Table of inverse transforms

$F(s)$	$f(t)$	
$\frac{a}{s}$	a	
$\frac{1}{s+a}$	e^{-at}	
$\frac{n!}{s^{n+1}}$	t^n	(n a positive integer)
$\frac{1}{s^n}$	$\frac{t^{n-1}}{(n-1)!}$	(n a positive integer)
$\frac{a}{s^2+a^2}$	$\sin at$	
$\frac{s}{s^2+a^2}$	$\cos at$	
$\frac{a}{s^2-a^2}$	$\sinh at$	
$\frac{s}{s^2-a^2}$	$\cosh at$	

Theorem 1

The first shift theorem can be stated as follows.

If $F(s)$ is the Laplace transform of $f(t)$ then $F(s+a)$ is the Laplace transform of $e^{-at}f(t)$.

Solution of differential equations by Laplace transforms

To solve a differential equation by Laplace transforms, we go through four distinct stages

(a) Rewrite the equation in terms of Laplace transforms.
(b) Insert the given initial conditions.
(c) Rearrange the equation algebraically to give the transform of the solution.
(d) Determine the inverse transform to obtain the particular solution.

Transforms of derivatives

$$L\{f'(t)\} = -f(0) + sL\{f(t)\}$$

$$L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

$$L\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'0) - f''(0)$$

we denote the Laplace transform of x by $\bar{x}$,

$$\bar{x} = L\{x\} = L\{f(t)\} = F(s).$$

$$L\{x\} = \bar{x}$$
$$L\{\dot{x}\} = s\bar{x} - x_0$$
$$L\{\ddot{x}\} = s^2\bar{x} - sx_0 - x_1$$
$$L\{\dddot{x}\} = s^3\bar{x} - s^2x_0 - sx_1 - x_2$$
$$L\{\ddddot{x}\} = s^4\bar{x} - s^3x_0 - s^2x_1 - sx_2 - x_3$$

Solution of first-order differential equations

Example 1

Solve the equation $\frac{dx}{dt} - 2x = 4$ given that at $t = 0$, $x = 1$.

We go through the four stages.

(a) Rewrite the equation in Laplace transforms, using the last notation

$$L\{x\} = \bar{x}; \quad L\{\dot{x}\} = \ldots\ldots\ldots\ldots$$
$$L\{4\} = \ldots\ldots\ldots\ldots$$

$$L\{\dot{x}\} = s\bar{x} - x_0; \quad L\{4\} = \frac{4}{s}$$

Then the equation becomes $(s\bar{x} - x_0) - 2\bar{x} = \frac{4}{s}$

(b) Insert the initial condition that at $t = 0$, $x = 1$, i.e. $x_0 = 1$

$$\therefore \; s\bar{x} - 1 - 2\bar{x} = \frac{4}{s}$$

(c) Now we rearrange this to give an expression for $\bar{x}$

$$\bar{x} = \ldots\ldots\ldots\ldots$$

$$\bar{x} = \frac{s+4}{s(s-2)}$$

(d) Finally, we take inverse transforms to obtain x.

$\frac{s+4}{s(s-2)}$ in partial fractions gives ………. $\frac{3}{s-2} - \frac{2}{s}$

Because

$$\frac{s+4}{s(s-2)} \equiv \frac{A}{s} + \frac{B}{s-2} \qquad \therefore \; s + 4 = A(s-2) + Bs$$

(1) Put $(s-2) = 0$, i.e. $s = 2$ $\quad \therefore \; 6 = B(2) \quad \therefore \; B = 3$

(2) Put $s = 0$ $\quad \therefore \; 4 = A(-2) \quad \therefore \; A = -2$

$$\therefore \; \bar{x} = \frac{s+4}{s(s-2)} = \frac{3}{s-2} - \frac{2}{s}$$

Therefore, taking inverse transforms

$$x = L^{-1}\left\{\frac{s+4}{s(s-2)}\right\} = L^{-1}\left\{\frac{3}{s-2} - \frac{2}{s}\right\} = \ldots\ldots$$

$$x = 3e^{2t} - 2$$

Example 2

Solve the equation $\frac{dx}{dt}+2x=10e^{3t}$ given that at $t=0$, $x=6$.

(a) Convert the equations to Laplace transforms, i.e.

$$(s\bar{x}-x_0)+2\bar{x}=\frac{10}{s-3}$$

(b) Insert the initial condition, $x_0=6$

$$s\bar{x}-6+2\bar{x}=\frac{10}{s-3}$$

(c) Rearrange to obtain $\bar{x}=$ ..

$$\bar{x}=\frac{6s-8}{(s+2)(s-3)}$$

(d) Taking inverse transforms to obtain x

$$x=L^{-1}\left\{\frac{6s-8}{(s+2)(s-3)}\right\}=.$$

$$x=4e^{-2t}+2e^{3t}$$

Because

$$\frac{6s-8}{(s+2)(s-3)}\equiv\frac{A}{s+2}+\frac{B}{s-3}$$

$$\therefore\ 6s-8=A(s-3)+B(s+2)$$

(1) Put $(s-3)=0$, i.e. $s=3$ $\quad\therefore\ 10=B(5)$ $\quad\therefore\ B=2$

(2) Put $(s+2)=0$, i.e. $s=-2$. $\quad\therefore\ -20=A(-5)$ $\quad\therefore\ A=4$

$$\therefore\ \bar{x}=\frac{6s-8}{(s+2)(s-3)}=\frac{4}{s+2}+\frac{2}{s-3}$$

$$\therefore\ x=L^{-1}\left\{\frac{4}{s+2}+\frac{2}{s-3}\right\}=4e^{-2t}+2e^{3t}$$

Example 3

Solve the equation $\frac{dx}{dt}-4x=2e^{2t}+e^{4t}$, given that at $t=0$, $x=0$.

(a) $(s\bar{x}-x_0)-4\bar{x}=\frac{2}{s-2}+\frac{1}{s-4}$

(b) $x_0=0\quad\therefore\ s\bar{x}-4\bar{x}=\frac{2}{s-2}+\frac{1}{s-4}$

(c) $\therefore\ \bar{x}=\frac{2}{(s-2)(s-4)}+\frac{1}{(s-4)^2}$

(d) $\frac{2}{(s-2)(s-4)}\equiv\frac{A}{s-2}+\frac{B}{s-4}\quad\therefore\ 2=A(s-4)+B(s-2)$

Putting $(s-2)=0$, i.e. $s=2$ $\quad\therefore\ 2=A(-2)$ $\quad\therefore\ A=-1$

Putting $(s-4)=0$, i.e. $s=4$ $\quad\therefore\ 2=B(2)$ $\quad\therefore\ B=1$

$$\therefore\ \bar{x}=\frac{1}{s-4}-\frac{1}{s-2}+\frac{1}{(s-4)^2}$$

$$\therefore\ x=e^{4t}-e^{2t}+te^{4t}$$

Solution of second-order differential equations

Example 1

Solve the equation $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 2e^{3t}$, given that at $t = 0$, $x = 5$ and $\frac{dx}{dt} = 7$.

(a) We rewrite the equation in terms of its transforms, remembering that

$$L\{x\} = \bar{x}$$
$$L\{\dot{x}\} = s\bar{x} - x_0$$
$$L\{\ddot{x}\} = s^2\bar{x} - sx_0 - x_1$$

The equation becomes ....

$$\left(s^2\bar{x} - sx_0 - x_1\right) - 3(s\bar{x} - x_0) + 2\bar{x} = \frac{2}{s-3}$$

(b) Insert the initial conditions. In this case $x_0 = 5$ and $x_1 = 7$

$$\therefore\ \left(s^2\bar{x} - 5s - 7\right) - 3(s\bar{x} - 5) + 2\bar{x} = \frac{2}{s-3}$$

(c) Rearrange to obtain $\bar{x} = \dots$

$$\bar{x} = \frac{5s^2 - 23s + 26}{(s-1)(s-2)(s-3)}$$

Because

$$s^2\bar{x} - 5s - 7 - 3s\bar{x} + 15 + 2\bar{x} = \frac{2}{s-3}$$
$$(s^2 - 3s + 2)\bar{x} - 5s + 8 = \frac{2}{s-3}$$
$$(s-1)(s-2)\bar{x} = \frac{2}{s-3} + 5s - 8 = \frac{2 + 5s^2 - 23s + 24}{s-3}$$
$$\therefore\ \bar{x} = \frac{5s^2 - 23s + 26}{(s-1)(s-2)(s-3)}$$

(d) Now for partial fractions

$$\frac{5s^2 - 23s + 26}{(s-1)(s-2)(s-3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3}$$
$$\therefore\ 5s^2 - 23s + 26 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2)$$

So that $A = \dots\dots\dots\dots$; $B = \dots\dots\dots\dots$; $C = \dots\dots\dots\dots$

$$A = 4; \quad B = 0; \quad C = 1$$

$$\therefore\ \bar{x} = \frac{4}{s-1} + \frac{1}{s-3}$$
$$\therefore\ x = \dots\dots\dots\dots$$

$$x = 4e^t + e^{3t}$$

Example 2

Solve $\ddot{x} + 5\dot{x} + 6x = 4t$, given that at $t = 0$, $x = 0$ and $\dot{x} = 0$.

As usual we begin $(s^2\bar{x} - sx_0 - x_1) + 5(s\bar{x} - x_0) + 6\bar{x} = \frac{4}{s^2}$

$$x_0 = 0;\ x_1 = 0 \quad \therefore\ (s^2 + 5s + 6)\bar{x} = \frac{4}{s^2}$$

$$\therefore\ \bar{x} = \frac{4}{s^2(s+2)(s+3)}$$

The s^2 in the denominator can be awkward, so we introduce a useful trick and detach one factor s outside the main expression, thus

$$\bar{x} = \frac{1}{s}\left\{\frac{4}{s(s+2)(s+3)}\right\} = \frac{1}{s}\left\{\frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+3}\right\}$$

Applying the 'cover up' rule to the expressions within the brackets

$$\bar{x} = \frac{1}{s}\left\{\frac{4}{6}\cdot\frac{1}{s} - \frac{2}{(s+2)} + \frac{4}{3}\cdot\frac{1}{s+3}\right\}$$

Now we bring the external $\frac{1}{s}$ back into the fold

$$\bar{x} = \frac{2}{3}\cdot\frac{1}{s^2} - \frac{2}{s(s+2)} + \frac{4}{3}\cdot\frac{1}{s(s+3)}$$

and the second and third terms can be expressed in simple partial fractions so that

$$\bar{x} = \frac{2}{3}\cdot\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+2} + \frac{4}{9}\cdot\frac{1}{s} - \frac{4}{9}\cdot\frac{1}{s+3}$$

which can now be simplified into

$$\bar{x} = \frac{2}{3}\cdot\frac{1}{s^2} - \frac{5}{9}\cdot\frac{1}{s} + \frac{1}{s+2} - \frac{4}{9}\cdot\frac{1}{s+3}$$

$$\boxed{x = \frac{2}{3}t - \frac{5}{9} + e^{-2t} - \frac{4}{9}e^{-3t}}$$

Example 1

Solve the pair of simultaneous equations

$$\dot{y} - x = e^t$$
$$\dot{x} + y = e^{-t}$$

given that at $t = 0$, $x = 0$ and $y = 0$.

(a) We first express both equations in Laplace transforms.

$$(s\bar{y} - y_0) - \bar{x} = \frac{1}{s-1}$$
$$(s\bar{x} - x_0) + \bar{y} = \frac{1}{s+1}$$

(b) Then we insert the initial conditions, $x_0 = 0$ and $y_0 = 0$.

$$\therefore\ \left.\begin{aligned} s\bar{y} - \bar{x} &= \frac{1}{s-1} \\ s\bar{x} + \bar{y} &= \frac{1}{s+1} \end{aligned}\right\}$$

(c) We now solve these for $\bar{x}$ and $\bar{y}$ by the normal algebraic method. Eliminating $\bar{y}$ we have

$$s\bar{y} - \bar{x} = \frac{1}{s-1}$$

$$s\bar{y} + s^2\bar{x} = \frac{s}{s+1}$$

$$\therefore \quad (s^2+1)\bar{x} = \frac{2}{s+1} - \frac{1}{s-1} = \frac{s^2-2s-1}{(s+1)(s-1)}$$

$$\therefore \quad \bar{x} = \frac{s^2-2s-1}{(s-1)(s+1)(s^2+1)}$$

Representing this in partial fractions gives ....

$$\boxed{\bar{x} = -\frac{1}{2}\cdot\frac{1}{s-1} - \frac{1}{2}\cdot\frac{1}{s+1} + \frac{s}{s^2+1} + \frac{1}{s^2+1}}$$

Because

$$\bar{x} = \frac{s^2-2s-1}{(s-1)(s+1)(s^2+1)} \equiv \frac{A}{s-1} + \frac{B}{s+1} + \frac{Cs+D}{s^2+1}$$

$$\therefore \quad s^2-2s-1 = A(s+1)(s^2+1) + B(s-1)(s^2+1) + (s-1)(s+1)(Cs+D)$$

Putting $s=1$ and $s=-1$ gives $A=-\frac{1}{2}$ and $B=-\frac{1}{2}$.

Putting $s=1$ and $s=-1$ gives $A=-\frac{1}{2}$ and $B=-\frac{1}{2}$.
Comparing coefficients of s^3 and the constant terms gives $C=1$ and $D=1$.

$$\therefore \quad \bar{x} = \frac{1}{2}\cdot\frac{1}{s-1} - \frac{1}{2}\cdot\frac{1}{s+1} + \frac{s+1}{s^2+1}$$

$$\therefore \quad x = \ldots\ldots\ldots\ldots$$

$$\boxed{x = -\tfrac{1}{2}e^{t} - \tfrac{1}{2}e^{-t} + \cos t + \sin t}$$

We now revert to equations (1) and eliminate $\bar{x}$ to obtain $\bar{y}$ and hence y, in the same way. Do this on your own.

$$y = \ldots\ldots\ldots\ldots$$

$$\boxed{y = \tfrac{1}{2}e^{t} + \tfrac{1}{2}e^{-t} - \cos t + \sin t}$$

Example 2

Solve the equations

$$2\dot{y} - 6y + 3x = 0$$
$$3\dot{x} - 3x - 2y = 0$$

given that at $t = 0$, $x = 1$ and $y = 3$.

Expressing these in Laplace transforms, we have

$$2(s\bar{y} - y_0) - 6\bar{y} + 3\bar{x} = 0$$
$$3(s\bar{x} - x_0) - 3\bar{x} - 2\bar{y} = 0$$

Then we insert the initial conditions and simplify, obtaining

$$3\bar{x} + (2s - 6)\bar{y} = 6 \qquad (1)$$
$$(3s - 3)\bar{x} - 2\bar{y} = 3 \qquad (2)$$

(a) *To find* $\bar{x}$

(1) $\qquad 3\bar{x} + (2s - 6)\bar{y} = 6$

(2) $\times (s - 3)$ $\qquad (s - 3)(3s - 3)\bar{x} - (2s - 6)\bar{y} = 3(s - 3)$

Adding, $\qquad [(s - 3)(3s - 3) + 3]\bar{x} = 3s - 9 + 6$

$$\therefore \quad (3s^2 - 12s + 12)\bar{x} = 3s - 3$$
$$(s^2 - 4s + 4)\bar{x} = s - 1$$

$$\therefore \quad \bar{x} = \frac{s - 1}{(s - 2)^2} \equiv \frac{A}{s - 2} + \frac{B}{(s - 2)^2} = \frac{A(s - 2) + B}{(s - 2)^2}$$

$\therefore \quad s - 1 = A(s - 2) + B \quad$ giving $\quad A = 1 \quad$ and $\quad B = 1$

$$\therefore \quad \bar{x} = \frac{1}{s - 2} + \frac{1}{(s - 2)^2} \qquad \therefore \quad x = e^{2t} + te^{2t}$$

(b) Going back to equations (1) and (2), we can find y.

$$y = \tfrac{1}{2}\{6e^{2t} + 3te^{2t}\}$$

Because, eliminating $\bar{x}$ we get

$$\bar{y} = \frac{6s - 9}{2(s - 2)^2} \equiv \frac{1}{2}\left\{\frac{A}{s - 2} + \frac{B}{(s - 2)^2}\right\} = \frac{1}{2}\left\{\frac{A(s - 2) + B}{(s - 2)^2}\right\}$$

$$\therefore \quad 6s - 9 = A(s - 2) + B \qquad \therefore \quad A = 6; \quad B = 3$$

$$\therefore \quad \bar{y} = \frac{1}{2}\left\{\frac{6}{s - 2} + \frac{3}{(s - 2)^2}\right\} \qquad \therefore \quad y = \tfrac{1}{2}\{6e^{2t} + 3te^{2t}\}$$