Inverse transforms

Here we have the reverse process, i.e. given a Laplace transform, we have to find the function of t to which it belongs.

For example, we know that $\frac{a}{s^2+a^2}$ is the Laplace transform of $\sin at$, so we can now write $L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at$, the symbol L^{-1} indicating the inverse transform and **not** a reciprocal.

(a) $L^{-1}\left\{\frac{1}{s-2}\right\} = \ldots\ldots$ (c) $L^{-1}\left\{\frac{4}{s}\right\} = \ldots\ldots$ (a) $L^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t}$;

(b) $L^{-1}\left\{\frac{s}{s^2+25}\right\} \ldots\ldots$ (d) $L^{-1}\left\{\frac{12}{s^2-9}\right\} = .$ (b) $L^{-1}\left\{\frac{s}{s^2+25}\right\} = \cos 5t$;

(c) $L^{-1}\left\{\frac{4}{s}\right\} = 4$

(d) $L^{-1}\left\{\frac{12}{s^2-9}\right\} = 4\sinh 3t$

But what about $L^{-1}\left\{\frac{3s+1}{s^2-s-6}\right\}$? This certainly did not appear in our list of standard transforms.

In considering $L^{-1}\left\{\frac{3s+1}{s^2-s-6}\right\}$, it happens that we can write $\frac{3s+1}{s^2-s-6}$ as the sum of two simpler functions $\frac{1}{s+2}+\frac{2}{s-3}$ which, of course, makes all the difference, since we can now proceed

$$L^{-1}\left\{\frac{3s+1}{s^2-s-6}\right\} = L^{-1}\left\{\frac{1}{s+2}+\frac{2}{s-3}\right\}$$

which we immediately recognise as ....

$e^{-2t} + 2e^{3t}$

The two simpler expressions $\frac{1}{s+2}$ and $\frac{2}{s-3}$ are called the *partial fractions* of $\frac{3s+1}{s^2-s-6}$, and the ability to represent a complicated algebraic fraction in terms of its partial fractions is the key to much of this work. Let us take a closer look at the rules.

Rules of partial fractions

1. The numerator must be of lower degree than the denominator. This is usually the case in Laplace transforms. If it is not, then we first divide out.
2. Factorise the denominator into its prime factors. These determine the shapes of the partial fractions.
3. A linear factor $(s+a)$ gives a partial fraction $\frac{A}{s+a}$ where A is a constant to be determined.
4. A repeated factor $(s+a)^2$ gives $\frac{A}{(s+a)}+\frac{B}{(s+a)^2}$.
5. Similarly $(s+a)^3$ gives $\frac{A}{(s+a)}+\frac{B}{(s+a)^2}+\frac{C}{(s+a)^3}$.
6. A quadratic factor (s^2+ps+q) gives $\frac{Ps+Q}{s^2+ps+q}$.
7. Repeated quadratic factors $(s^2+ps+q)^2$ give $\frac{Ps+Q}{s^2+ps+q}+\frac{Rs+T}{(s^2+ps+q)^2}$.

So $\frac{s-19}{(s+2)(s-5)}$ has partial fractions of the form $\boxed{\frac{A}{s+2}+\frac{B}{s-5}}$

and $\frac{3s^2-4s+11}{(s+3)(s-2)^2}$ has partial fractions of the form . $\boxed{\frac{A}{s+3}+\frac{B}{(s-2)}+\frac{C}{(s-2)^2}}$

Be careful of the repeated factor.

Example 1

To determine $L^{-1}\left\{\frac{5s+1}{s^2-s-12}\right\}$.

(a) First we check that the numerator is of lower degree than the denominator. In fact, this is so.

(b) Factorise the denominator $\frac{5s+1}{s^2-s-12}=\frac{5s+1}{(s-4)(s+3)}$.

(c) Then the partial fractions are of the form .... $\boxed{\frac{A}{s-4}+\frac{B}{s+3}}$

We therefore have the identity

$$\frac{5s+1}{s^2-s-12} \equiv \frac{A}{s-4} + \frac{B}{s+3}$$

If we multiply through both sides by the denominator $s^2 - s - 12 \equiv (s-4)(s+3)$ we have

$$5s+1 \equiv A(s+3) + B(s-4)$$

This is also an identity and true for any value of s we care to substitute – our job is now to find the values of A and B.

We now substitute convenient values for s

(a) Let $(s-4)=0$, i.e. $s=4$ $\therefore$ $21 = A(7) + B(0)$ $\therefore$ $A = 3$

(b) Let $(s+3)=0$, i.e. $s=-3$ and we get ....

$$\boxed{B = 2}$$

$$\therefore \frac{5s+1}{s^2-s-12} \equiv \frac{3}{s-4} + \frac{2}{s+3}$$

$$\therefore L^{-1}\left\{\frac{5s+1}{s^2-s-12}\right\} = \text{...........}$$

$$\boxed{3e^{4t} + 2e^{-3t}}$$

Example 2

Determine $L^{-1}\left\{\frac{9s-8}{s^2-2s}\right\}$.

Working as before, $f(t) = \ldots$

$$L\{f(t)\} = \frac{9s-8}{s^2-2s}.$$

(a) Numerator of first degree; denonominator of second degree. Therefore rule satisfied.

(b) $\frac{9s-8}{s(s-2)} \equiv \frac{A}{s} + \frac{B}{s-2}$.

(c) Multiply by $s(s-2)$. $\therefore$ $9s - 8 = A(s-2) + Bs$.

(d) Put $s = 0$. $-8 = A(-2) + B(0)$ $\therefore$ $A = 4$.

(e) Put $s - 2 = 0$, i.e. $s = 2$. $10 = A(0) + B(2)$ $\therefore$ $B = 5$.

$$\therefore f(t) = L^{-1}\left\{\frac{4}{s} + \frac{5}{s-2}\right\} = 4 + 5e^{2t}$$

Example 3

Express $F(s) = \frac{s^2 - 15s + 41}{(s+2)(s-3)^2}$ in partial fractions and hence determine its inverse transform.

$\frac{s^2 - 15s + 41}{(s+2)(s-3)^2}$ has partial fractions of the form ....

$$\boxed{\frac{A}{s+2} + \frac{B}{s-3} + \frac{C}{(s-3)^2}}$$

Now we multiply throughout by $(s+2)(s-3)^2$ and get

$$s^2 - 15s + 41 \equiv A(s-3)^2 + B(s+2)(s-3) + C(s+2)$$

Putting $(s-3) = 0$ and then $(s+2) = 0$ we obtain .... $\boxed{A = 3 \text{ and } C = 1}$

Now that we have run out of 'crafty' substitutions, we equate coefficients of the highest power of s on each side, i.e. the coefficients of s^2. This gives ....

$$1 = A + B \quad \therefore\ 1 = 3 + B \quad \therefore\ B = -2$$

So $\dfrac{s^2 - 15s + 41}{(s+2)(s-3)^2} = \dfrac{3}{s+2} - \dfrac{2}{s-3} + \dfrac{1}{(s-3)^2}$

Now $L^{-1}\left\{\dfrac{3}{s+2}\right\} = \ldots\ldots\ldots\ldots$ and $L^{-1}\left\{\dfrac{2}{s-3}\right\} = \ldots\ldots$

$$3e^{-2t} \text{ and } 2e^{3t}$$

But what about $L^{-1}\left\{\dfrac{1}{(s-3)^2}\right\}$?

We remember that $L^{-1}\left\{\dfrac{1}{s^2}\right\} = \ldots\ldots\ldots\ldots$

and that by Theorem 1, if $L\{f(t)\} = F(s)$ then $L\{e^{-at}f(t)\} = F(s+a)$.

$\therefore\ \dfrac{1}{(s-3)^2}$ is like $\dfrac{1}{s^2}$ with s replaced by $(s-3)$ i.e. $a = -3$.

$$\therefore\ L^{-1}\left\{\frac{1}{(s-3)^2}\right\} = te^{3t}$$

$$\therefore\ L^{-1}\left\{\frac{s^2 - 15s + 41}{(s+2)(s-3)^2}\right\} = 3e^{-2t} + 2e^{3t} + te^{3t}$$

The 'cover up' rule

While we can always find A, B, C, etc., there are many cases where we can use the 'cover up' methods and write down the values of the constant coefficients almost on sight. However, this method only works when the denominator of the original fraction has non-repeated, linear factors. The following examples illustrate the method.

Example 1

We know that $F(s) = \dfrac{9s-8}{s(s-2)}$ has partial fractions of the form $\dfrac{A}{s} + \dfrac{B}{s-2}$.

By the 'cover up' rule, the constant A, that is the coefficient of $\dfrac{1}{s}$, is found by temporarily covering up the factor s in the denominator of $F(s)$ and finding the limiting value of what remains when s (the factor covered up) tends to zero.

Therefore A = coefficient of $\dfrac{1}{s} = \lim\limits_{s\to 0}\left\{\dfrac{9s-8}{s-2}\right\} = 4$. That is $A = 4$.

Similarly, B, the coefficient of $\dfrac{1}{s-2}$, is obtained by covering up the factor $(s-2)$ in the denominator of $F(s)$ and finding the limiting value of what remains when $(s-2) \to 0$, that is $s \to 2$.

Therefore B = coefficient of $\dfrac{1}{s-2} = \lim\limits_{s\to 2}\left\{\dfrac{9s-8}{s}\right\} = 5$. That is $B = 5$.

So that

$$\frac{9s-8}{s(s-2)} = \frac{4}{s} + \frac{5}{s-2}$$

Example 2

$$F(s) = \frac{s+17}{(s-1)(s+2)(s-3)} \equiv \frac{A}{s-1} + \frac{B}{s+2} + \frac{C}{s-3}.$$

A: cover up $(s-1)$ in $F(s)$ and find

$$\lim_{s\to 1}\left\{\frac{s+17}{(s+2)(s-3)}\right\} = \frac{18}{-6} \quad \therefore A = -3$$

Similarly

$$B = \lim_{s\to -2}\left\{\frac{s+17}{(s-1)(s-3)}\right\} = \frac{15}{(-3)(-5)} = 1 \qquad \therefore B = 1$$

$$C = \lim_{s\to 3}\left\{\frac{s+17}{(s-1)(s+2)}\right\} = \frac{20}{(2)(5)} = 2 \qquad \therefore C = 2$$

$$\therefore F(s) = \frac{1}{s+2} + \frac{2}{s-3} - \frac{3}{s-1}$$

So $f(t) = e^{-2t} + 2e^{3t} - 3e^{t}$

Table of inverse transforms

$F(s)$	$f(t)$	
$\frac{a}{s}$	a	
$\frac{1}{s+a}$	e^{-at}	
$\frac{n!}{s^{n+1}}$	t^n	(n a positive integer)
$\frac{1}{s^n}$	$\frac{t^{n-1}}{(n-1)!}$	(n a positive integer)
$\frac{a}{s^2+a^2}$	$\sin at$	
$\frac{s}{s^2+a^2}$	$\cos at$	
$\frac{a}{s^2-a^2}$	$\sinh at$	
$\frac{s}{s^2-a^2}$	$\cosh at$	

Theorem 1

The first shift theorem can be stated as follows.

If $F(s)$ is the Laplace transform of $f(t)$ then $F(s+a)$ is the Laplace transform of $e^{-at}f(t)$.

Solution of differential equations by Laplace transforms

To solve a differential equation by Laplace transforms, we go through four distinct stages

(a) Rewrite the equation in terms of Laplace transforms.

(b) Insert the given initial conditions.

(c) Rearrange the equation algebraically to give the transform of the solution.

(d) Determine the inverse transform to obtain the particular solution.

Transforms of derivatives

$$L\{f'(t)\} = -f(0) + sL\{f(t)\}$$

$$L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

$$L\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'0) - f''(0)$$

we denote the Laplace transform of x by $\bar{x}$,

$$\bar{x} = L\{x\} = L\{f(t)\} = F(s).$$

$$L\{x\} = \bar{x}$$
$$L\{\dot{x}\} = s\bar{x} - x_0$$
$$L\{\ddot{x}\} = s^2\bar{x} - sx_0 - x_1$$
$$L\{\dddot{x}\} = s^3\bar{x} - s^2x_0 - sx_1 - x_2$$
$$L\{\ddddot{x}\} = s^4\bar{x} - s^3x_0 - s^2x_1 - sx_2 - x_3$$