EXAMPLE: find the **I.Z.T** using long division ?

$$
\begin{array}{r|l}
 & z + 4z^2 + 13z^3 + \dots\dots\dots \\
\hline
1 - 4z + 3z^2 & z \\
 & \underline{z - 4z^2 + 3z^3} \\
 & \quad 4z^2 - 3z^3 \\
 & \quad \underline{4z^2 - 16z^3 + 12z^4} \\
 & \qquad 13z^3 - 12z^4 \\
 & \qquad \dots\dots\dots\dots \\
 & \qquad \dots\dots\dots\dots
\end{array}
$$

ie.,

$$X(z) = z + 4z^2 + 13z^3 + \dots\dots\dots$$

$$\therefore x(n) = [\dots\dots\dots 13, 4, 1, \underset{\uparrow}{0}]$$

EXAMPLE: find the **I.Z.T** using long division ?

$$F(z) = \frac{z}{z - 0.5}$$

$$
\begin{array}{r|l}
 & 1 + 0.5z^{-1} + 0.25z^{-2} + \cdots \\
\hline
z - 0.5 & z \\
 & \underline{z - 0.5} \\
 & \quad 0.5 \\
 & \quad \underline{0.5 - 0.25z^{-1}} \\
 & \qquad 0.25z^{-1} \\
 & \qquad 0.25z^{-1} - 0.125z^{-2}
\end{array}
$$

$$
\begin{array}{r|l}
 & \frac{1}{z} + \frac{2}{z^2} + \frac{4}{z^3} \\
\hline
z-2 & 1 \\
 & 1 - \frac{2}{z} \\
\hline
 & \frac{2}{z} \\
 & \frac{2}{z} - \frac{4}{z^2} \\
\hline
 & \frac{4}{z^2} \\
 & \frac{4}{z^2} - \frac{8}{z^3} \\
\hline
 & \frac{8}{z^3}
\end{array}
$$

$$
\begin{aligned}
\frac{1}{z-2} &= \frac{1}{z} + \frac{2}{z^2} + \frac{4}{z^3} + \frac{8}{z^4} + \ldots \\
&= z^{-1} + 2z^{-2} + 4z^{-3} + 8z^{-4} + \ldots + 2^{k-1} z^{-k} + \ldots \\
&= \{2^{k-1}\} z^{-k}
\end{aligned}
$$

$$
Z^{-1} \frac{1}{z-2} = \{2^{k-1}\}
$$

EXAMPLE// USING Partial Fraction Expansion Method SOLVE

$$
X(z) = \frac{z}{3z^2 - 4z + 1}
$$

for the following ROCs:

a) $|z| > 1$ b) $|z| < \frac{1}{3}$ c) $\frac{1}{3} < |z| < 1$

Solution:

We have $\frac{X(z)}{z} = \frac{1}{3z^2 - 4z + 1} = \frac{1}{3[z^2 - \frac{4}{3}z + \frac{1}{3}]}$

$$\frac{X(z)}{z} = \frac{1}{3(z - \frac{1}{3})(z-1)} = \frac{A1}{(z-\frac{1}{3})} + \frac{A2}{(z-1)}$$

where $A1 = \frac{X(z)}{z}(z - \frac{1}{3})\Big|_{z=1/3} = \frac{\cancel{(z-\frac{1}{3})}}{3\cancel{(z-\frac{1}{3})}(z-1)}\Big|_{z=1/3} = -\frac{1}{2}$

and $A2 = \frac{X(z)}{z}(z-1)\Big|_{z=1} = \frac{\cancel{(z-1)}}{3(z-\frac{1}{3})\cancel{(z-1)}}\Big|_{z=1} = \frac{1}{2}$

$$\therefore \frac{X(z)}{z} = \frac{-1/2}{(z-\frac{1}{3})} + \frac{1/2}{(z-1)}$$

or $X(z) = -\frac{1}{2}\frac{z}{(z-\frac{1}{3})} + \frac{1}{2}\frac{z}{(z-1)}$

a) ROC $|z| > 1$ Here the ROC is outside a circle of radius = 1 (magnitude of the largest pole). This indicates that the sequence x(n) is a sum of two right hand sequences.

$$\therefore x(n) = -\frac{1}{2}\left(\frac{1}{3}\right)^n u(n) + \frac{1}{2}u(n)$$

b) ROC $|z| < \frac{1}{3}$ Here the ROC is inside a circle of radius = $\frac{1}{3}$ (magnitude of the smallest pole). This indicates that the sequence x(n) is a sum of two left hand sequences

$$\therefore \quad x(n) = \frac{1}{2}\left(\frac{1}{3}\right)^n u(-n-1) - \frac{1}{2}u(-n-1)$$

c) ROC $\frac{1}{3} < |z| < 1$ Here the ROC falls in a ring defined by the boundaries $|z| = 1/3$ and $|z| = 1$

So the first term of X(z) (with pole = 1/3) corresponds to a RH sequence and the second term (with pole = 1) corresponds to a LH sequence.

$$\therefore \quad x(n) = -\frac{1}{2}\left(\frac{1}{3}\right)^n u(n) - \frac{1}{2}u(-n-1)$$

EXAMPLE// USING **Partial Fraction** Expansion Method SOLVE

Find the inverse Z transform of

$$X(z) = \frac{z+1}{3z^2 - 4z + 1} \qquad \text{ROC} : |z| > 1$$

Solution:

We have
$$\frac{X(z)}{z} = \frac{z+1}{z(3z^2 - 4z + 1)} = \frac{z+1}{3z\left[z^2 - \frac{4}{3}z + \frac{1}{3}\right]}$$

$$= \frac{z+1}{3z\left(z - \frac{1}{3}\right)(z-1)} = \frac{A1}{z} + \frac{A2}{\left(z - \frac{1}{3}\right)} + \frac{A3}{(z-1)}$$

$$A_1 = \left(\frac{X(z)}{z}\right)z\Big|_{z=0} = \frac{(z+1)z}{3z(z-\frac{1}{3})(z-1)}\Big|_{z=0} = 1$$

$$A_2 = \left(\frac{X(z)}{z}\right)(z-\tfrac{1}{3})\Big|_{z=\frac{1}{3}} = \frac{(z+1)(z-\frac{1}{3})}{3z(z-\frac{1}{3})(z-1)}\Big|_{z=\frac{1}{3}} = -2$$

$$A_3 = \left(\frac{X(z)}{z}\right)(z-1)\Big|_{z=1} = \frac{(z+1)(z-1)}{3z(z-\frac{1}{3})(z-1)}\Big|_{z=1} = 1$$

$$\therefore \frac{X(z)}{z} = \frac{1}{z} + \frac{-2}{(z-\frac{1}{3})} + \frac{1}{(z-1)}$$

ie., $X(z) = 1 - \frac{2z}{(z-\frac{1}{3})} + \frac{z}{(z-1)}$ with ROC $|z| > 1$.

Inversion gives the time sequence x(n) as

$$x(n) = \partial(n) - 2.\left(\frac{1}{3}\right)^n u(n) + u(n)$$

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EXAMPLE : FIND THE INVERSE Z TRANSFORM OF

$$\frac{2z}{z-1} + \frac{3z}{z-2}.$$

Solution From Table

$$\mathcal{Z}^{-1}\left\{\frac{z}{z-1}\right\} = u_n$$

$$\mathcal{Z}^{-1}\left\{\frac{z}{z-2}\right\} = 2^n \quad (r = 2)$$

So

$$\mathcal{Z}^{-1}\left\{\frac{2z}{z-1} + \frac{3z}{z-2}\right\} = 2u_n + 3 \times 2^n$$

Example. Find the inverse Z-transform of $\frac{1}{Z-2}$

Solution. $F(z) = \frac{1}{z-2}$

Case I. If $\left|\frac{2}{z}\right| < 1$, $F(z) = \frac{1}{z}\frac{1}{1-2z^{-1}}$

$$= z^{-1}(1-2z^{-1})^{-1} = z^{-1}[1+2z^{-1}+2^2z^{-2}+\ldots]$$

$$= z^{-1}+2z^{-2}+2^2z^{-3}+\ldots$$

$$\{f(k)\} = \{2^{k-1}\}, \qquad k \geq 1$$

Case II. If $\left|\frac{z}{2}\right| < 1$

$$F(z) = \frac{1}{z-2} = -\frac{1}{2}\frac{1}{1-\frac{z}{2}} = \frac{-1}{2}\left(1-\frac{z}{2}\right)^{-1} = -\frac{1}{2}\left[1+\frac{z}{2}+\frac{z^2}{2^2}+\ldots\right]$$

$$\{f(k)\} = \{-2^{k-1}\}, \quad k \leq 0$$

Note. The inverse Z-transform can only be settled when region of convergence (ROC) is given.

Example //

Find the inverse Z –transform of USING Partial Fraction Expansion Method

$$\frac{z}{z^2+7z+10}$$

Let $F(z) = \frac{z}{z^2+7z+10}$

Then $\frac{F(z)}{z} = \frac{1}{z^2+7z+10} = \frac{1}{(z+2)(z+5)}$

Now, consider $\frac{1}{(z+2)(z+5)} = \frac{A}{z+2} + \frac{B}{z+5}$

$$= \frac{1}{3}\frac{1}{z+2} - \frac{1}{3}\frac{1}{z+5}$$

Therefore, $F(z) = \frac{1}{3}\frac{z}{z+2} - \frac{1}{3}\frac{z}{z+5}$

INVERTING WE GET

$$= \frac{1}{3}(-2)^n - \frac{1}{3}(-5)^n$$

CONVOLUTION THEOREM to Find Inverse Z-Transform

The inverse Z-transform can be calculated using the convolution theorem. In the convolution integration method, the given Z-transform X(z) is first split into X1(z) and X2(z) such that X(z)=X1(z)X2(z)

The signals x1(n) and x2(n) are then obtained by taking the inverse Z-transform of X1(z) and X2(z) respectively. Finally, the function x(n) is obtained by performing the convolution of x1(n) and x2(n) in the time domain

As from the definition of Z-transform of convolution of two signals, we have

$$Z[x_1(n) * x_2(n)] = X_1(z)X_2(z) = X(z)$$

Therefore, the inverse Z-transform is obtained as,

$$x(n) = Z^{-1}[X(z)] = Z^{-1}[Z\{x_1(n) * x_2(n)\}]$$

$$\therefore Z^{-1}[X(z)] = x(n) = x_1(n) * x_2(n) = \sum_{k=0}^{n} x_1(k)x_2(n-k)$$

Numerical Example

Using the convolution method, find the Z-transform of

$$X(z) = \frac{z^2}{(z-3)(z-4)}$$

Solution /Let

$$X(z) = X_1(z)X_2(z) = \frac{z}{(z-3)}\,\frac{z}{(z-4)}$$

Taking the inverse Z-transform of X1(z) and X2(z) respectively as –

$$Z^{-1}[X_1(z)] = x_1(n) = Z^{-1}\left[\frac{z}{(z-3)}\right] = 3^n u(n)$$

Similarly

$$Z^{-1}[X_2(z)] = x_2(n) = Z^{-1}\left[\frac{z}{(z-4)}\right] = 4^n u(n)$$

Now, using the convolution method for finding inverse Z-transform, we have

$$Z^{-1}[X(z)] = x(n) = x_1(n) * x_2(n) = \sum_{k=0}^{n} x_1(k)x_2(n-k)$$

$$\therefore x(n) = \sum_{k=0}^{n} 3^k u(k)4^{n-k}u(n-k) = \sum_{k=0}^{n} 3^k u(k)\left(\frac{4^n}{4^k}\right)u(n-k)$$

$$\Rightarrow x(n) = 4^n \sum_{k=0}^{n} \left(\frac{3^k}{4^k}\right) = 4^n \sum_{k=0}^{n} \left(\frac{3}{4}\right)^k$$

$$\Rightarrow x(n) = 4^n \left[\frac{1-\left(\frac{3}{4}\right)^{n+1}}{1-\left(\frac{3}{4}\right)}\right] = 4^{n+1}\left[1-\left(\frac{3}{4}\right)^{n+1}\right]$$

$$\therefore x(n) = 4^{n+1}u(n) - 3^{n+1}u(n)$$

Example/ **Using the convolution property of Z-transform, find the Z-transform of the following signal.**

$$x(n) = \left(\frac{1}{3}\right)^n u(n) * \left(\frac{1}{5}\right)^n u(n)$$

Solution

Let signal is

$$x(n) = x_1(n) * x_2(n)$$

$$\therefore x_1(n) = \left(\frac{1}{3}\right)^n u(n)$$

Taking Z-transform, we get,

$$Z[x_1(n)] = X_1(z) = Z\left[\left(\frac{1}{3}\right)^n u(n)\right]$$

$$X_1(z) = \frac{z}{\left(z - \frac{1}{3}\right)};\ \mathrm{ROC} \to |z| > \frac{1}{3}$$

Similarly

$$Z[x_2(n)] = X_2(z) = Z\left[\left(\frac{1}{5}\right)^n u(n)\right]$$

$$X_2(z) = \frac{z}{\left(z - \frac{1}{5}\right)};\ \mathrm{ROC} \to |z| > \frac{1}{5}$$

Now, using the convolution property of Z-transform

$\left[\text{i.e.},\ x_1(n) * x_2(n) \overset{ZT}{\leftrightarrow} X_1(z)X_2(z)\right]$,we get,

$$Z[x(n)] = X_1(z)X_2(z)$$

$$\therefore Z\left[\left(\frac{1}{3}\right)^n u(n) * \left(\frac{1}{5}\right)^n u(n)\right] = \frac{z}{\left(z - \frac{1}{3}\right)} \frac{z}{\left(z - \frac{1}{5}\right)}$$

The ROC of the Z-transform of the given sequence is

$$\mathrm{ROC} \to \left[|z| > \frac{1}{3}\right] \cap \left[|z| > \frac{1}{5}\right] = |z| > \frac{1}{3}$$

$$\therefore \left(\frac{1}{3}\right)^n u(n) * \left(\frac{1}{5}\right)^n u(n) \overset{ZT}{\leftrightarrow} \frac{z^2}{\left(z - \frac{1}{3}\right)\left(z - \frac{1}{5}\right)};\ \mathrm{ROC} \to |z| > \frac{1}{3}$$

Example (*Convolution*) Find the inverse ***Z***- transform of.

$$\frac{z}{z-1}\frac{z}{z-4}$$

Solution Note that

$$\mathcal{Z}^{-1}\left\{\frac{z}{z-1}\right\}=u_n \quad \text{and} \quad \mathcal{L}^{-1}\left\{\frac{z}{z-4}\right\}=4^n$$

Hence, using convolution

$$\mathcal{Z}^{-1}\left\{\frac{z}{z-1}\frac{z}{z-4}\right\}=u_n * 4^n=\sum_{k=0}^{n}u_k 4^{n-k}$$

Writing out this sequence for $n=0,1,2,3,\ldots$

$$\begin{array}{cccc} 1, & (1+4), & 1+4+16, & 1+4+16+64,\ldots \\ (n=0) & (n=1) & (n=2) & (n=3) \end{array}$$

We see that the *n*th term is a geometric series with *n* + 1 terms and first term 1 and common ratio 4 From the formula for the sum for *n* terms of a geometric progression,

$$Sn=\frac{a(r^n-1)}{r-1}$$ **where *a* is the first term, *r* is the common ratio and *n* is the number of terms. Therefore, for the *n* th term of the above sequence, we get:**

$$\frac{4^{n+1}-1}{4-1} = \frac{4^{n+1}-1}{3}.$$

So we have found

$$Z^{-1}\left\{\frac{z}{z-1}\frac{z}{z-4}\right\} = \frac{4^{n+1}-1}{3}.$$

SOLUTION OF DIFFERENCE EQUATIONS

Solution of first order linear constant coefficient difference equations. **To solve a difference equation, we have to take the Z - transform of both sides of the difference equation using the property**

$$Z\{f_{n+k}\} = z^k\{ F(z) - f_0 - (f_1 / z) - \dots - (f_{k-1} / z^{k-1}) \} \quad (k > 0)$$

Using the initial conditions, we get an algebraic equation of the form
$F(z) = \phi(z)$.
By taking the inverse Z-transform, we get the required solution f_n of the given difference equation

Exmaple Solve the difference equation $y_{n+1} + y_n = 1, y_0 = 0$, **by Z** - transform method. Given equation is $y_{n+1} + y_n = 1$ ---------- (1)

Let Y(z) be the Z -transform of $\{y_n\}$.

Taking the Z - transforms of both sides of (1), we get

$$Z\{y_{n+1}\} + Z\{y_n\} = Z\{1\}.$$

$$\text{ie, } z\{Y(z) - y_0\} + Y(z) = z/(z-1).$$

Using the given condition, it reduces to

$$(z+1)\,Y(z) = \frac{z}{z-1}$$

$$\text{i.e, } Y(z) = \frac{z}{(z-1)\,(z+1)}$$

$$\text{or} \qquad Y(z) = \frac{1}{2}\left\{\frac{z}{z-1} - \frac{z}{z+1}\right\}$$

On taking inverse Z-transforms, we obtain

$$y_n = (1/2)\{1 - (-1)^n\}$$

Example

Solve $y_{n+2} + y_n = 1,\ y_0 = y_1 = 0$, using Z-transforms.

Consider $y_{n+2} + y_n = 1$ ------------- (1)

Taking Z- transforms on both sides, we get

$$Z\{y_{n+2}\} + Z\{y_n\} = Z\{1\}$$

$$z^2\left\{Y(z) - y_0 - \frac{y_1}{z}\right\} + Y(z) = \frac{z}{z-1}$$

$$(z^2+1)\ Y(z) = \frac{z}{z-1}$$

$$\text{or } Y(z) = \frac{z}{(z-1)(z^2+1)}$$

$$\text{Now,} \qquad \frac{Y(z)}{z} = \frac{1}{(z-1)(z^2+1)} = \frac{A}{z-1} + \frac{Bz+C}{z^2+1}$$

$$= \frac{1}{2}\left(\frac{1}{z-1} - \frac{z}{z^2+1} - \frac{1}{z^2+1}\right)$$

$$\text{Therefore,} \qquad Y(z) = \frac{1}{2}\left(\frac{z}{z-1} - \frac{z^2}{z2+1} - \frac{z}{z2+1}\right)$$

Using Inverse Z-transform, we get

y_n =(½){1 - cos (nπ/ 2) - sin (nπ/ 2)}

Example: Consider the first order difference equation

$$y_{n+1} - 3y_n = 4 \qquad n = 0, 1, 2, \ldots$$

The equation could be solved in a step-by-step or recursive manner, provided that y_0 is known because

$$y_1 = 4 + 3y_0 \qquad y_2 = 4 + 3y_1 \qquad y_3 = 4 + 3y_2$$

IF WE NOW THAT $y_0 = 1$ THEN

$y_1 = 7$ $y_2 = 25$ $y_3 = 79$ direct method

$$y_{n+1} - 3y_n = 4 \qquad n = 0, 1, 2, \ldots$$

with initial condition $y_0 = 1$
We multiply both sides of (1) by $Z^{-n}$and sum each side over all positive integer values of n and zero. We obtain

$$\sum_{n=0}^{\infty}(y_{n+1} - 3y_n)z^{-n} = \sum_{n=0}^{\infty} 4z^{-n}$$

OR

$$\sum_{n=0}^{\infty} y_{n+1}z^{-n} - 3\sum_{n=0}^{\infty} y_n z^{-n} = 4\sum_{n=0}^{\infty} z^{-n} \qquad (2)$$

The three terms in (2) are clearly recognisable as z-transforms The right-hand side is the z-transform of the constant sequence {4, 4, . . .} which is

$$\frac{4z}{z-1}.$$

If $Y(z) = \sum_{n=0}^{\infty} y_n z^{-n}$

$$\sum_{n=0}^{\infty} y_{n+1}z^{-n} = z\,Y(z) - zy_0 \text{ (by the left shift theorem).}$$

$$z\,Y(z) - zy_0 - 3\,Y(z) = \frac{4z}{z-1}$$

$$(z-3)Y(z)-z = \frac{4z}{(z-1)}$$

$$(z-3)Y(z) = \frac{4z}{z-1}+z = \frac{z^2+3z}{z-1}$$

$$Y(z) = \frac{z^2+3z}{(z-1)(z-3)}$$

$$Y(z) = z\frac{(z+3)}{(z-1)(z-3)}$$

$$= z\left(\frac{-2}{z-1}+\frac{3}{z-3}\right) \qquad \text{(in partial fractions)}$$

SO

$$Y(z) = \frac{-2z}{z-1}+\frac{3z}{z-3}$$

$$y_n = -2\mathbb{Z}^{-1}\{\frac{z}{z-1}\} + 3\ \mathbb{Z}^{-1}\{\frac{z}{z-3}\}$$

$$y_n = -2+3\times 3^n = -2+3^{n+1} \qquad n = 0,1,2,\ldots$$

Checking the solution From this solution

$$y_0 = -2+3 = 1 \qquad \text{(as given)}$$

$$y_1 = -2+3^2 = 7$$

$y_{2=25}$ ، $y_{3=79}$

Example. ***Solve the difference equation***

$$y_{k+1} - 2y_{k-1} = 0, \quad k \geq 1 , \quad y_{(0)} = 1$$

Solution. $y_{k+1} - 2y_{k-1} = 0$ ......(1)

Taking the Z-transform of both sides of (1), we get

$$Z[\, y_{k+1} - 2y_{k-1}\,] = 0$$

$$Z[\, y_{k+1}\,] - 2Z[\, y_{k-1}\,] = 0$$

$$z\,Y(z) - y_0\, z - 2Y(z) = 0 \qquad (y0 = 1$$

$$(z-2)\,Y(z) - z = 0$$

$$Y(z) = \frac{z}{z-2}$$

$$\{y_{(k)}\} = Z^{-1}\left[\frac{z}{z-2}\right] = Z^{-1}\left[\frac{1}{1-2z^{-1}}\right]$$

$$= Z^{-1}\,[\,1-2z^{-1}\,]^{-1} = 1 + 2z^{-1} + (2z^{-1})^2 + \ldots$$

$$= \{2^k\}, \qquad k \geq 0$$

ANS($y_{k=2^k}$)

Example Solve the difference equation

$$y_n + 2y_{n-1} = 2u_n$$

for $n \geqslant 0$ given $y_{-1} = 1$.

Solution We take the z transform of both sides of the difference equation

$$y_n + 2y_{n-1} = 2u_n$$

and using the right shift property to find

$$\mathcal{Z}\{y_{n-1}\} = z^{-1}Y(z) + y_{-1}$$

$$Y(z) + 2\left(z^{-1}Y(z) + y_{-1}\right) = \frac{2z}{z-1}$$

As $y_{-1} = 1$,

$$Y(z)\left(1 + 2z^{-1}\right) = \frac{2z}{z-1} - 2$$

$$Y(z) = \frac{2z^2}{(z-1)(z+2)} - \frac{2z}{(z+2)}$$

To take the inverse transform we need to express the first terms using partial fractions. Using the 'cover up' rule we get

$$Y(z) = \frac{2z}{3(z-1)} + \frac{4z}{3(z+2)} - \frac{2z}{(z+2)} = \frac{2z}{3(z-1)} - \frac{2z}{3(z+2)}.$$

$$\frac{2z}{(z-1)(z+2)} = \frac{2}{3(z-1)} + \frac{4}{3(z+2)}$$

Taking inverse transforms we find

$$y_n = \tfrac{2}{3}u_n - \tfrac{2}{3}(-2)^n$$

Check: **To check that we have the correct solution we can substitute in a couple of values for n and see that we get the same value from the difference equation as from the explicit formula found From the explicit formula and using u_0 = 1 (by definition of the unit step function),**

n = 0 gives

$$y_0 = \tfrac{2}{3}u_0 - \tfrac{2}{3}(-2)^0 = \tfrac{2}{3} - \tfrac{2}{3} = 0$$

From the difference equation,

$$y_n + 2y_{n-1} = 2u_n, \quad \text{where}$$

y_{-1} = 1, n = 0 gives

y_0 + 2y−1 = 2u_0

Substituting y_{-1} = 1 gives y_0 = 0 as before. From the explicit formula,

n **= 1** gives

$$y_1 = \frac{2}{3}u_1 - \frac{2}{3}(-2)^1 = \frac{2}{3} + \frac{4}{3} = 2$$

From the difference equation, ***n*** **= 1** gives

$$y_1 + 2y_0 = 2u_1$$

substituting $y0 = 0$ gives $y1 = 2$, confirming the result of the explicit formula.

Example/ Using partial fractions to find the inverse transform

$$\mathcal{Z}^{-1}\left\{\frac{z^2}{(z-1)(z-0.5)}\right\}.$$

Solution Notice that most of the values of the transform in Table have a factor of Z in the numerator. We write

$$\frac{z^2}{(z-1)(z-0.5)} = z\left(\frac{z}{(z-1)(z-0.5)}\right)$$

We use the 'cover up' rule to write

$$\frac{z}{(z-1)(z-0.5)} = \frac{1}{0.5(z-1)} - \frac{1}{z-0.5}$$

$$= \frac{2}{z-1} - \frac{1}{z-0.5}$$

So

$$\frac{z^2}{(z-1)(z-0.5)} = \frac{2z}{z-1} - \frac{z}{z-0.5}$$

and using Table we find

$$\mathcal{Z}^{-1}\left\{\frac{2z}{z-1}-\frac{z}{z-0.5}\right\}=2u_n-(0.5)^n$$

The transfer function and impulse response function

$$ay_n+by_{n-1}+cy_{n-2}=f_n.$$

$$Y(z)=\frac{z^2}{az^2+bz+c}$$

$$H(z)=z^2/(az^2+bz+c)$$

and $\mathcal{Z}^{-1}\{H(z)\}=h_n$, where h_n is the impulse response function.

Example Find the transfer function and impulse response of the system described by the following difference equation:

$$3y_n+4y_{n-1}=f_n$$

Solution To find the transfer function replace f_n by δn and take the z transform of the resulting equation assuming zero initial conditions:

$$4y_n + 3y_{n-1} = \delta_n.$$

Taking the z transform of both sides of the equation we get

$$4Y + 3(z^{-1}Y + y_{-1}) = 1.$$

As $y_{-1} = 0$,

$$Y = \frac{z}{4z+3} = H(z) = \frac{z/4}{z + (3/4)}$$

To find the **impulse response sequence** we take the inverse transform

of the transfer function to find

$$h_n = \mathcal{Z}^{-1}\left\{\frac{z/4}{z + \frac{3}{4}}\right\} = \frac{1}{4}\left(-\frac{3}{4}\right)^n$$

INVERSE OF Z-TRANSFORM BY DIVISION

Example *Find* $Z^{-1} \dfrac{1}{z-2}$

$$\frac{1}{z-2} = \frac{1}{z} + \frac{2}{z^2} + \frac{4}{z^3} + \frac{8}{z^4} + \ldots$$
$$= z^{-1} + 2z^{-2} + 4z^{-3} + 8z^{-4} + \ldots + 2^{k-1} z^{-k} + \ldots$$
$$= \{2^{k-1}\} z^{-k}$$
$$Z^{-1} \frac{1}{z-2} = \{2^{k-1}\}$$

Example: Find I.Z.T, $h[n]$ using the long division

$$H(z) = \frac{1+2z^{-1}-5z^{-2}+6z^{-3}}{1-3z^{-1}+2z^{-2}}, \quad |z| > 2$$

Solution:

$$\begin{array}{r|l} & 2 + 3z^{-1} \\ \hline 1 - 3z^{-1} + 2z^{-2} & 1 + 2z^{-1} - 5z^{-2} + 6z^{-3} \\ & 3z^{-1} - 9z^{-2} + 6z^{-3} \\ \hline & 1 - z^{-1} + 4z^{-2} \\ & 2 - 6z^{-1} + 4z^{-2} \\ \hline & -1 + 5z^{-1} \end{array}$$

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