so

$$m(t) = Me^{kt}.$$

If at some later time T we find that there are M_T grams, then

$$m(T) = M_T = Me^{kT}.$$

Then

$$\ln\left(\frac{M_T}{M}\right) = kT,$$

hence

$$k = \frac{1}{T}\ln\left(\frac{M_T}{M}\right).$$

This gives us k and determines the mass at any time:

$$m(t) = Me^{\ln(M_T/M)t/T}.$$

We obtain a more convenient formula for the mass if we choose the time of the second measurement more carefully. Suppose we make the second measurement at that time $T = H$ at which exactly half of the mass has radiated away. At this time, half of the mass remains, so $M_T = M/2$ and $M_T/M = 1/2$. Now the expression for the mass becomes

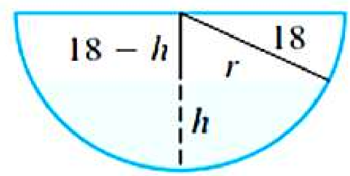
$$m(t) = Me^{\ln(1/2)t/H},$$

or

$$m(t) = Me^{-\ln(2)t/H}.$$

This number H is called the half-life of the element. Although we took it to be the time needed for half of the original amount M to decay, in fact, between any times t_1 and $t_1 + H$, exactly half of the mass of the element present at t_1 will radiate away. To see this, write

$$\begin{aligned} m(t_1 + H) &= Me^{-\ln(2)(t_1+H)/H} \\ &= Me^{-\ln(2)t_1/H}e^{-\ln(2)H/H} = e^{-\ln(2)}m(t_1) \\ &= \tfrac{1}{2}m(t_1). \end{aligned}$$



the rate of discharge of a fluid flowing through an opening at the bottom of a container is given by

$$\frac{dV}{dt} = -kAv,$$

in which $V(t)$ is the volume of fluid in the container at time t, $v(t)$ is the discharge velocity of fluid through the opening, A is the cross sectional area of the opening (assumed constant), and k is a constant determined by the viscosity of the fluid, the shape of the opening, and the fact that the cross-sectional area of fluid pouring out of the opening is slightly less than that of the opening itself. In practice, k must be determined for the particular fluid, container, and opening, and is a number between 0 and 1.

We also need Torricelli's law, which states that $v(t)$ is equal to the velocity of a free-falling particle released from a height equal to the depth of the fluid at time t. (Free-falling means that the particle is influenced by gravity only). Now the work done by gravity in moving the particle from its initial point by a distance $h(t)$ is $mgh(t)$, and this must equal the change in the kinetic energy, $(\frac{1}{2})mv^2$. Therefore,

$$v(t) = \sqrt{2gh(t)}.$$

Putting these two equations together yields

$$\frac{dV}{dt} = -kA\sqrt{2gh(t)}.$$

Suppose we have a hemispherical tank of water. The tank has radius 18 feet, and water drains through a circular hole of radius 3 inches at the bottom. How long will it take the tank to empty?

Equation contains two unknown functions, $V(t)$ and $h(t)$, so one must be eliminated. Let $r(t)$ be the radius of the surface of the fluid at time t and consider an interval of time from t_0 to $t_1 = t_0 + \Delta t$. The volume ΔV of water draining from the tank in this time equals the volume of a disk of thickness Δh (the change in depth) and radius $r(t^*)$, for some t^* between t_0 and t_1. Therefore

$$\Delta V = \pi [r(t^*)]^2 \Delta h$$

so

$$\frac{\Delta V}{\Delta t} = \pi [r(t^*)]^2 \frac{\Delta h}{\Delta t}.$$

In the limit as $t \to 0$,

$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}.$$

Putting this into equation

$$\pi r^2 \frac{dh}{dt} = -kA\sqrt{2gh}.$$

Now V has been eliminated, but at the cost of introducing $r(t)$.

$$r^2 = 18^2 - (18 - h)^2 = 36h - h^2$$

so

$$\pi (36h - h^2) \frac{dh}{dt} = -kA\sqrt{2gh}.$$

This is a separable differential equation, which we write as

$$\pi \frac{36h - h^2}{h^{1/2}} \, dh = -kA\sqrt{2g} \, dt.$$

Take g to be 32 feet per second per second. The radius of the circular opening is 3 inches, or $\frac{1}{4}$ feet, so its area is $A = \pi/16$ square feet. For water, and an opening of this shape and size, the experiment gives $k = 0.8$. The last equation becomes

$$(36h^{1/2} - h^{3/2}) \, dh = -(0.8)\left(\frac{1}{16}\right)\sqrt{64} \, dt,$$

or

$$(36h^{1/2} - h^{3/2}) \, dh = -0.4 \, dt.$$

A routine integration yields

$$24h^{3/2} - \frac{2}{5}h^{5/2} = -\frac{2}{5}t + c,$$

or

$$60h^{3/2} - h^{5/2} = -t + k.$$

Now $h(0) = 18$, so

$$60(18)^{3/2} - (18)^{5/2} = k.$$

Thus $k = 2268\sqrt{2}$ and $h(t)$ is implicitly determined by the equation

$$60h^{3/2} - h^{5/2} = 2268\sqrt{2} - t.$$

The tank is empty when $h = 0$, and this occurs when $t = 2268\sqrt{2}$ seconds, or about 53 minutes, 28 seconds. □

Homogeneous equations – by substituting $y = vx$

In a homogeneous differential equation the total degree in x and y for the terms involved is the same.

For example, in the differential equation:

$$\frac{dy}{dx} = \frac{x+3y}{2x}$$

the terms in x and y are both of degree 1.

To solve this equation requires a change of variable using the equation:

$$y = v(x)x$$

Homogeneous equations – by substituting $y = vx$

To solve:

$$\frac{dy}{dx} = \frac{x+3y}{2x}$$

let

$$y = v(x)x$$

to yield:

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \quad \text{and} \quad \frac{x+3y}{2x} = \frac{1+3v}{2}$$

That is:

$$\boxed{x\frac{dv}{dx} = \frac{1+v}{2}}$$

which can now be solved using the separation of variables method.

A first-order differential equation is homogeneous if it has the form

$$y' = f\left(\frac{y}{x}\right).$$

Consider

$$xy' = \frac{y^2}{x} + y.$$

Write this as

$$y' = \left(\frac{y}{x}\right)^2 + \frac{y}{x}.$$

Let $y = ux$. Then

$$u'x + u = u^2 + u,$$

or

$$u'x = u^2.$$

Write this as

$$\frac{1}{u^2}\ du = \frac{1}{x}\ dx$$

and integrate to obtain

$$-\frac{1}{u} = \ln|x| + C.$$

Then

$$u(x) = \frac{-1}{\ln|x| + C},$$

the general solution of the transformed equation. The general solution of the original equation is

$$y = \frac{-x}{\ln|x| + C}.$$

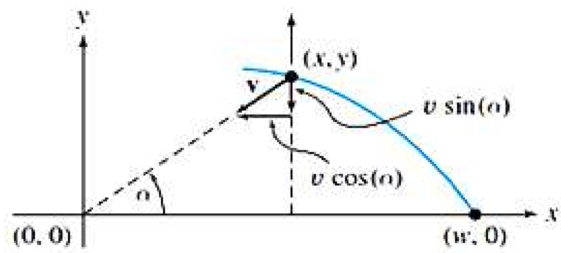
Suppose a person jumps into a canal of constant width w and swims toward a fixed point directly opposite the point of entry into the canal. The person's speed is v and the water current's speed is s. Assume that, as the swimmer makes his way across, he always orients to point toward the target. We want to determine the swimmer's trajectory.

Figure 1.13 shows a coordinate system drawn so that the swimmer's destination is the origin and the point of entry into the water is $(w, 0)$. At time t the swimmer is at the point $(x(t), y(t))$. The horizontal and vertical components of his velocity are, respectively,

$$x'(t) = -v\cos(\alpha) \quad \text{and} \quad y'(t) = s - v\sin(\alpha),$$

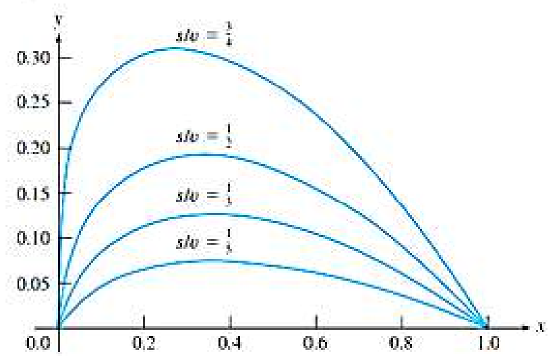
with α the angle between the positive x axis and $(x(t), y(t))$ at time t. From these equations,

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{s - v\sin(\alpha)}{-v\cos(\alpha)} = \tan(\alpha) - \frac{s}{v}\sec(\alpha).$$



From Figure

$$\tan(\alpha) = \frac{y}{x} \quad \text{and} \quad \sec(\alpha) = \frac{1}{x}\sqrt{x^2 + y^2}.$$



Therefore

$$\frac{dy}{dx} = \frac{y}{x} - \frac{s}{v}\frac{1}{x}\sqrt{x^2 + y^2}.$$

Write this as the homogeneous equation

$$\frac{dy}{dx} = \frac{y}{x} - \frac{s}{v}\sqrt{1 + \left(\frac{y}{x}\right)^2}$$

and put $y = ux$ to obtain

$$\frac{1}{\sqrt{1+u^2}}\,du = -\frac{s}{v}\frac{1}{x}\,dx.$$

Integrate to get

$$\ln\left|u + \sqrt{1+u^2}\right| = -\frac{s}{v}\ln|x| + C.$$

Take the exponential of both sides of this equation:

$$\left|u + \sqrt{1+u^2}\right| = e^C e^{-(s\ln|x|)/v}.$$

We can write this as

$$u + \sqrt{1+u^2} = Kx^{-s/v}.$$

This equation can be solved for u. First write

$$\sqrt{1+u^2} = Kx^{-s/v} - u$$

and square both sides to get

$$1+u^2 = K^2 e^{-2s/v} - 2Kue^{-s/v} + u^2.$$

Now u^2 cancels and we can solve for u:

$$u(x) = \frac{1}{2}Kx^{-s/v} - \frac{1}{2}\frac{1}{K}x^{s/v}.$$

Finally, put $u = y/x$ to get

$$y(x) = \frac{1}{2}Kx^{1-s/v} - \frac{1}{2}\frac{1}{K}x^{1+s/v}.$$

To determine K, notice that $y(w) = 0$, since we put the origin at the point of destination. Thus,

$$\frac{1}{2}Kw^{1-s/v} - \frac{1}{2}\frac{1}{K}w^{1+s/v} = 0$$

and we obtain

$$K = w^{s/v}.$$

Therefore,

$$y(x) = \frac{w}{2}\left[\left(\frac{x}{w}\right)^{1-s/v} - \left(\frac{x}{w}\right)^{1+s/v}\right].$$