Numerical Techniques

INTRODUCTION

we shall deal with the methods for solving the equations. Sometimes, a rough approximation of a root can be found by graph and more accurate results by the following methods :

(*i*) Newton Raphson method or successive substitution method.

(*ii*) Rule of false position (*Regula falsi*).

(*iii*) Iteration method.

SOLUTION OF THE EQUATIONS GRAPHICALLY

(*i*) Find a small interval (*a*, *b*) between which the root lies. *f* (*a*) and f (*b*) are of opposite sign. Let $f(x) = 0$ and $f(a) = -ve$ and $f(b) = +ve$

(*ii*) Prepare a table of the different values of *x* between *a* and *b*, for $y = f(x)$.

(*iii*) Plot these points and join them to get smooth curve.

(*iv*) The real root of the equation $f(x) = 0$ is the abscissa where the curve cuts the *x*-axis.

Example 1. Find graphically the positive root of the equation.

$$x^3 - 6x - 13 = 0$$

Solution.

$$f(x) = x^3 - 6x - 13 = 0 \quad \ldots\ldots.(1)$$

$$f(3) = 27 - 18 - 13 = -4 = -ve$$

$$f(4) = 64 - 24 - 13 = 27 = +ve$$

The root of (1) lies between 3 and 4 as f (3) and f (4) are opposite in sign.

(1) is written as $f(x) = x^3 - 6x - 13 = 0$

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$y = x^3$

$y = 6x + 13$

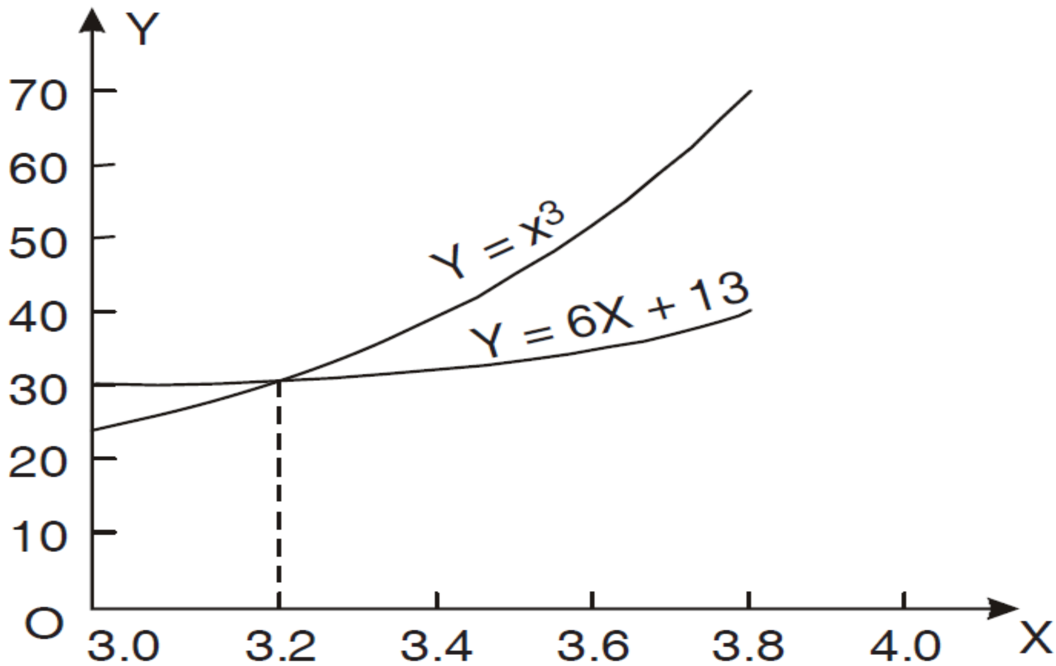
Let us draw two curves for $y = x^3$ and $y = 6x + 13$.

$y = x^3$

x	3	3.2	3.4	3.6	3.8	4.0
y	27	32.8	39.3	46.7	54.9	64

$y = 6x + 13$

x	3	3.2	3.4	3.6	3.8	4
y	31	32.2	33.4	34.6	35.8	37



Let the origin be (3, 0).

The graphs of $y = x3$ and $y = 6x + 13$ are sketched in the figure.

The abscissa of the point of intersection of two curves is 3.2.

The root of the given equation is [3.2].

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EXERCISE

1-Draw the graph of y = x^3 and y = – 2x + 20 and find approximate the solution of the equation
+ 2x - 20 = 0. Ans. - 2.47x^3

2. Solve graphically x^3 - 2x - 5 = 0 Ans. 2.099

NEWTON-RAPHSON METHOD
OR SUCCESSIVE SUBSTITUTION METHOD

By this method, we get closer approximation of the root of an equation if we already know its approximate root.

$$a_1 = a_0 - \frac{f(a_0)}{f'(a_0)}$$

$$a_2 = a_1 - \frac{f(a_1)}{f'(a_1)} \quad , \quad a_3 = a_2 - \frac{f(a_2)}{f'(a_2)}$$,and so on

Example 2. Starting with x_0 =3, *find a root of* $x^3 - 3x - 5 = 0$, correct to three decimal *places. Use* *Newton-Raphson* *method.*

SOLUTION

$$f(x) = x^3 - 3x - 5 = 0, \quad f'(x) = 3x^2 - 3$$

$$f(3) = 27 - 9 - 5 = 13, \quad f'(3) = 27 - 3 = 24$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3 - \frac{f(3)}{f'(3)} = 3 - \frac{13}{24} = 3 - 0.5417 = 2.4583$$

$$x_2 = 2.4583 - \frac{f(2.4583)}{f'(2.4583)} = 2.4583 - \frac{2.4812}{15.1297} = 2.4583 - 0.1640 = 2.2943$$

$$x_3 = 2.2943 - \frac{f(2.2943)}{f'(2.2943)} = 2.2943 - \frac{0.1939}{12.7914} = 2.2791$$

$$f(2.2791) = 0.0010$$

the required root = 2.2791 **Ans.**

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Example 3. Find the real root of the following equation, correct three decimal places using Newton-Raphson method.

SOLUTION:

$$x^3 - 2x - 5 = 0$$

$$x^3 - 2x - 5 = 0$$

$$f(x) = x^3 - 2x - 5$$

$$f(2) = 8 - 4 - 5 = -1$$

$$f(2.5) = (2.5)^3 - 2(2.5) - 5 = +5.625$$

Since $f(2)$ and $f(2.5)$ are, of opposite sign, the root of (1) lies between 2 and 2.5 ; $f(2)$ is near to zero than $f(2.5)$, so 2 is better appropriate root than 2.5.

$$f'(x) = 3x^2 - 2 \qquad f'(2) = 12 - 2 = 10$$

Let 2 be an approximate root of (1). By Newton-Raphson method

$$a_1 = a - \frac{f(a)}{f'(a)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{-1}{10} = 2.1$$

$$f(2.1) = (2.1)^3 - 2(2.1) - 5 = 9.261 - 4.2 - 5 = 0.061$$

$$f'(2.1) = 3(2.1)^2 - 2 = 11.23$$

$$a_2 = 2.1 - \frac{f(2.1)}{f'(2.1)} = 2.1 - \frac{0.061}{11.23} = 2.1 - 0.00543 = 2.0945$$

$$f(2.09457) = (2.09457)^3 - 2(2.09457) - 5$$

$$= 9.1893 - 4.18914 - 5 = 0.00016$$

$$f'(2.09457) = 3(2.09457)^2 - 2 = 13.16167 - 2 = 11.16167$$

$$a_3 = 2.09457 - \frac{f(2.09457)}{f'(2.09457)} = 2.09457 - \frac{0.00016}{11.16167} = 2.09457 + 0.000014 = 2.09456$$

As $a_3 = a_2$ correct upto four places of decimal, hence the root of (1) is 2.0945.

Ans.

Example 4. Find an interval of length 1, in which the root of $f(x) = 3x^3 - 4x^2 - 4x - 7 = 0$ lies. Take the middle point of this interval as the starting approximation and iterate two times, using the **Newton- Raphson** method.

Solution. $$f(x) = 3x^3 - 4x^2 - 4x - 7 = 0$$

$$f(2) = 24 - 16 - 8 - 7 = -7$$

$$f(3) = 81 - 36 - 12 - 7 = +26$$

The root of (1) lies between 2 and 3 as $f(2)$ and $f(3)$ are of opposite sign. The middle point of this interval is 2.5.

$$f(2.5) = 46.875 - 25 - 10 - 7 = 4.875$$

$$f'(x) = 9x^2 - 8x - 4$$

$$f'(2.5) = 56.25 - 20 - 4 = 32.25$$

Newton-Raphson

$$a_1 = a - \frac{f(a)}{f'(a)} \quad \text{and}$$

$$a_1 = 2.5 - \frac{f(2.5)}{f'(2.5)} = 2.5 - \frac{4.875}{32.25} = 2.5 - 0.15 = 2.35$$

$$f(2.35) = 38.93 - 22.09 - 9.4 - 7 = 0.44$$

$$f'(2.35) = 49.7 - 18.8 - 4 = 26.9$$

$$a_2 = 2.35 - \frac{f(2.35)}{f'(2.35)} = 2.35 - \frac{0.44}{26.9} = 2.35 - 0.016 = 2.334$$

$$f(2.334) = 38.14 - 21.79 - 9.34 - 7 = 0.01 \text{ which is nearly zero.}$$

the required root is 2.334 AUC...DAWAH **Ans.**

Example 5. By using Newton-Raphson's method, find the root of x^4 - x - 10 = 0, which is near to x = 2 correct to three places of decimal.

Solution. $f(x) = x^4 - x - 10 = 0, \quad f'(x) = 4x^3 - 1$

$$f(2) = 16 - 2 - 10 = 4$$

$$f'(2) = 32 - 1 = 31$$

By Newton-Raphson's method

$$a_1 = a - \frac{f(a)}{f'(a)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{4}{31} = 2 - .129 = 1.871$$

$$f(1.871) = (1.871)^4 - 1.871 - 10 = 12.25 - 1.871 - 10 = 0.379$$

$$f'(1.871) = 4(1.871)^3 - 1 = 4 \times 6.5497 - 1 = 25.1988$$

$$a_2 = 1.871 - \frac{f(1.871)}{f'(1.871)} = 1.871 - \frac{0.379}{25.1988} = 1.871 - 0.0150 = 1.856$$

$$f(1.856) = (1.856)^4 - (1.856) - 10 = 11.8662 - 11.856 = 0.0102$$

$$f'(1.856) = 4(1.856)^3 - 1 = 4 \times 6.3934 - 1 = 24.5736$$

$$a_3 = (1.856) - \frac{f(1.856)}{f'(1.856)} = 1.856 - \frac{0.0102}{24.5736} = 1.856 - 0.00042$$

$$= 1.8556$$

$$f(1.8556) = (1.8556)^4 - (1.8556) - 10 = 0.00038$$

$$f'(1.8556) = 4(1.8556)^3 - 1 = 24.5572$$

$$a_4 = 1.8556 - \frac{f(1.8556)}{f'(1.8556)} = 1.8556 - \frac{0.00038}{24.5572} = 1.8556 - 0.00002 = 1.85558$$

$$= 1.8556 \text{ (say)}$$

As $a_4 = a_3$ correct upto four places of decimals, so the correct root of the given equation is 1.8556. **Ans.**

Example 6. *Determine the root of* $x^4 + x^3 - 7x^2 - X + 5 = 0$ *which lies between 2 and 3 correct to three decimal places.*

Solution. $f(x) = x^4 + x^3 - 7x^2 - x + 5 = 0$

$$f(2) = 16 + 8 - 28 - 2 + 5 = -1$$

$$f(3) = 81 + 27 - 63 - 3 + 5 = +47.$$

ROOT LIES Between 2 AND 3
Take X1=2

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{f(2)}{f'(2)} \qquad \left[\begin{array}{l} f'(x) = 4x^3 + 3x^2 - 14x - 1 \\ f'(2) = 32 + 12 - 28 - 1 = 15 \end{array}\right]$$

$$x_2 = 2 - \frac{-1}{15} = 2\frac{1}{15} = 2.067$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.067 - \frac{f(2.067)}{f'(2.067)}$$

$$= 2.067 - \frac{-0.0028}{18.422} = 2.067 + 0.0001519 = 2.0671519 \qquad \textbf{Ans.}$$

Exercise

Solve the following equations by Newton's method:

1. $x^3 - 2x - 5 = 0$ — **Ans.** 2.0946
2. $x^3 - 2x + 0.5 = 0$ — **Ans.** 0.2578
3. $3x^3 + 8x^2 + 8x + 5 = 0$ *(A.M.I.E.T.E., Summer 1995)* — **Ans.** −1.67
4. $x^3 - 5x + 3 = 0$ AUC ...DAWAH — **Ans.** 0.6565

RULE OF FALSE POSITION (REGULA FALSI)

Let f (x) = 0 .
Let y = f (x) be represented by the curve AB
The curve AB cuts the x-axis at P

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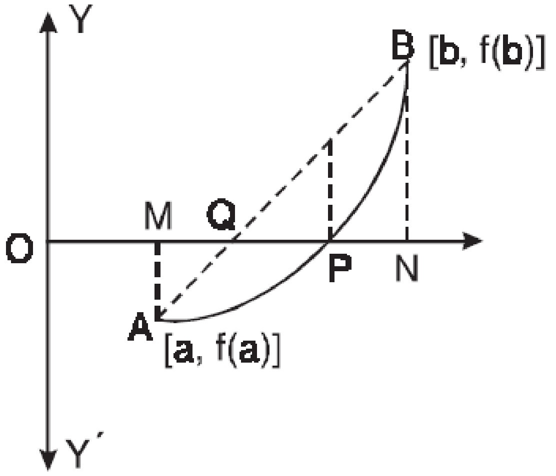
The real root of (1) is OP

The false position of the curve AB is taken as the chord AB

The chord AB cuts the x-axis at Q. The approximate root of f (x) = 0 is OQ

By this method, we find OQ

Let A [a, f(a)] , B [b, f (b)] be the extremities of the chord AB The equation of the chord AB is:



$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

To find OQ, put $y = 0$, $\quad -f(a) = \dfrac{f(b) - f(a)}{b - a}(x - a)$

$$(x - a) = \frac{-(b - a) f(a)}{f(b) - f(a)} \quad \text{or} \quad x = a + \frac{(a - b) f(a)}{f(b) - f(a)}$$

$$x = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

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Example 7. Find an approximate value of the root of the equation $x^3 + x - 1 = 0$ near x = 1, using the method of false position (regula falsi) two times ?

Solution. f (x) =x^3+ x - 1 = 0

f (1) = 1 + 1 – 1 = + 1

f (.5) = $(0.5)^3$+ (0.5) – 1 = – 0.375

The root lies between 0.5 and 1

Let x_1 = 0.5 and x_2 = 1

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} \quad \text{or} \quad x_3 = \frac{0.5 f(1) - 1 f(0.5)}{f(1) - f(0.5)}$$

$$= \frac{0.5\,(1) - 1\,(-0.375)}{1 + 0.375} = 0.6363$$

Now $f(0.6363) = -0.1061$ and $f(1) = 1$

∴ Root lies between .6363 and 1.

$$x_3 = 0.6363, \quad x_2 = 1$$

$$x_4 = \frac{0.6363 f(1) - 1 f(0.6363)}{f(1) - f(0.6363)} = \frac{0.6363 - 1\,(-0.1061)}{1 + 0.1061} = 0.6712$$

New, $f(0.6712) = -0.0264$ and $f(1) = 1$

$$x_5 = \frac{0.6712 f(1) - 1 f(0.6712)}{f(1) - f(0.6712)} = \frac{0.6712 - (-0.0264)}{1 - (-0.0264)}$$

$$= 0.6797$$ **Ans.**

Example 8. *Find by the method of Regula Falsi a root of the equation*

$x^3 + x^2 - 3x - 3 = 0$ lying between 1 and 2.

Solution. $f(x) = x^3 + x^2 - 3x - 3 = 0$

$$f(1) = 1+1-3-3 = -4 = -ve$$

$$f(2) = 8+4-6-3 = +3 = +ve$$

The root lies between 1 and 2 as $f(1)$ is –ve and $f(2)$ is +ve.

By Regula Falsi method:

$$x_1 = \frac{1\,f(2) - 2\,f(1)}{f(2) - f(1)} = \frac{1\times 3 - 2\times -4}{3-(-4)} = \frac{11}{7} = 1.571$$

$$f(1.571) = (1.571)^3 + (1.571)^2 - 3\,(1.571) - 3$$

$$= 3.877 + 2.468 - 4.713 - 3 = -1.368 = -ve$$

The root lies between 1.571 and 2 as $f(1.571)$ is –ve and $f(2)$ is + ve.

$$x_2 = \frac{1.571\,f(2) - 2\,f(1.571)}{f(2) - f(1.571)}$$

$$= \frac{1.571\times 3 - 2\times(-1.368)}{3-(-1.368)} = \frac{4.713+2.736}{4.368} = 1.705$$

$$f(1.705) = (1.705)^3 + (1.705)^2 - 3\,(1.705) - 3 = 4.956 + 2.907 - 5.115 - 3$$

$$= -0.252 = -ve.$$

The root lies between 1.705 and 2 as $f(1.705)$ is –ve and $f(2)$ is +ve.

$$x_3 = \frac{1.705\,f(2) - 2\,f(1.705)}{f(2) - f(1.705)} = \frac{1.705\times 3 - 2\times(-0.252)}{3-(-0.252)} = 1.728$$

Ans.

Example 9. *Find the approximate value, correct to three places of decimals, of the* real root which lies between*(- 2) and(- 3)* of the equation x^3 - 3x + 4 = 0, using the method of false position three times in succession

Solution. f(x)= x^3 - 3x + 4 = 0

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$$x_1 = -2, \; x_2 = -3$$

$$f(x_1) = f(-2) = (-2)^3 - 3(-2) + 4 = -8 + 6 + 4 = 2.$$

$$f(x_2) = f(-3) = (-3)^3 - 3(-3) + 4 = -27 + 9 + 4 = -14$$

$$x = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{-2 f(-3) - (-3) f(-2)}{f(-3) - f(-2)} = \frac{-2(-14) - (-3)(2)}{(-14) - (2)} = \frac{28 + 6}{-16} = \frac{34}{-16}$$

$$= -2.125$$

$$f(-2.125) = (-2.125)^3 - 3(-2.125) + 4 = -9.596 + 6.375 + 4 = +0.779$$

$$f(-3) = -14 \quad \text{and} \quad f(-2.125) = +0.779$$

$\therefore$ Root lies between -2.125 and -3.

$$x = \frac{(-2.125) f(-3) - (-3) f(-2.125)}{f(-3) - f(-2.125)} = \frac{(-2.125)(-14) - (-3)(0.779)}{(-14) - (0.779)}$$

$$x = \frac{(-2.171) f(-3) - (-3) f(-2.171)}{f(-3) - f(-2.171)} = \frac{(-2.171)(-14) - (-3)(.293)}{-14 - .293}$$

$$= \frac{30.494 + .879}{-14.293} = \frac{31.273}{-14.293} = -2.188$$

Numerical Techniques

SOLUTION OF LINEAR SYSTEMS

Here we shall discuss two methods for solving the linear systems *i.e.*, Gauss-Seidel and Crout's methods

Gauss method

By this method elimination of unknown is done more systematically and we have a check to detect the errors.

Example Solve the following simultaneous equations

$$2x + 3y + z = 13$$

$$x - y - 2z = -1$$

$$3x + y + 4z = 15$$

Step 1. We write the equation, first, which has unity as coefficient of *x*, otherwise divide the equation by the coefficient of **X** to make it unity. Thus

$$x - y - 2z = -1$$

$$2x + 3y + z = 13$$

$$3x + y + 4z = 15$$

Step 2. To eliminate X, subtract suitable multiples of first equation from the remaining equations, and we get

$$x - y - 2z = -1$$

$$5y + 5z = 15 \qquad \text{....R2 - 2R1=R2}$$

$$4y + 10z = 18 \qquad \text{....R3 – 3R1=R3}$$

Step 3. The coefficient of y is made unity in none of the resulting equations and we have

$$x - y - 2z = -1$$

$$y + z = 3 \qquad \text{....R2/5 = R2}$$

$$4y + 10z = 18$$

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Step 4. To eliminate *y*, subtract suitable multiple of second equation from the third. Thus we have

$x \;-y\; -2z = -1$,,, **R3-4R2=R3**

$y + z = 3$,,, **4y+10z -4y-4z=18-12**

6 z = 6 … **6z=6**

Step 5. Start from bottom and substitute.

6 z = 6 or Z = 1

y + z = 3 or y + 1 = 3 or y = 2

x - y - 2 z = - 1 or x - 2 - 2 = - 1 or X = 3

X-Y-2Z=-1,,,,,,,,,,,,,3-2-2*1= -1 ,,,,OK

EXAMPLE . Solve the following system of equations:

$$\begin{aligned} 2x + 4y + 6z &= 22 \\ 3x + 8y + 5z &= 27 \\ -x + y + 2z &= 2 \end{aligned}$$

Solution First, we transform this system into an equivalent system in which the
coefficient of *x* in the first equation is 1:

$$\begin{aligned} 2x + 4y + 6z &= 22 \\ 3x + 8y + 5z &= 27 \\ -x + y + 2z &= 2 \end{aligned} \qquad \textbf{(4a)}$$

$$\begin{aligned} x + 2y + 3z &= 11 \\ 3x + 8y + 5z &= 27 \\ -x + y + 2z &= 2 \end{aligned} \qquad \textbf{(4b)}$$

Multiply the first equation in (4a) by $\frac{1}{2}$.

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Next, we eliminate the variable x from all equations except the first:

$$\begin{aligned} x + 2y + 3z &= 11 \\ 2y - 4z &= -6 \\ -x + y + 2z &= 2 \end{aligned} \qquad (4c)$$

Replace the second equation in (4b) by the sum of −3 × the first equation + the second equation:

$$\begin{aligned} -3x - 6y - 9z &= -33 \\ 3x + 8y + 5z &= 27 \\ \hline 2y - 4z &= -6 \end{aligned}$$

-3R1+R2=R2

$$\begin{aligned} x + 2y + 3z &= 11 \\ 2y - 4z &= -6 \\ 3y + 5z &= 13 \end{aligned} \qquad (4d)$$

Replace the third equation in (4c) by the sum of the first equation + the third equation:

$$\begin{aligned} x + 2y + 3z &= 11 \\ -x + y + 2z &= 2 \\ \hline 3y + 5z &= 13 \end{aligned}$$

R1+R3=R3

Then we transform System (4d) into yet another equivalent system, in which the coefficient of y in the second equation is 1:

$$\begin{aligned} x + 2y + 3z &= 11 \\ y - 2z &= -3 \\ 3y + 5z &= 13 \end{aligned} \qquad (4e)$$

Multiply the second equation in (4d) by $\frac{1}{2}$.

$$\frac{R2}{2}$$

We now eliminate y from all equations except the second, using operation 3 of the elimination method:

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$$\begin{aligned} x \qquad + 7z &= 17 \\ y - 2z &= -3 \\ 3y + 5z &= 13 \end{aligned} \tag{4f}$$

Replace the first equation in (4e) by the sum of the first equation + (−2) × the second equation:

$$\begin{aligned} x + 2y + 3z &= 11 \\ -2y + 4z &= 6 \\ \hline x \qquad + 7z &= 17 \end{aligned}$$

R1-2R2=R1

$$\begin{aligned} x \qquad + 7z &= 17 \\ y - 2z &= -3 \\ 11z &= 22 \end{aligned} \tag{4g}$$

Replace the third equation in (4f) by the sum of (−3) × the second equation + the third equation:

$$\begin{aligned} -3y + 6z &= 9 \\ 3y + 5z &= 13 \\ \hline 11z &= 22 \end{aligned}$$

-3R2+R3=R3

Multiplying the third equation by $\frac{1}{11}$ in (4g) leads to the system

$$\begin{aligned} x \qquad + 7z &= 17 \\ y - 2z &= -3 \\ z &= 2 \end{aligned}$$

Eliminating z from all equations except the third (try it!) then leads to the system

$$\begin{aligned} x &= 3 \\ y &= 1 \\ z &= 2 \end{aligned} \tag{4h}$$

In its final form, the solution to the given system of equations can be easily read off!
We have $x = 3$, $y = 1$, and $z = 2$. Geometrically, the point (3, 1, 2) is the intersection
of the three planes described by the three equations comprising the given system.

Augmented Matrices

Observe from the preceding example that the variables x, y, and z play no significant role in each step of the reduction process, except as a reminder of the position of each coefficient in the system. With the aid of matrices, which are rectangular arrays of numbers, we can eliminate writing the variables at each step of the reduction and thus save ourselves a great deal of work. For example, the system

$$\begin{aligned} 2x + 4y + 6z &= 22 \\ 3x + 8y + 5z &= 27 \\ -x + y + 2z &= 2 \end{aligned} \qquad \textbf{(4a)}$$

may be represented by the matrix

EXAMPLE Write the augmented matrix corresponding to each equivalent system given in (4a) through (4h).

Solution The required sequence of augmented matrices follows.

Equivalent System		Augmented Matrix	
a. $\begin{aligned} 2x + 4y + 6z &= 22 \\ 3x + 8y + 5z &= 27 \\ -x + y + 2z &= 2 \end{aligned}$		$\left[\begin{array}{ccc\|c} 2 & 4 & 6 & 22 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{array}\right]$	**(7a)**
b. $\begin{aligned} x + 2y + 3z &= 11 \\ 3x + 8y + 5z &= 27 \\ -x + y + 2z &= 2 \end{aligned}$	**R1/2**	$\left[\begin{array}{ccc\|c} 1 & 2 & 3 & 11 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{array}\right]$	**(7b)**

c. $$\begin{aligned} x + 2y + 3z &= 11 \\ 2y - 4z &= -6 \\ -x + y + 2z &= 2 \end{aligned}$$

R2-3R1=R2

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 11 \\ 0 & 2 & -4 & -6 \\ -1 & 1 & 2 & 2 \end{array}\right] \quad (7c)$$

d. $$\begin{aligned} x + 2y + 3z &= 11 \\ 2y - 4z &= -6 \\ 3y + 5z &= 13 \end{aligned}$$

R1+R3=R3

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 11 \\ 0 & 2 & -4 & -6 \\ 0 & 3 & 5 & 13 \end{array}\right] \quad (7d)$$

e. $$\begin{aligned} x + 2y + 3z &= 11 \\ y - 2z &= -3 \\ 3y + 5z &= 13 \end{aligned}$$

R2/2

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 11 \\ 0 & 1 & -2 & -3 \\ 0 & 3 & 5 & 13 \end{array}\right] \quad (7e)$$

f. $$\begin{aligned} x \quad + 7z &= 17 \\ y - 2z &= -3 \\ 3y + 5z &= 13 \end{aligned}$$

R1-2R2=R1

$$\left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -2 & -3 \\ 0 & 3 & 5 & 13 \end{array}\right] \quad (7f)$$

g. $$\begin{aligned} x \quad + 7z &= 17 \\ y - 2z &= -3 \\ 11z &= 22 \end{aligned}$$

R3-3R2=R3

$$\left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 11 & 22 \end{array}\right] \quad (7g)$$

h. $$\begin{aligned} x &= 3 \\ y &= 1 \\ z &= 2 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \quad (7h)$$

The augmented matrix in (7h) is an example of a matrix in row-reduced form. In general, an augmented matrix with *m* rows and *n* columns (called an *m x n* matrix) is in **row-reduced form** if it satisfies the following conditions.

Row-Reduced Form of a Matrix

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1-Each row consisting entirely of zeros lies below all rows having nonzero entries

.2-The first nonzero entry in each (nonzero) row is 1 (called a leading 1)

3-In any two successive (nonzero) rows, the leading 1 in the lower row lies to the right of the leading 1 in the upper row

4-If a column in the coefficient matrix contains a leading 1, then the other entries in that column are zeros.

EXAMPLE Solve the system of linear equations given by

$$\begin{aligned} 2y + 3z &= 7 \\ 3x + 6y - 12z &= -3 \\ 5x - 2y + 2z &= -7 \end{aligned}$$

Solution Using the Gauss–Jordan elimination method, we obtain the following sequence of equivalent augmented matrices:

$$\left[\begin{array}{ccc|c} 0 & 2 & 3 & 7 \\ 3 & 6 & -12 & -3 \\ 5 & -2 & 2 & -7 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 3 & 6 & -12 & -3 \\ 0 & 2 & 3 & 7 \\ 5 & -2 & 2 & -7 \end{array}\right]$$

$$\xrightarrow{\frac{1}{3}R_1} \left[\begin{array}{ccc|c} 1 & 2 & -4 & -1 \\ 0 & 2 & 3 & 7 \\ 5 & -2 & 2 & -7 \end{array}\right]$$

$$\xrightarrow{R_3 - 5R_1} \left[\begin{array}{ccc|c} 1 & 2 & -4 & -1 \\ 0 & 2 & 3 & 7 \\ 0 & -12 & 22 & -2 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 2 & -4 & -1 \\ 0 & 1 & \frac{3}{2} & \frac{7}{2} \\ 0 & -12 & 22 & -2 \end{array}\right]$$

$$\xrightarrow[R_3 + 12R_2]{R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -7 & -8 \\ 0 & 1 & \frac{3}{2} & \frac{7}{2} \\ 0 & 0 & 40 & 40 \end{array}\right]$$

$$\xrightarrow{\frac{1}{40}R_3} \left[\begin{array}{ccc|c} 1 & 0 & -7 & -8 \\ 0 & 1 & \frac{3}{2} & \frac{7}{2} \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[R_2 - \frac{3}{2}R_3]{R_1 + 7R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

The solution to the system is given by $x = -1$, $y = 2$, and $z = 1$; this may be verified by substitution into the system. ■