Z- Transform

The Z-transform can be defined as either a *one-sided* or *two-sided* transform.

The Laplace transform and its discrete time counterpart the Z-transform are essential mathematical tools for system design and analysis, and for monitoring the stability of a system. In this section we derive the Z- transform from the Laplace transform a discrete time signal.

The Fourier transform for Continuous time a periodic signal is:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

And the inverse F.T is

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(\omega)e^{-j\omega t}\,d\omega$$

Where the complex variables s = σ + j ω, so that **ds = jdω**, so that the Fourier transform equation and its inverse F. T. can be written as:

$$X(s) = \int_{-0}^{\infty} x(t)e^{-st}\,dt \qquad (1)$$

And the inverse is

$$x(t) = \frac{1}{2\pi}\int_{\sigma 1-j\infty}^{\sigma 1+j\infty} x(s)e^{-st}\,ds \qquad (2)$$

Where σ1 is selected so that *X*(s) is analytic (no singularities) for s > σ1.

The Z-transform can be derived from equation 1 by sampling the continuous time input signal x (t). For sampled signal x (m Ts), normally denoted as x(m) assuming the sampling periods Ts =1, the Laplace transform equation 1 becomes :

$$X(e^{s}) = \sum_{m=0}^{\infty} x(m)e^{-sm} \qquad (3)$$

Substituting the variable (e^s) in equation 3 with the variable Z, we obtain the one-sided Z transform equation:

$$X(z) = \sum_{m=0}^{\infty} x(m)\, z^{-m} \qquad (4)$$

Where $\mathbf{z = e^s = e^{\sigma+j\omega}}$

The two-sided Z-transform is defined as;

$$X(z) = \sum_{m=-\infty}^{\infty} x(m)\, z^{-m} \qquad (5)$$

Note that for a one-sided signal, x (m) = 0 for m < 0, equation4 and 5 are equivalent.

The z-transform is a very important tool in describing and analyzing digital systems. It also offers the techniques for digital filter design and frequency analysis of digital signals. The z-transform of a causal sequence x (n), designated by X(z) or Z(x(n)), is defined as:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}.$$

$$X(z) = Z(x(n)) = \sum_{n=0}^{\infty} x(n) z^{-n} = x(0)z^{-0} + x(1)z^{-1} + x(2)z^{-2} + \ldots$$

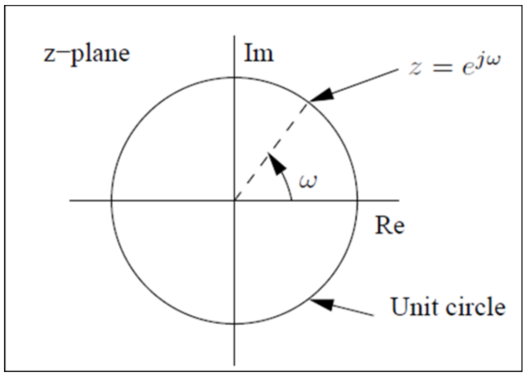
The z-transform can also be thought of as an operator Z{·} that transforms a sequence to a function:

$$\mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = X(z).$$

In both cases z is a continuous complex variable. We may obtain the Fourier transform from the z-transform by making the substitution $z = e^{j\omega}$. This corresponds to restricting $|z| = 1$. Also, with $z = re^{j\omega}$,

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n](re^{j\omega})^{-n} = \sum_{n=-\infty}^{\infty} \left(x[n]r^{-n}\right) e^{-j\omega n}.$$

That is, the z-transform is the Fourier transform of the sequence x[*n*] r^{-n}. For r = 1 this becomes the Fourier transform of *x*[*n*]. The Fourier transform therefore corresponds to the z-transform evaluated on the unit circle:

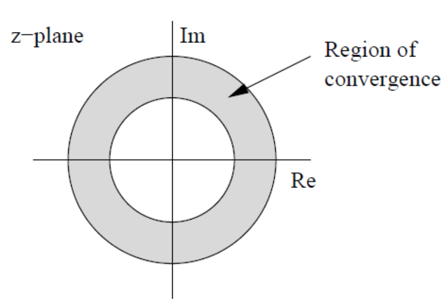


The inherent periodicity in frequency of the Fourier transform is captured naturally under this interpretation. The Fourier transform does not converge for all sequences the infinite sum may not always be finite. Similarly, the z-transform does not converge for all sequences or for all values of z. The set of values of z for which the z-transform converges is called the **region of convergence (ROC)**.

The Fourier transform of x[n] exists if the sum $\sum_{n=-\infty}^{\infty}|x[n]|$ converges.
However, the z-transform of *x*[*n*] is just the Fourier transform of the sequence x[*n*] r^{-n}. The z-transform therefore exists (or converges) if

$$\sum_{n=-\infty}^{\infty} |x[n]||z|^{-n} < \infty$$

For the existence of the z-transform, the ROC therefore consists of a ring in the z-plane:



In specific cases the inner radius of this ring may include the origin, and the outer radius may extend to infinity.
If the ROC includes the unit circle $|z| = 1$, then the Fourier transform will converge. Most useful z-transforms can be expressed in the form

$$X(z) = \frac{P(z)}{Q(z)},$$

Where *P (z)* and *Q (z)* are polynomials in z, the values of z for which $P(z) = 0$ are called the **zeros** of *X (z)*, and the values with $Q(z) = 0$ are called the **poles**. The zeros and poles completely specify *X (z)* to within a multiplicative constant.
Where, z is the complex variable. Here, the summation taken from $n = 0$ to $n = \infty$ is according to the fact that for most situations, the digital signal x (n) is the causal sequence, that is, $x(n) = 0$ for $n \leq 0$. For non-causal system, the summation starts at $n = -\infty$. The region of convergence is defined based on the particular sequence x (n) being applied. The z-transforms for common sequences are summarized below:

Line No.	$x(n),\ n \geq 0$	z-Transform $X(z)$	Region of Convergence
1	$x(n)$	$\sum_{n=0}^{\infty} x(n) z^{-n}$	
2	$\delta(n)$	1	$\lvert z\rvert > 0$
3	$au(n)$	$\frac{az}{z-1}$	$\lvert z\rvert > 1$
4	$nu(n)$	$\frac{z}{(z-1)^2}$	$\lvert z\rvert > 1$
5	$n^2u(n)$	$\frac{z(z+1)}{(z-1)^3}$	$\lvert z\rvert > 1$
6	$a^n u(n)$	$\frac{z}{z-a}$	$\lvert z\rvert > \lvert a\rvert$
7	$e^{-na}u(n)$	$\frac{z}{(z-e^{-a})}$	$\lvert z\rvert > e^{-a}$
8	$na^n u(n)$	$\frac{az}{(z-a)^2}$	$\lvert z\rvert > \lvert a\rvert$
9	$\sin(an)u(n)$	$\frac{z\sin(a)}{z^2 - 2z\cos(a) + 1}$	$\lvert z\rvert > 1$
10	$\cos(an)u(n)$	$\frac{z[z-\cos(a)]}{z^2 - 2z\cos(a) + 1}$	$\lvert z\rvert > 1$
11	$a^n \sin(bn)u(n)$	$\frac{[a\sin(b)]z}{z^2 - [2a\cos(b)]z + a^2}$	$\lvert z\rvert > \lvert a\rvert$
12	$a^n \cos(bn)u(n)$	$\frac{z[z-a\cos(b)]}{z^2 - [2a\cos(b)]z + a^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
13	$e^{-an}\sin(bn)u(n)$	$\frac{[e^{-a}\sin(b)]z}{z^2 - [2e^{-a}\cos(b)]z + e^{-2a}}$	$\lvert z\rvert > e^{-a}$
14	$e^{-an}\cos(bn)u(n)$	$\frac{z[z-e^{-a}\cos(b)]}{z^2 - [2e^{-a}\cos(b)]z + e^{-2a}}$	$\lvert z\rvert > e^{-a}$
15	$2\lvert A\rvert\lvert P\rvert^n \cos(n\theta + \phi)u(n)$ where P and A are complex constants defined by $P = \lvert P\rvert\angle\theta, A = \lvert A\rvert\angle\phi$	$\frac{Az}{z-P} + \frac{A^*z}{z-P^*}$	

The Z-Plane and the Unit Circle

The Z-transform $\mathbf{z = e^{s} = e^{\sigma+j\omega} = e^{\sigma}\, e^{j\omega} = r\, e^{j\omega}}$ are complex variables with real and imaginary parts. Figures (a) and (b) shows the S-Plane of the Laplace transform and the Z-Plane of Z-transform. In the S-Plane the vertical jω-axis is the frequency axis, and the horizontal σ-axis gives the exponential rate of decay, or the rate of growth, of the amplitude of complex sinusoid as also shown in figure (c).

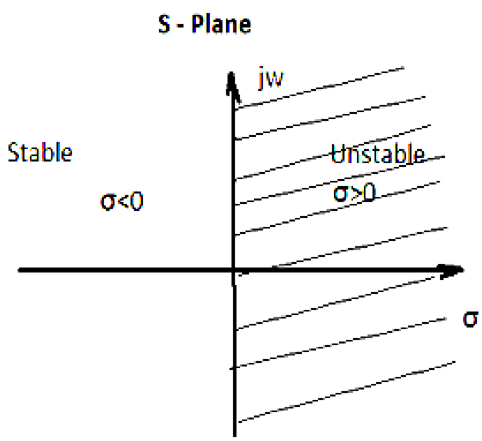


Fig. (a)

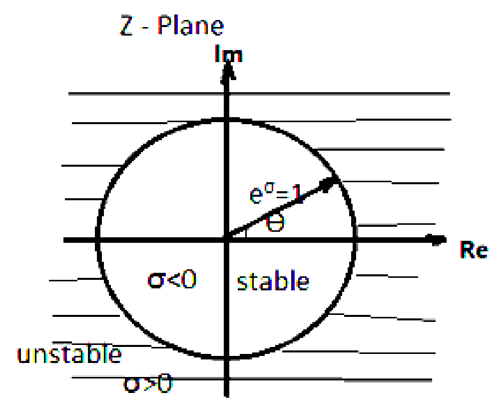


Fig. (b)

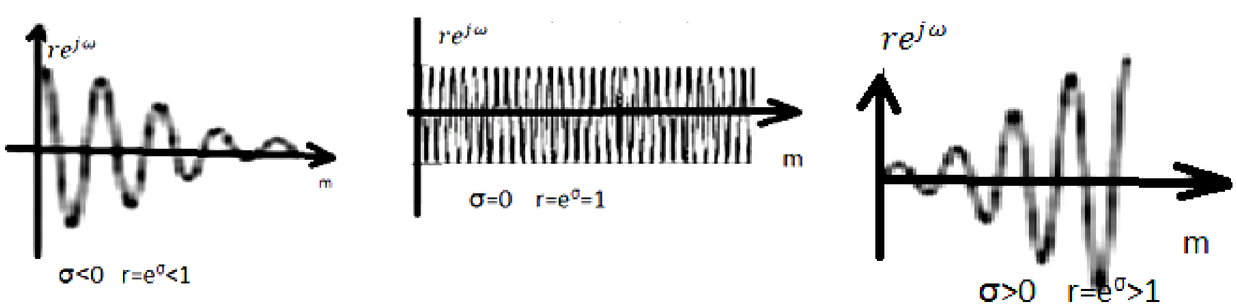


Fig. (C)