Z- Transform

The Z-transform can be defined as either a *one-sided* or *two-sided* transform.

Example: right-sided exponential sequence

Consider the signal $x[n] = a^n u[n]$. This has the z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n.$$

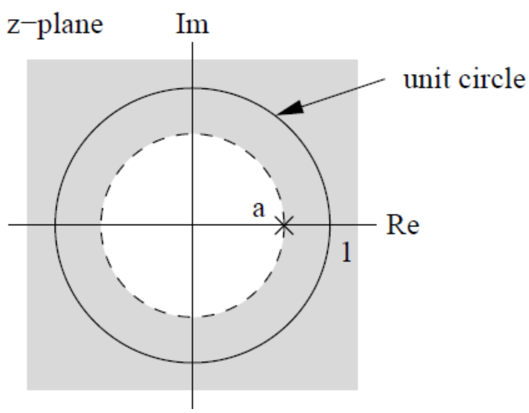
Convergence requires that

$$\sum_{n=0}^{\infty} |az^{-1}|^n < \infty,$$

Which is only the case if $|az^{-1}| < 1$, or equivalently $|z| > |a|$. In the ROC, the series converges to

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \qquad |z| > |a|,$$

Since it is just a geometric series, the z-transform has a region of convergence for any finite value of a.



The Fourier transform of x[n] only exists if the ROC includes the unit circle, which requires that $|a| < 1$. On the other hand, if $|a| > 1$ then the ROC does not include the unit circle and the Fourier transform does not exist. This is consistent with the fact that for these values of *(a) the* sequence $a^n u[n]$ is exponentially growing, and the sum therefore does not converge.

Example1 : left-sided exponential sequence

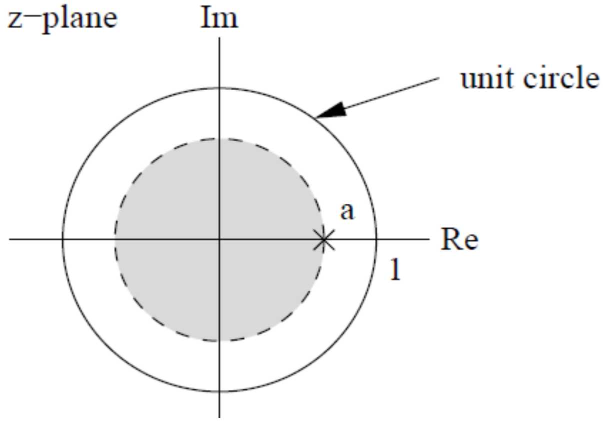
Now consider the sequence $x[n] = -a^n u[-n-1]$. This sequence is left-sided because it is nonzero only for $n \le -1$. The z - transform is

$$X(z) = \sum_{n=-\infty}^{\infty} -a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n.$$

For $|a^{-1}z| < 1$, or $|z| < |a|$, the series converges to

$$X(z) = 1 - \frac{1}{1-a^{-1}z} = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \qquad |z| < |a|.$$



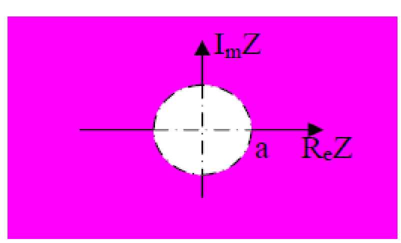
Note that the expression for the z-transform (and the pole zero plot) is exactly the same as for the right-handed exponential sequence—*only the region of convergence is different*. Specifying the ROC is therefore critical when dealing with the z-transform.

Example(2): Find Z.T including region of convergence of $x(n) = a^n u(n)$

$$X(Z) = \sum_{n=0}^{\infty} a^n Z^{-n} = \sum_{n=0}^{\infty} (a Z^{-1})^n = \frac{1}{1-aZ^{-1}} = \frac{Z}{Z-a} \quad , \quad |a Z^{-1}| \langle 1$$

Or $|Z| > |a|$

The region of convergence (ROC) is <u>outside</u> the unit circle only.



Example: sum of two exponentials

The signal $x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$ is the sum of two real exponentials. The z-transform is

$$X(z) = \sum_{n=-\infty}^{\infty} \left\{ \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n] \right\} z^{-n}$$

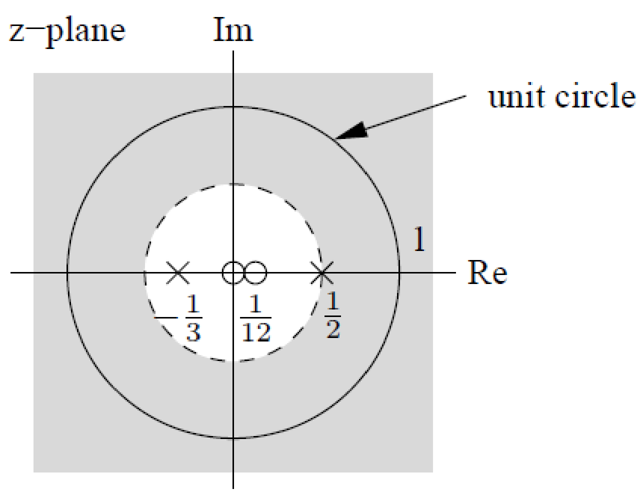
$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u[n] z^{-n} + \sum_{n=-\infty}^{\infty} \left(-\frac{1}{3}\right)^n u[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n + \sum_{n=0}^{\infty} \left(-\frac{1}{3} z^{-1}\right)^n .$$

From the example for the right-handed exponential sequence, the first term in this sum converges for $|z| > 1/2$, and the second for $|z| > 1/3$. The combined transform X(z) therefore converges in the intersection of these regions, namely when $|z| > 1/2$. In this case

$$X(z) = \frac{1}{1 - \frac{1}{2} z^{-1}} + \frac{1}{1 + \frac{1}{3} z^{-1}} = \frac{2z(z - \frac{1}{12})}{(z - \frac{1}{2})(z + \frac{1}{3})}.$$

The pole-zero plot and region of convergence of the signal is



Power series expansion

If the z-transform is given as a power series in the form

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$= \ldots + x[-2]z^{2} + x[-1]z^{1} + x[0] + x[1]z^{-1} + x[2]z^{-2} + \ldots,$$

Then any value in the sequence can be found by identifying the coefficient of the appropriate power of z^{-1}.

Example: finite-length sequence

The z-transform

$$X(z) = z^{2}(1 - \frac{1}{2}z^{-1})(1 + z^{-1})(1 - z^{-1})$$

can be multiplied out to give

$$X(z) = z^{2} - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}.$$

By inspection, the corresponding sequence is therefore

$$x[n] = \begin{cases} 1 & n = -2 \\ -\frac{1}{2} & n = -1 \\ -1 & n = 0 \\ \frac{1}{2} & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

or equivalently

$$x[n] = 1\delta[n+2] - \frac{1}{2}\delta[n+1] - 1\delta[n] + \frac{1}{2}\delta[n-1].$$

Example: power series expansion by long division

Consider the transform

$$X(z) = \frac{1}{1 - az^{-1}}, \qquad |z| > |a|.$$

Since the ROC is the exterior of a circle, the sequence is right-sided. We therefore *divide to get a power series in powers of z^{-1}:*

$$\begin{array}{r@{\,}l}
 & 1 + az^{-1} + a^2z^{-2} + \cdots \\ \cline{2-2}
1 - az^{-1} \big) & 1 \\
 & 1 - az^{-1} \\ \cline{2-2}
 & az^{-1} \\
 & az^{-1} - a^2z^{-2} \\ \cline{2-2}
 & a^2z^{-2} + \cdots
\end{array}$$

or

$$\frac{1}{1 - az^{-1}} = 1 + az^{-1} + a^2z^{-2} + \cdots .$$

Therefore $x[n] = a^n u[n]$.

Example: power series expansion for left-sided sequence

Consider instead the z-transform

$$X(z) = \frac{1}{1 - az^{-1}}, \qquad |z| < |a|.$$

Because of the ROC, the sequence is now a left-sided one. Thus we *divide to obtain a series in powers of z:*

$$X(z) = \frac{z}{z - a} = \frac{z}{-a + z}$$

$$\begin{array}{r@{\,}l}
 & -a^{-1}z - a^{-2}z^2 - \cdots \\ \cline{2-2}
-a + z \big) & z \\
 & z - a^{-1}z^2 \\ \cline{2-2}
 & az^{-1}
\end{array}$$

Thus $x[n] = -a^n u[-n-1]$.