Civil Engineering Department
Analysis Engineering
Lecture No. 7+8

Third Stage
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The Fourier Transform

Fourier Transform is a very powerful tool that can able to convert any Time Domain Signal (periodic non periodic) as a function of Frequency called **Frequency Domain**

we can also convert from frequency domain to time domain by **inverse fourier transform**

The time domain description tells you what sound you hear every instant, while the frequency domain description tells you, roughly,what instruments are involved in the ways & how they are played.

Fourier Series (FS) applies only to periodic signals, with a period T_0 and a fundamental frequency:
$f_o = 1/To$. As $To \to \infty$,
The signal becomes **non-periodic** and its FS will tend to the ***Fourier transform (FT)***, F (.), which is normally defined as a Fourier transform pair since the time signal can be obtained by the inverse transformation F^{-1} (.):

$$X(f) = F\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt \quad \text{................ (1)}$$

$$x(t) = F^{-1}\{X(f)\} = \int_{-\infty}^{\infty} X(f) e^{+j2\pi ft} df \quad \text{............. (2)}$$

Where equation (1) called Fourier transform equation, and equation (2) called inverse transform of Fourier equation.

if $F(\omega)$ is the Fourier transform of $f(t)$, *i.e.*,

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} \, dt \quad \textbf{FT equation}$$

then

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} \, d\omega \quad \textbf{IFS equation}$$

The Fourier transform (***FT***) reveals the ***frequency content*** of the signal, known as the ***spectrum***, and the frequency behavior of the system, known as the ***transfer function* or *frequency response***. The Fourier transform $X(f)$ of ***the real time*** signal $x(t)$ is generally ***complex.***

It is normally plotted as magnitude $X(f)$ vs. frequency f (magnitude spectrum) and phase $\angle X(f)$ vs. frequency f (phase spectrum). For systems, these quantities are called the magnitude response and the phase response.

We'll be interested in signals defined for all t the Fourier transform of a signal f is the function

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

• F is a function of a real variable ω; the function value F(ω) is (in general) a complex number

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \cos \omega t \, dt - j \int_{-\infty}^{\infty} f(t) \sin \omega t \, dt$$

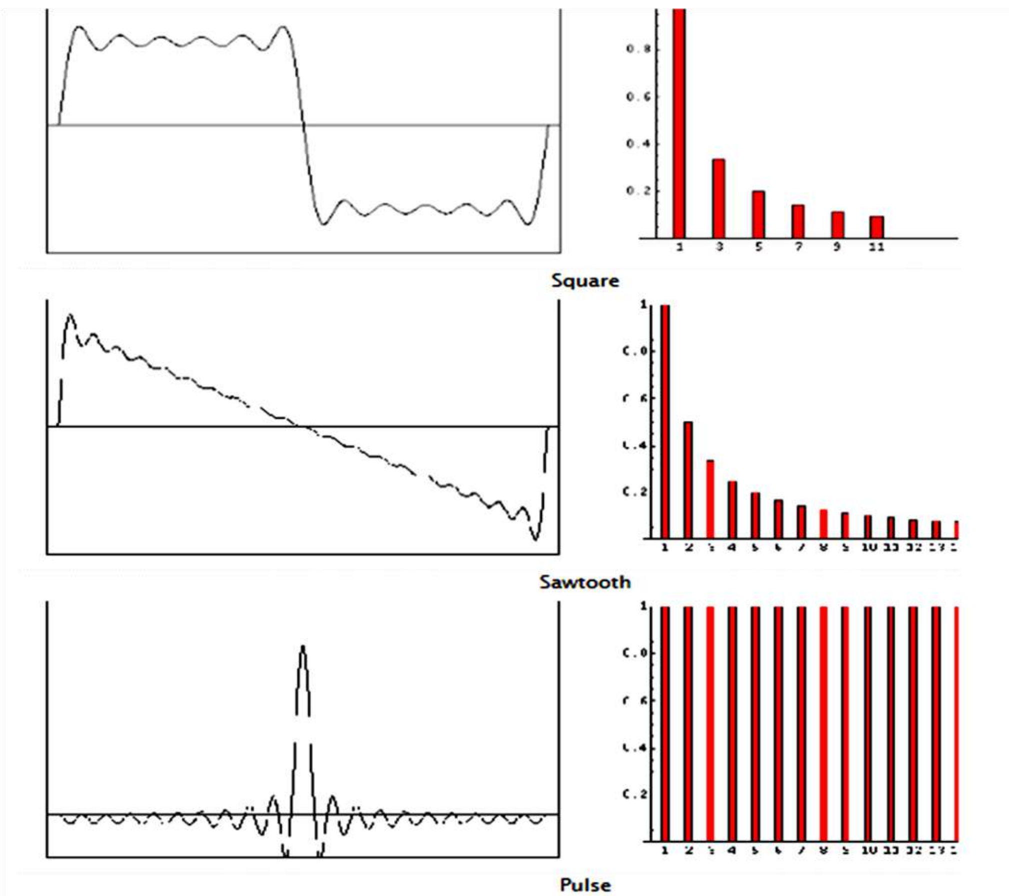
• $|F(\omega)|$ is called the amplitude spectrum of f; $\angle F(\omega)$ is the phase spectrum of f

• Notation: F = (f) means F is the Fourier transform of f; use uppercase letters for the transforms (e.g., x (t) and X (ω), h (t) and H (ω), etc.)

in fact
any field of physical science that uses sinusoidal signals, such as engineering, physics, applied mathematics, and chemistry.

- This equation ($f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n \cos n\omega t + b_n \sin n\omega t\}$) is representing the Fourier series (FS) equation.
- This equation ($F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$) is representing the Fourier Transform (FT) equation.
- This equation ($f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$) is representing the inverse Fourier transform equation.

let's take an example -



Examples

1- One-sided decaying exponential

$$f(t) = \begin{cases} 0 & t < 0 \\ e^{-t} & t \geq 0 \end{cases}$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\; e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-t}\; e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-(1+j\omega)t} dt = \frac{-1}{1+j\omega} e^{-(1+j\omega)t}\Big]_{0}^{\infty}$$

$$= \mathbf{\frac{1}{1+j\omega}}$$

Fourier transform is

$$\int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt = \frac{1}{j\omega + 1} = F(j\omega)$$

For $f(t) = \begin{cases} 0 & t < 0 \\ e^{-at} & t \geq 0 \end{cases}$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\; e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-at}\; e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-(a+j\omega)t} dt = \frac{-1}{a+j\omega} e^{-(a+j\omega)t}\Big]_{0}^{\infty}$$

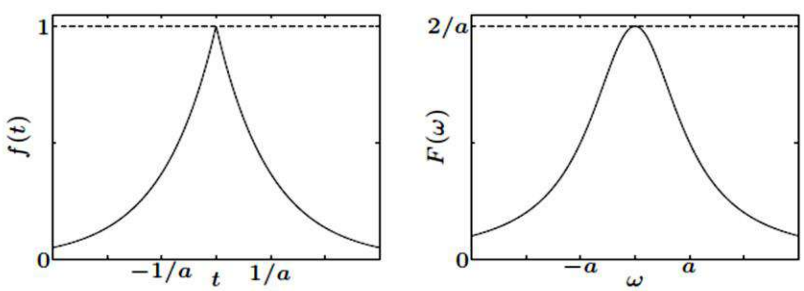
$$= \mathbf{\frac{1}{a+j\omega}}$$

If a = 1, become the Fourier transform equal $\frac{1}{1+j\omega}$

2- double-sided exponential:

$f(t) = e^{-a|t|}$ (with $a > 0$)

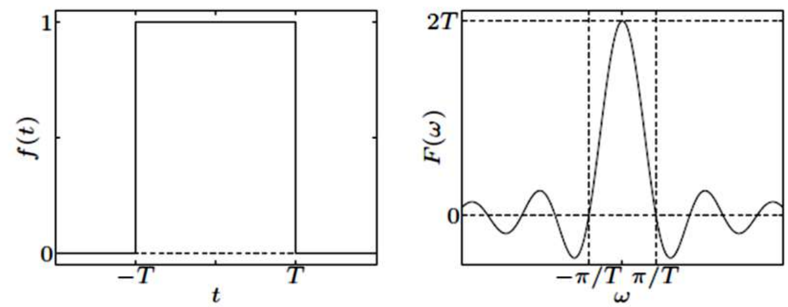
$$\begin{aligned} F(\omega) = \int_{-\infty}^{\infty} e^{-a|t|} e^{-j\omega t}\, dt &= \int_{-\infty}^{0} e^{at} e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt \\ &= \frac{1}{a - j\omega} + \frac{1}{a + j\omega} \\ &= \frac{2a}{a^2 + \omega^2} \end{aligned}$$



3-

rectangular pulse: $f(t) = \begin{cases} 1 & -T \le t \le T \\ 0 & |t| > T \end{cases}$

$$F(\omega) = \int_{-T}^{T} e^{-j\omega t}\, dt = \frac{-1}{j\omega}\left(e^{-j\omega T} - e^{j\omega T}\right) = \frac{2\sin \omega T}{\omega}$$



The inverse Fourier transform

if $F(\omega)$ is the Fourier transform of $f(t)$, *i.e.*,

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt$$

then

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{j\omega t}\, d\omega$$

let's check

$$\begin{aligned} \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} F(\omega) e^{j\omega t}\, d\omega &= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty}\left(\int_{\tau=-\infty}^{\infty} f(\tau) e^{-j\omega\tau}\right) e^{j\omega t} d\omega \\ &= \frac{1}{2\pi}\int_{\tau=-\infty}^{\infty} f(\tau)\left(\int_{\omega=-\infty}^{\infty} e^{-j\omega(\tau - t)} d\omega\right) d\tau \\ &= \int_{-\infty}^{\infty} f(\tau)\delta(\tau - t) d\tau \\ &= f(t) \end{aligned}$$

Example

Determine the inverse transform, if the Fourier transform F (ω) are given as;

$$F(\omega) = \begin{cases} 1 & |\omega| \le \omega_c \\ 0 & \omega_c < |\omega| \le \pi \end{cases}$$

Sol:

$$f(n) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{j\omega n} d\omega$$

$$f(n) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega = \frac{\sin \omega_c n}{\pi n} \qquad n \ne 0$$

For n=0, we have

$$f(0) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} d\omega = \frac{\omega_c}{\pi}$$

Hence

$$f(n) = \begin{cases} \dfrac{\omega_c}{\pi} & n = 0 \\ \dfrac{\omega_c}{\pi}\dfrac{\sin\omega_c n}{\omega_c n} & n \neq 0 \end{cases}$$