متسلسلة فورييه

تفكيك دالة دورية إلى مجموع الموجات الجيبية البسيطة

1- Fourier Concept

2- Fourier series , and Fourier coefficient

3- Fourier transform

4- Inverse Fourier transform

5- Fast Fourier Transform (FFT)

1- Fourier Concept

في الرياضيات، متسلسلة فورييه (بالإنجليزية: Fourier series) هي طريقة تتيح كتابة أي دالة رياضية دورية في شكل متسلسلة أو مجموع من دوال الجيب وجيب التمام مضروب بمعامل معين

يعزى اسمها إلى العالم الفرنسي جوزيف فورييه تقديرا لأعماله الفذة في المتسلسلات المثلثية.

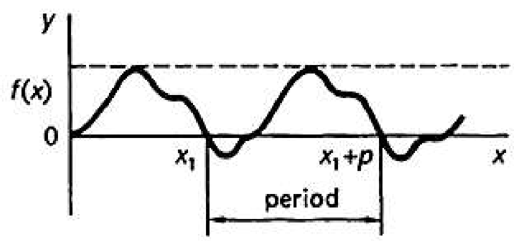
سميت هذه المتسلسلات هكذا نسبة إلى العالم الفرنسي جوزيف فورييه (١٧٦٨ء١٨٣٠)، الذي حقق تطورات مهمة في دراسة المتسلسلات المثلثية، بعد أن تعرض لبحث بدائي من طرف كل من ليونهارت أويلر ولورن دالمبير ودانييل برنولي. أبدع فورييه هذه المعادلات من أجل حلحلة معادلة الحرارة في صفيحة معدنية، ناشرا نتائجه الأولى في عام ١٨٠٧ ، في عمل له

Fourier series

A Fourier series is a representation of a function as a series of constants times sine and/or cosine functions of different frequencies.

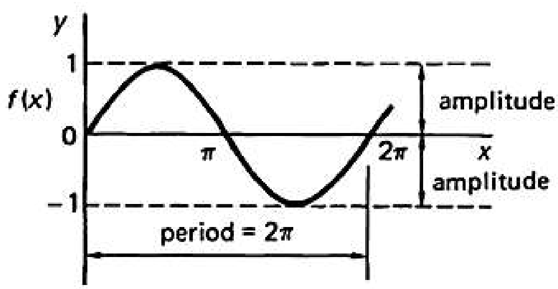
Periodic functions

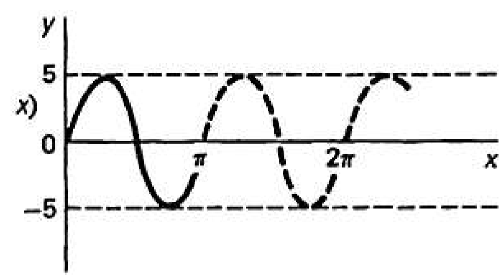
A function $f(x)$ is said to be *periodic* if its function values repeat at regular intervals of the independent variable. The regular interval between repetitions is the *period* of the oscillations.



$y = \sin x$

$y = 5\sin 2x$





Harmonics

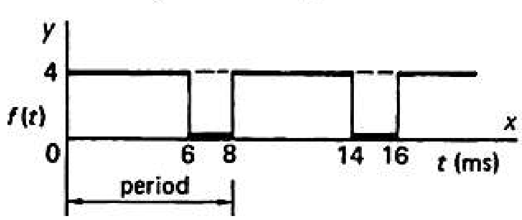
A function $f(x)$ is sometimes expressed as a series of a number of different sine components. The component with the largest period is the *first harmonic*, or *fundamental* of $f(x)$.

$y = A_1 \sin x$ is the first harmonic or fundamental

$y = A_2 \sin 2x$ is the second harmonic

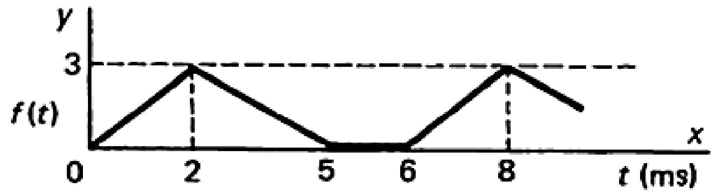
$y = A_3 \sin 3x$ is the third harmonic, etc.

Non-sinusoidal periodic functions

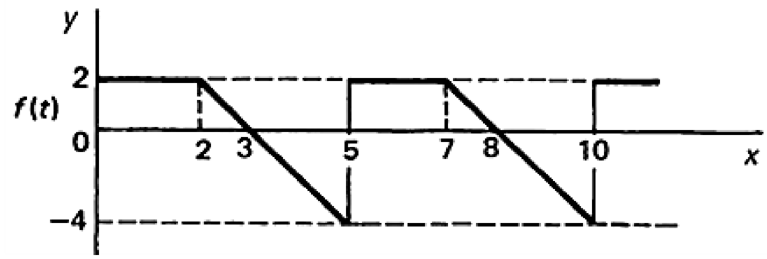


period = 8 ms

$$f(x)=\begin{cases} 4 & 0 \le x < 6 \\ 0 & 6 < x < 8 \end{cases}$$

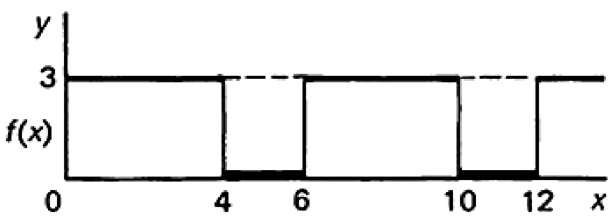


period = 6 ms;



period = 5 ms

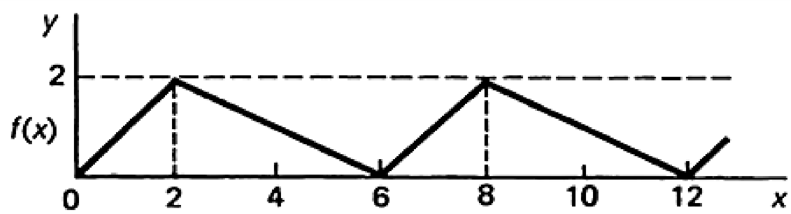
Analytic description of a periodic function



(a) Between $x = 0$ and $x = 4$, $y = 3$, i.e. $f(x) = 3$ $\quad 0 < x < 4$

(b) Between $x = 4$ and $x = 6$, $y = 0$, i.e. $f(x) = 0$ $\quad 4 < x < 6$

$$f(x) = \begin{cases} 3 & 0 < x < 4 \\ 0 & 4 < x < 6 \end{cases}$$

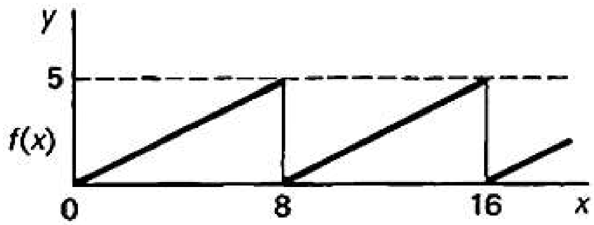


(a) Between $x = 0$ and $x = 2$, $y = x$ $\quad$ i.e. $f(x) = x$ $\quad 0 < x < 2$

(b) Between $x = 2$ and $x = 6$, $y = -\frac{x}{2} + 3$, i.e. $f(x) = 3 - \frac{x}{2}$ $\quad 2 < x < 6$

$$f(x) = \begin{cases} x & 0 < x < 2 \\ 3 - \frac{x}{2} & 2 < x < 6 \end{cases}$$

$f(x + 6) = f(x)$.



$$f(x) = \frac{5x}{8} \qquad 0 < x < 8$$
$$f(x+8) = f(x)$$

Integrals of periodic functions

The integrals are those of sines, cosines and their combinations where the integration is over a single period from $-\pi$ to π

$$\int_{-\pi}^{\pi} \mathrm{d}x = \left[x\right]_{-\pi}^{\pi} = 2\pi$$

$$\int_{-\pi}^{\pi} \cos nx \, \mathrm{d}x = \left[\frac{\sin nx}{n}\right]_{-\pi}^{\pi} \qquad (n \neq 0)$$
$$= \frac{\sin n\pi}{n} - \frac{\sin(-n\pi)}{n}$$
$$= 0 \quad \text{because } \sin n\pi = 0$$

sin(-x)= -sin(x)

$$\int_{-\pi}^{\pi} \sin nx \, \mathrm{d}x = 0$$

$$\int_{-\pi}^{\pi} \sin nx \, \mathrm{d}x = \left[-\frac{\cos nx}{n}\right]_{-\pi}^{\pi} \qquad (n \neq 0)$$
$$= -\frac{\cos n\pi}{n} + \frac{\cos(-n\pi)}{n}$$
$$= 0 \quad \text{because } \cos(-x) = \cos x$$

$$\int_{-\pi}^{\pi} \cos^2 nx \, \mathrm{d}x = \int_{-\pi}^{\pi} \frac{\cos 2nx + 1}{2} \, \mathrm{d}x \quad \text{because } \cos 2A = 2\cos^2 A - 1$$
$$= \left[\frac{\sin 2nx}{4n} + \frac{x}{2}\right]_{-\pi}^{\pi} \qquad (n \neq 0)$$
$$= \frac{\sin 2n\pi}{4n} + \frac{\pi}{2} - \frac{\sin(-2n\pi)}{4n} - \frac{(-\pi)}{2}$$
$$= \pi$$

$$\int_{-\pi}^{\pi} \sin^2 nx\,dx = \int_{-\pi}^{\pi} \frac{1-\cos 2nx}{2}\,dx \quad \text{because } \cos 2A = 1 - 2\sin^2 A$$
$$= \left[\frac{x}{2} - \frac{\sin 2nx}{4n}\right]_{-\pi}^{\pi} \qquad (n \neq 0)$$
$$= \frac{\pi}{2} - \frac{\sin 2n\pi}{4n} - \frac{(-\pi)}{2} + \frac{\sin(-2n\pi)}{4n}$$
$$= \pi$$

Orthogonal functions

If two different functions $f(x)$ and $g(x)$ are defined on the interval $a \leq x \leq b$ and

$$\int_a^b f(x)g(x)\,dx = 0$$

the two functions are **orthogonal**

$$\int_{-\pi}^{\pi} \cos mx \cos nx\,dx = 0 \quad \text{for } m \neq n$$
$$\int_{-\pi}^{\pi} \sin mx \sin nx\,dx = 0 \quad \text{for } m \neq n$$

and

$$\int_{-\pi}^{\pi} \cos mx \sin nx\,dx = 0$$

Given that certain conditions are satisfied then it is possible to write a periodic function of period 2π as a series expansion of the orthogonal periodic functions

if $f(x)$ is defined on the

interval $-\pi \leq x \leq \pi$ where $f(x + 2n\pi) = f(x)$ then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

a_n and b_n are constants called the *Fourier coefficients*. ***or Euler Coefficients***

To derive the Fourier series equation, Let *f(x)* periodic function, have interval time α_n, $f(x) = c_n \sin(x+\alpha_n)$, *where* $n = 1,2,3, \ldots. n$. we can obtain;

$$f(x)=c_0+c_1\sin(x+\alpha_1)+c_2\sin(2x+\alpha_2)+\cdots$$
$$+c_n\sin(nx+\alpha_n)+\cdots$$

But,

$$\boxed{\sin(a+b)=\sin(a)\cos(b)+\cos(a)\sin(b)}$$

i.e,

$$f(x)=c_0+c_1(\sin(x)\cos(\alpha_1)+\cos(x)\sin(\alpha_1))$$
$$+c_2(\sin(2x)\cos(\alpha_2)+\cos(2x)\sin(\alpha_2))+\ldots$$
$$+c_n(\sin(nx)\cos(\alpha_n)+\cos(nx)\sin(\alpha_n))+\cdots$$

Or,

$$f(x)=c_0+\{(c_1\sin(\alpha_1))\cos(x)+(c_2\sin(\alpha_2))\cos(2x)$$
$$+\ldots+(c_n\sin(\alpha_n))\cos(nx)+\cdots\}$$

$$+\{(c_1\cos(\alpha_1))\sin(x)+(c_2\cos(\alpha_2))\sin(2x)+\cdots$$
$$+(c_n\cos(\alpha_n))\sin(nx)+\cdots\}$$

Let,

$$a_0=c_0,\ a_1=c_1\sin(\alpha_1),\ \cdots,a_n=c_n\sin(\alpha_n)$$
$$b_1=c_1\cos(\alpha_1),\ b_2=c_2\cos(\alpha_2),\ \cdots,b_n=c_n\cos(\alpha_n)$$

So that,

$$f(x) = a_0 + a_1\cos(x) + a_2\cos(2x) + a_3\cos(3x) + \cdots$$
$$+a_n\cos(nx) + \cdots + b_1\sin(x) + b_2\sin(2x)$$
$$+b_3\sin(3x) + \cdots + b_n\sin(nx) + \cdots$$

Finally the periodic equation is,

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos(nx) + b_n\sin(nx)\right)$$

Which represent the Fourier series, Where,

if a_0 a_n and b_n

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx, \quad n \geq 1$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx; \quad n \geq 1$$

$$f(x) \cong \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(nx) + b_n\sin(nx)\right)$$

if a_0 a_n and b_n:

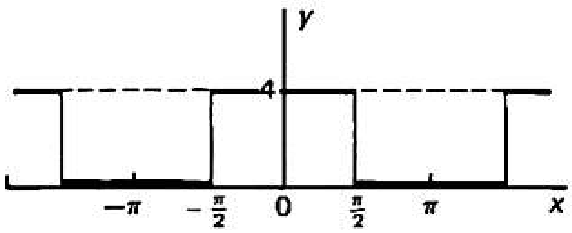
$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx, \quad n \ge 1$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx; \quad n \ge 1$$

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos(nx) + b_n \sin(nx)\right)$$

Example



Find the Fourier series for the function shown.

Consider one cycle between $x = -\pi$ and $x = \pi$.

The function can be defined by $f(x) = \begin{cases} 0 & -\pi < x < -\dfrac{\pi}{2} \\ 4 & -\dfrac{\pi}{2} < x < \dfrac{\pi}{2} \\ 0 & \dfrac{\pi}{2} < x < \pi \end{cases}$

$$f(x + 2\pi) = f(x).$$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n \cos nx + b_n \sin nx\}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x$$

$$a_0 = \frac{1}{\pi}\left\{\int_{-\pi}^{-\pi/2} 0\,\mathrm{d}x + \int_{-\pi/2}^{\pi/2} 4\,\mathrm{d}x + \int_{\pi/2}^{\pi} 0\,\mathrm{d}x\right\}$$

$$= \frac{1}{\pi}\Big[4x\Big]_{-\pi/2}^{\pi/2} \qquad \therefore\ a_0 = 4$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x$$

$$a_n = \frac{1}{\pi}\left\{\int_{-\pi}^{-\pi/2} (0)\cos nx\,\mathrm{d}x + \int_{-\pi/2}^{\pi/2} 4\cos nx\,\mathrm{d}x + \int_{\pi/2}^{\pi} (0)\cos nx\,\mathrm{d}x\right\}$$

$$a_n = \frac{8}{\pi n}\sin\frac{n\pi}{2}$$

If n is even $\qquad a_n = 0$

If $n = 1, 5, 9, \ldots \qquad a_n = \dfrac{8}{n\pi}$

If $n = 3, 7, 11, \ldots \qquad a_n = -\dfrac{8}{n\pi}$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

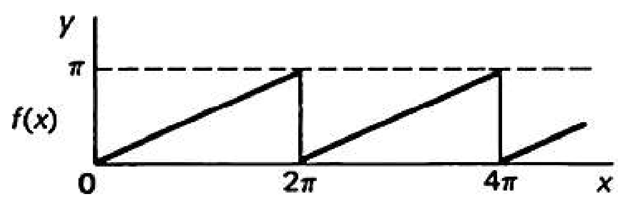
$$b_n = \frac{1}{\pi}\left\{\int_{-\pi}^{-\pi/2}(0)\sin nx\,dx + \int_{-\pi/2}^{\pi/2} 4\sin nx\,dx + \int_{\pi/2}^{\pi}(0)\sin nx\,dx\right\}$$

$$= \frac{4}{\pi}\int_{-\pi/2}^{\pi/2}\sin nx\,dx = \frac{4}{\pi}\left[\frac{-\cos nx}{n}\right]_{-\pi/2}^{\pi/2}$$

$$= -\frac{4}{n\pi}\left\{\cos\frac{n\pi}{2} - \cos\left(\frac{-n\pi}{2}\right)\right\} = 0 \qquad \therefore\ b_n = 0$$

$$f(x) = 2 + \frac{8}{\pi}\left\{\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \frac{1}{7}\cos 7x + \ldots\right\}$$

Example



Determine the Fourier series to represent the periodic function shown.

$$f(x) = \frac{x}{2} \qquad 0 < x < 2\pi \qquad f(x + 2\pi) = f(x) \quad \text{i.e. period} = 2\pi.$$

$$\text{(a)}\ \ a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{x}{2}\right)dx = \frac{1}{4\pi}\left[x^2\right]_0^{2\pi}$$

$$= \pi \qquad \therefore\ a_0 = \pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{x}{2}\right)\cos nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\cos nx\,dx$$

integrating by parts $\int u\,dv = uv - \int v\,du$ (Integration by parts)

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} x\cos nx\,\mathrm{d}x = \frac{1}{2\pi}\left\{\left[\frac{x\sin nx}{n}\right]_0^{2\pi} - \frac{1}{n}\int_0^{2\pi}\sin nx\,\mathrm{d}x\right\}$$

$$= \frac{1}{2\pi}\left\{(0-0) - \frac{1}{n}(0)\right\} = 0 \qquad \therefore\ a_n = 0$$

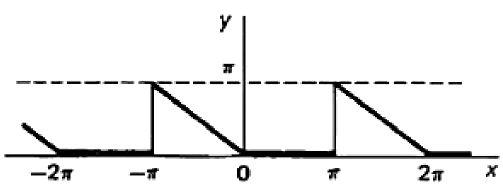
(c) $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x$

$$b_n = -\frac{1}{n}$$

$$f(x) = \frac{\pi}{2} - \left\{\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \dots\right\}$$

Example

Find the Fourier series for the function defined by



$$f(x) = -x \qquad -\pi < x < 0$$
$$f(x) = 0 \qquad 0 < x < \pi$$
$$f(x + 2\pi) = f(x).$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x$$

$$a_0 = \frac{\pi}{2}$$

(a) $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \frac{1}{\pi}\int_{-\pi}^{0}(-x)\,\mathrm{d}x = \frac{1}{\pi}\left[-\frac{x^2}{2}\right]_{-\pi}^{0} = \frac{\pi}{2}$

(b) To find a_n

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x \qquad a_n = -\frac{2}{\pi n^2}\quad (n \text{ odd});\quad 0 \quad (n \text{ even})$$

Because

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x = \frac{1}{\pi}\int_{-\pi}^{0}(-x)\cos nx\,\mathrm{d}x$$

$$= -\frac{1}{\pi}\int_{-\pi}^{0} x\cos nx\,\mathrm{d}x$$

$$= -\frac{1}{\pi}\left\{\left[x\frac{\sin nx}{n}\right]_{-\pi}^{0} - \frac{1}{n}\int_{-\pi}^{0}\sin nx\,\mathrm{d}x\right\}$$

$$= -\frac{1}{\pi}\left\{(0-0) - \frac{1}{n}\left[\frac{-\cos nx}{n}\right]_{-\pi}^{0}\right\} = -\frac{1}{\pi n^2}\{1-\cos n\pi\}$$

But $\cos n\pi = 1$ (n even) or -1 (n odd)

$$\therefore\ a_n = -\frac{2}{\pi n^2} \quad (n \text{ odd}) \quad \text{or } 0 \quad (n \text{ even})$$

(c) b_n $\qquad b_n = -\frac{1}{n}$ (n even) or $\frac{1}{n}$ (n odd)

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0}(-x)\sin nx\,dx$$

$$= -\frac{1}{\pi}\int_{-\pi}^{0} x\sin nx\,dx$$

$$= -\frac{1}{\pi}\left\{\left[x\left(\frac{-\cos nx}{n}\right)\right]_{-\pi}^{0} + \frac{1}{n}\int_{-\pi}^{0}\cos nx\,dx\right\}$$

$$= -\frac{1}{\pi}\left\{\frac{\pi\cos n\pi}{n} + \frac{1}{n}\left[\frac{\sin nx}{n}\right]_{-\pi}^{0}\right\} = -\frac{\cos n\pi}{n}$$

$$\therefore\ b_n = -\frac{1}{n} \quad (n \text{ even});\quad \frac{1}{n} \quad (n \text{ odd})$$

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \dots\right)$$
$$+\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \dots\right)$$

Example

Let $f(x) = x$ for $-\pi \le x \le \pi$. We will write the Fourier series of f on $[-\pi, \pi]$. The coefficients are:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx = 0,$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(nx)\,dx$$

$$= \left[\frac{1}{n^2\pi}\cos(nx) + \frac{x}{n\pi}\sin(nx)\right]_{-\pi}^{\pi} = 0,$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx$$

$$= \left[\frac{1}{n^2\pi}\sin(nx) - \frac{x}{n\pi}\cos(nx)\right]_{-\pi}^{\pi}$$

$$= -\frac{2}{n}\cos(n\pi) = \frac{2}{n}(-1)^{n+1},$$

since $\cos(n\pi) = (-1)^n$ if n is an integer. The Fourier series of x on $[-\pi, \pi]$ is

$$\sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin(nx) = 2\sin(x) - \sin(2x) + \frac{2}{3}\sin(3x) - \frac{1}{2}\sin(4x) + \frac{2}{5}\sin(5x) - \cdots.$$

In this example the constant term and cosine coefficients are all zero, and the Fourier series contains only sine terms. □

Example:

Find the Fourier Series for the function defined by:

$$\boldsymbol{f(x) = |x| \,;\; -\pi \le x \le \pi}$$

Solution $\quad |x| = \begin{cases} -x & ; x<0 \\ x & ; x\ge 0 \end{cases}$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\left(\int_{-\pi}^{0} -x dx + \int_{0}^{\pi} x dx\right) = \frac{\pi}{2} \;;$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\cos(nx)dx$$

$$= \frac{1}{\pi}\left(\int_{-\pi}^{0} -x\cos(nx)dx + \int_{0}^{\pi} x\cos(nx)dx\right) = \frac{2(\cos(n\pi)-1)}{\pi n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = 0$$

وبالتالي فإن متسلسلة فورير للدالة المعطاة هي

$$\boxed{\frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{\pi n^2}(\cos(n\pi)-1)\cos(nx) \quad \forall\, x}$$

EXAMPLE :

Find the Fourier Series for the function defined :

$$f(x)=\begin{cases}0 & ;\ -\pi \le x < 0\\ \sin(x) & ;\ 0\le x\le \pi\end{cases}$$

Solution

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\left(\int_{-\pi}^{0} 0dx+\int_{0}^{\pi}\sin(x)dx\right)=\frac{1}{\pi}\ ;$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx=\frac{1}{\pi}\int_{0}^{\pi}\sin(x)\cos(nx)dx$$

$$=-\frac{\cos[(1-n)x]}{2\pi(1-n)}-\frac{\cos[(1+n)x]}{2\pi(1+n)}\bigg|_0^{\pi}=\frac{1+\cos(n\pi)}{\pi\left(1-n^2\right)}\ ;n\neq 1$$

في حالة $n=1$ نستخدم

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx=\frac{1}{\pi}\int_{0}^{\pi}\sin(x)\sin(nx)dx$$

$$=\frac{\sin[(1-n)x]}{2\pi(1-n)}-\frac{\sin[(1+n)x]}{2\pi(1+n)}\bigg|_0^{\pi}=0\ ;\ n\neq 1$$

$$a_1=\frac{1}{\pi}\int_{0}^{\pi}\sin(x)\cos(x)dx=0;$$

في حالة $n=1$ نستخدم

$$b_1=\frac{1}{\pi}\int_{0}^{\pi}\sin^2(x)dx=\frac{1}{2\pi}\int_{0}^{\pi}(1-\cos(2x))dx=\frac{1}{2}$$

Fourier series is :

$$\boxed{\frac{1}{\pi}+\frac{1}{2}\sin(x)+\frac{1}{\pi}\sum_{n=2}^{\infty}\frac{1+\cos(n\pi)}{1-n^2}\cos(nx)}$$