Example

We will find the Fourier series of $f(x) = x^4$ on $[-1, 1]$. Since f is an even function, $x^4 \sin(n\pi x)$ is odd and we know immediately that all the sine coefficients b_n are zero. For the other coefficients, compute:

$$a_0 = \int_{-1}^{1} x^4 dx = 2\int_0^1 x^4 dx = \frac{2}{5},$$

and

$$a_n = \int_{-1}^{1} x^4 \cos(n\pi x) dx$$
$$= 2\int_0^1 x^4 \cos(n\pi x) dx = 8\frac{n^2\pi^2 - 6}{\pi^4 n^4}(-1)^n.$$

The Fourier series of x^4 on $[-1, 1]$ is

$$\frac{1}{5} + \sum_{n=1}^{\infty} 8\frac{n^2\pi^2 - 6}{\pi^4 n^4}(-1)^n \cos(n\pi x).$$

To again make the point about convergence, notice that $f(0) = 0$ in this example, but the Fourier series at $x = 0$ is

$$\frac{1}{5} + \sum_{n=1}^{\infty} 8\frac{n^2\pi^2 - 6}{\pi^4 n^4}(-1)^n.$$

It is not clear whether or not this series sums to the function value 0.

Example

Let $f(x) = x^3$ for $-4 \le x \le 4$. Because f is odd on $[-4, 4]$, its Fourier cosine coefficients are all zero. Its Fourier sine coefficients are

$$b_n = \frac{1}{4}\int_{-4}^{4} x^3 \sin\left(\frac{n\pi x}{4}\right) dx$$
$$= \frac{1}{2}\int_0^4 x^3 \sin\left(\frac{n\pi x}{4}\right) dx = (-1)^{n+1} 128\frac{n^2\pi^2 - 6}{n^3\pi^3}.$$

The Fourier series of x^3 on $[-4, 4]$ is

$$\sum_{n=1}^{\infty} (-1)^{n+1} 128\frac{n^2\pi^2 - 6}{n^3\pi^3} \sin\left(\frac{n\pi x}{4}\right).$$

conclusions:

If f is even on $[-L, L]$, then its Fourier series on this interval is

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right),$$

in which

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx \quad \text{for} \quad n = 0, 1, 2, \ldots.$$

If f is odd on $[-L, L]$, then its Fourier series on this interval is

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad \text{for} \quad n = 1, 2, \ldots.$$

The Trigonometric Fourier series

If $x(t)$ is a ***periodic signal*** with fundamental period (T_o), then we can expand it as follows:

$$x(t) = a_o + a_1\cos(\omega_o t) + a_2\cos(2\omega_o t) + \cdots + b_1\sin(\omega_o t) + b_2\sin(2\omega_o t) + \cdots$$

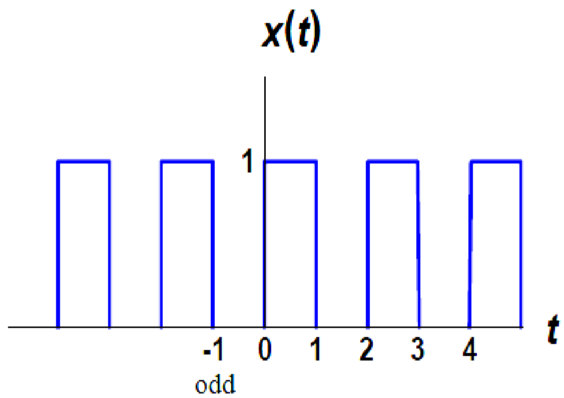
$$= a_o + \sum_{n=1}^{\infty}\left[a_n\cos(n\omega_o t) + b_n\sin(n\omega_o t)\right]$$

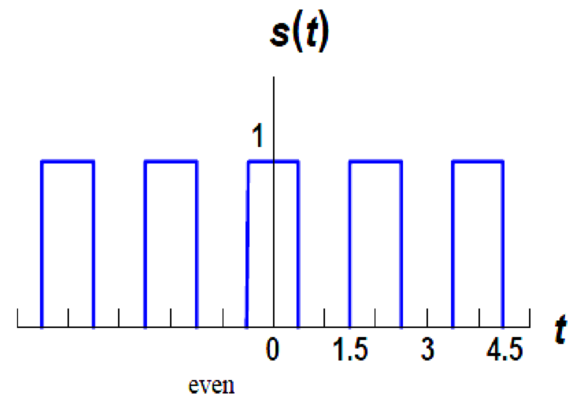
where $\omega_o = 2\pi f_o = 2\pi / T_o$,

$$a_o = \frac{1}{T_o}\int_0^{T_o} x(t)dt$$ (**the constant term**)

$$a_n = \frac{2}{T_o}\int_0^{T_o} x(t)\cos(n\omega_o t)dt, \quad b_n = \frac{2}{T_o}\int_0^{T_o} x(t)\sin(n\omega_o t)dt.$$

Odd functions have **no cosines**; ***even*** functions have ***no sines***.





$$x(t) = \frac{1}{2} + \frac{2}{\pi}[\sin(\omega_o t) + \frac{1}{3}\sin(3\omega_o t) + \frac{1}{5}\sin(5\omega_o t) + \ldots],$$

$$s(t) = \frac{1}{2} + \frac{2}{\pi}[\cos(\omega_o t) - \frac{1}{3}\cos(3\omega_o t) + \frac{1}{5}\cos(5\omega_o t) + \ldots],$$

$$\omega_o = \frac{2\pi}{T_o} = \pi.$$

The Exponential (Complex) Fourier series:

Using **Euler** formula, we can write the trigonometric Fourier series in the exponential form:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \, e^{j\frac{2\pi n x}{L}}$$

where c_n is given by:

$$c_n = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\, e^{-j\frac{2\pi n x}{L}}\, dx$$

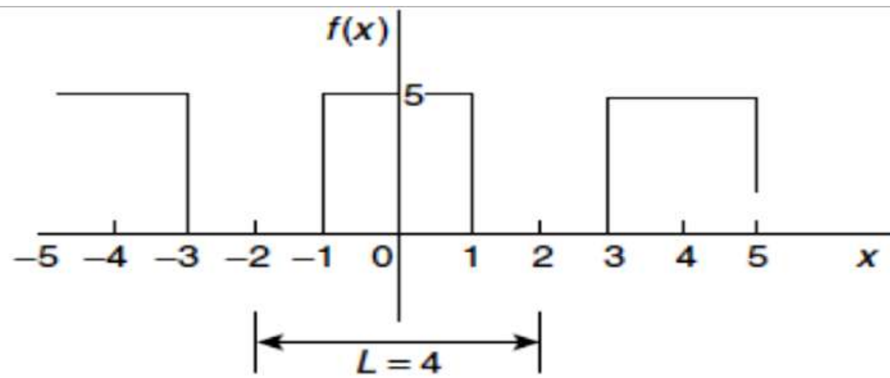
Relationship between Complex and Trigonometric Series Coefficients:

$$C_o = a_o, \quad C_n = \frac{1}{2}(a_n - jb_n), \quad \text{and} \quad C_{-n} = \frac{1}{2}(a_n + jb_n).$$

Problem 1. Determine the complex Fourier series for the function defined by:

$$f(x) = \begin{cases} 0, & \text{when } -2 \le x \le -1 \\ 5, & \text{when } -1 \le x \le 1 \\ 0, & \text{when } \quad 1 \le x \le 2 \end{cases}$$

The function is periodic outside this range of period 4.



$$f(x) = \sum_{n=-\infty}^{\infty} c_n \, e^{j\frac{2\pi nx}{L}}$$

where c_n is given by:

$$c_n = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\, e^{-j\frac{2\pi nx}{L}}\, dx$$

when $L = 4$.

$$c_n = \frac{1}{4}\left\{\int_{-2}^{-1} 0\,\mathrm{d}x + \int_{-1}^{1} 5\,\mathrm{e}^{-j\frac{2\pi nx}{4}}\,\mathrm{d}x + \int_{1}^{2} 0\,\mathrm{d}x\right\}$$

$$= \frac{1}{4}\int_{-1}^{1} 5\,\mathrm{e}^{-\frac{j\pi nx}{2}}\,\mathrm{d}x = \frac{5}{4}\left[\frac{\mathrm{e}^{-\frac{j\pi nx}{2}}}{-\frac{j\pi n}{2}}\right]_{-1}^{1}$$

$$= \frac{-5}{j2\pi n}\left[\mathrm{e}^{-\frac{j\pi nx}{2}}\right]_{-1}^{1} = \frac{-5}{j2\pi n}\left(\mathrm{e}^{-\frac{j\pi n}{2}} - \mathrm{e}^{\frac{j\pi n}{2}}\right)$$

$$= \frac{5}{\pi n}\left(\frac{\mathrm{e}^{j\frac{\pi n}{2}} - \mathrm{e}^{-j\frac{\pi n}{2}}}{2j}\right)$$

$$= \frac{5}{\pi n}\sin\frac{\pi n}{2}$$

Fourier series is given by:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n\,\mathrm{e}^{j\frac{2\pi nx}{L}} = \sum_{n=-\infty}^{\infty} \frac{5}{\pi n}\sin\frac{\pi n}{2}\;\mathrm{e}^{j\frac{\pi nx}{2}}$$

$$c_0 = a_0 = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\mathrm{d}x = \frac{1}{4}\int_{-1}^{1} 5\,\mathrm{d}x$$

$$= \frac{5}{4}[x]_{-1}^{1} = \frac{5}{4}[1-(-1)] = \frac{5}{2}$$

Since $c_n = \frac{5}{\pi n} \sin \frac{\pi n}{2}$, then

$$c_1 = \frac{5}{\pi} \sin \frac{\pi}{2} = \frac{5}{\pi}$$

$$c_2 = \frac{5}{2\pi} \sin \pi = 0$$

(in fact, all even terms will be zero since $\sin n\pi = 0$)

$$c_3 = \frac{5}{\pi n} \sin \frac{\pi n}{2} = \frac{5}{3\pi} \sin \frac{3\pi}{2} = -\frac{5}{3\pi}$$

By similar substitution,

$$c_5 = \frac{5}{5\pi} \qquad c_7 = -\frac{5}{7\pi}, \text{ and so on.}$$

Similarly,

$$c_{-1} = \frac{5}{-\pi} \sin \frac{-\pi}{2} = \frac{5}{\pi}$$

$$c_{-2} = -\frac{5}{2\pi} \sin \frac{-2\pi}{2} = 0 = c_{-4} = c_{-6}, \text{ and so on.}$$

$$c_{-3} = -\frac{5}{3\pi} \sin \frac{-3\pi}{2} = -\frac{5}{3\pi}$$

$$c_{-5} = -\frac{5}{5\pi} \sin \frac{-5\pi}{2} = \frac{5}{5\pi}, \text{ and so on.}$$

$$
\begin{aligned}
f(x) = \frac{5}{2} + \frac{5}{\pi}\,\mathrm{e}^{j\frac{\pi x}{2}} - \frac{5}{3\pi}\,\mathrm{e}^{j\frac{3\pi x}{2}} + \frac{5}{5\pi}\,\mathrm{e}^{j\frac{5\pi x}{2}} \\
- \frac{5}{7\pi}\,\mathrm{e}^{j\frac{7\pi x}{2}} + \cdots + \frac{5}{\pi}\,\mathrm{e}^{-j\frac{\pi x}{2}} \\
- \frac{5}{3\pi}\,\mathrm{e}^{-j\frac{3\pi x}{2}} + \frac{5}{5\pi}\,\mathrm{e}^{-j\frac{5\pi x}{2}} \\
- \frac{5}{7\pi}\,\mathrm{e}^{-j\frac{7\pi x}{2}} + \cdots \\
= \frac{5}{2} + \frac{5}{\pi}\left(\mathrm{e}^{j\frac{\pi x}{2}} + \mathrm{e}^{-j\frac{\pi x}{2}}\right) \\
- \frac{5}{3\pi}\left(\mathrm{e}^{j\frac{3\pi x}{2}} + \mathrm{e}^{-j\frac{3\pi x}{2}}\right) \\
+ \frac{5}{5\pi}\left(\mathrm{e}^{\frac{5\pi x}{2}} + \mathrm{e}^{-j\frac{5\pi x}{2}}\right) - \cdots
\end{aligned}
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