



<mark>3T+3E</mark>

AUC-CET-24-25-DAWAH

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EXAMPLE: find the **I.Z.T** using long division ?

$$\begin{array}{c|c} \frac{z+4z^2+13z^3+\dots}{1-4z+3z^2} \\ z \\ \frac{z-4z^2+3z^3}{4z^2-3z^3} \\ \frac{4z^2-16z^3+12z^4}{13z^3-12z^4} \end{array}$$

Engineering Analysis

Z-Transform

تحليلات هندسيه

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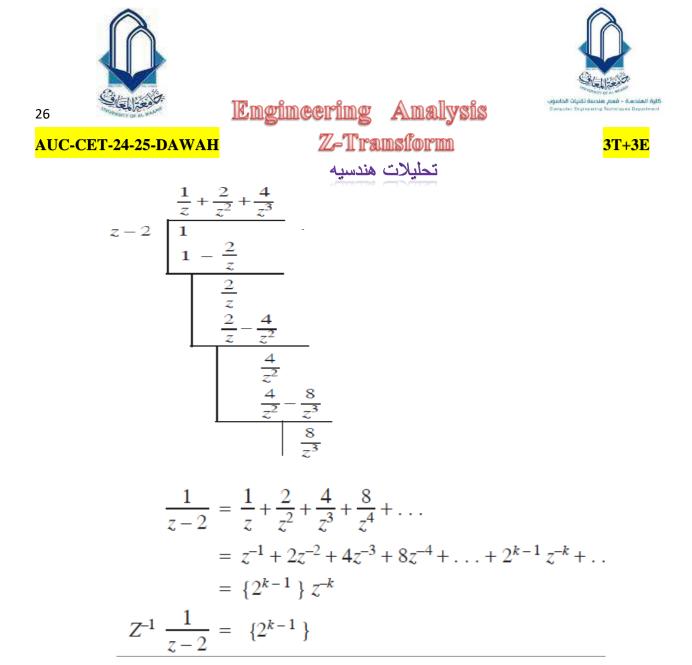
ie.,

x(z) = z + 4z² + 13z³ +
∴ x(n) = [.....13, 4,1,0]
$$\uparrow$$

EXAMPLE: find the **I.Z.T** using long division ?

$$F(z) = \frac{z}{z - 0.5}$$

$$z = 0.5 \underbrace{)z}_{z = -0.5} \underbrace{\frac{z - 0.5}{0.5}}_{0.5} \underbrace{0.5 - \underbrace{0.25z^{-1}}_{0.25z^{-1}}}_{0.25z^{-1}} \underbrace{0.25z^{-2}}_{0.25z^{-2}} \underbrace{0.25z^{-2}}_{0.25z^{-2}}}_{0.25z^{-1}} \underbrace{0.25z^{-2}}_{0.25z^{-2}} \underbrace{0.25z^{-2}}_{0.25z^{-2}}}_{0.25z^{-2}} \underbrace{0.25z^{-2}}_{0.25z^{-2}}}_{0.25z^{-2}} \underbrace{0.25z^{-2}}_{0.25z^{-2}}}_{0.25z^{-2}} \underbrace{0.25z^{-2}}_{0.25z^{-2}}}_{0.25z^{-2}} \underbrace{0.25z^{-2}}_{0.25z^{-2}}}_{0.25z^{-2}}}$$



EXAMPLE// USING Partial Fraction Expansion Method SOLVE

$$X(z) = \frac{z}{3z^2 \cdot 4z + 1}$$

for the following ROCs:
a) $|z| \ge 1$ b) $|z| \le \frac{1}{3}$ c) $\frac{1}{3} \le |z| \le 1$





Z-Transform تحليلات هندسيه

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Solution:

We have
$$\frac{X(z)}{z} = \frac{1}{3z^2 \cdot 4z + 1} = \frac{1}{3[z^2 \cdot \frac{4}{3}z + \frac{1}{3}]}$$

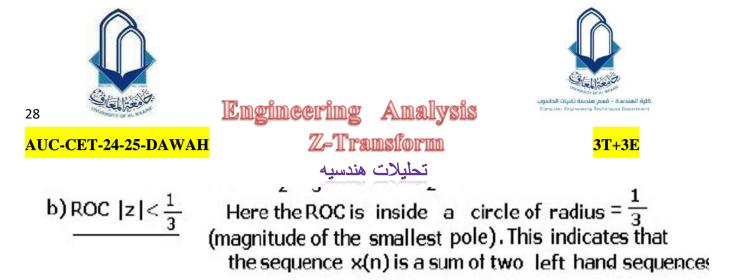
 $\frac{X(z)}{z} = \frac{1}{3(z - \frac{1}{3})(z - 1)} = \frac{A1}{(z - \frac{1}{3})} + \frac{A2}{(z - 1)}$
where $A1 = \frac{X(z)}{z}(z - \frac{1}{3})\Big|_{z = 1/3} = \frac{(z - \frac{1}{3})}{3(z - \frac{1}{3})(z - 1)}\Big|_{z = 1/3} = -\frac{1}{2}$
and $A2 = \frac{X(z)}{z}(z - 1)\Big|_{z = 1} = \frac{(z - 4)}{3(z - \frac{1}{3})(z - 1)}\Big|_{z = 1} = -\frac{1}{2}$

••
$$\frac{X(z)}{z} = \frac{-1/2}{(z - \frac{1}{3})} + \frac{1/2}{(z - 1)}$$

or $X(z) = -\frac{1}{2} \frac{z}{(z - \frac{1}{3})} + \frac{1}{2} \frac{z}{(z - 1)}$

a)
$$\underline{\text{ROC} |z| > 1}$$
 Here the ROC is outside a circle of radius = 1 (magnitude of the largest pole). This indicates that the sequence $x(n)$ is a sum of two right hand sequences.

• •
$$x(n) = -\frac{1}{2} \left(\frac{1}{3}\right)^n u(n) + \frac{1}{2} u(n)$$



•
$$x(n) = \frac{1}{2} \left(\frac{1}{3}\right)^n u(-n-1) - \frac{1}{2} u(-n-1)$$

 $\frac{c) \operatorname{ROC} \frac{1}{3} < |z| < 1}{|z| = 1/3} \quad \text{Here the ROC falls in a ring defined by the boundaries}} \\ |z| = 1/3 \text{ and } |z| = 1$

So the first term of X(z) (with pole = 1/3) corresponds to a RH sequence and the second term (with pole = 1) corresponds to a LH sequence.

•
$$x(n) = -\frac{1}{2} \left(\frac{1}{3}\right)^n u(n) - \frac{1}{2} u(-n-1)$$

EXAMPLE// USING Partial Fraction Expansion Method SOLVE

Find the inverse Z transform of

$$X(z) = \frac{z+1}{3z^2 - 4z + 1}$$
 ROC : $|z| > 1$

Solution:

We have
$$\frac{X(z)}{z} = \frac{z+1}{z(3z^2-4z+1)} = \frac{z+1}{3z[z^2-\frac{4}{3}z+\frac{1}{3}]}$$

$$= \frac{z+1}{3z(z-\frac{1}{3})(z-1)} = \frac{A1}{z} + \frac{A2}{(z-\frac{1}{3})} + \frac{A3}{(z-1)}$$





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Engineering Analysis Z-Transform AUC-CET-24-25-DAWAH تحلبلات هندسبه $A_{1} = \left(\frac{X(z)}{Z}\right) Z \bigg|_{z=0} = \frac{(z+1)z}{3z(z-\frac{1}{2})(z-1)}\bigg|_{z=0} = 1$ $A_{2} = \left(\frac{X(z)}{z}\right)(z - \frac{1}{3}) \bigg|_{z = \frac{1}{3}} = \left|\frac{(z+1)(z - \frac{1}{3})}{3z(z - \frac{1}{3})(z - 1)}\right|_{z = \frac{1}{3}} = -2$

$$A_{3} = \left(\frac{X(z)}{z}\right)(z-1) \left|_{z=1} = \frac{(z+1)(z-1)}{3z(z-\frac{1}{3})(z-1)}\right|_{z=1} = 1$$

$$\therefore \frac{X(z)}{z} = \frac{1}{z} + \frac{-2}{(z-\frac{1}{3})} + \frac{1}{(z-1)}$$

i.e., $X(z) = 1 - \frac{2z}{(z-\frac{1}{3})} + \frac{z}{(z-1)}$ with ROC $|z| > 1$.

Inversion gives the time sequence x(n) as $\mathbf{x}(\mathbf{n}) = \partial(\mathbf{n}) - 2 \cdot \left(\frac{1}{3}\right)^{\mathbf{n}} \mathbf{u}(\mathbf{n}) + \mathbf{u}(\mathbf{n})$

Inversion gives the time sequence x(n) as

$$\mathbf{x}(\mathbf{n}) = \partial(\mathbf{n}) - 2 \cdot \left(\frac{1}{3}\right)^{\mathbf{n}} \mathbf{u}(\mathbf{n}) + \mathbf{u}(\mathbf{n})$$

Inversion gives the time sequence x(n) as

$$\mathbf{x}(\mathbf{n}) = \partial(\mathbf{n}) - 2\left(\frac{1}{3}\right)^{\mathbf{n}} \mathbf{u}(\mathbf{n}) + \mathbf{u}(\mathbf{n})$$

EXAMPLE : FIND THE INVERSE Z TRANSFORM OF

$$\frac{2z}{z-1} + \frac{3z}{z-2}.$$

Solution From Table

$$Z^{-1}\left\{\frac{z}{z-1}\right\} = u_n$$
$$Z^{-1}\left\{\frac{z}{z-2}\right\} = 2^n \quad (r=2)$$

So

$$\mathcal{Z}^{-1}\left\{\frac{2z}{z-1} + \frac{3z}{z-2}\right\} = 2u_n + 3 \times 2^n$$





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Z-Transform تحليلات هندسيه

Engineering Analysis

Example. Find the inverse Z-transform of $\frac{1}{7-2}$

Solution.

$$F(z) = \frac{1}{z-2}$$
Case I. If $\left| \frac{2}{z} \right| < 1$, $F(z) = \frac{1}{z} \frac{1}{1-2z^{-1}}$

$$= z^{-1} (1-2z^{-1})^{-1} = z^{-1} [1+2z^{-1}+2^2z^{-2}+...]$$

$$= z^{-1}+2z^{-2}+2^2z^{-3}+...$$
 $\{f(k)\} = [2^{k-1}], \qquad k \ge 1$

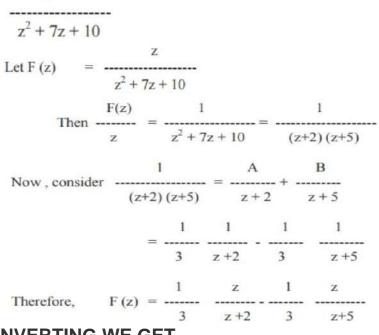
Case II. If
$$\left|\frac{z}{2}\right| < 1$$

 $F(z) = \frac{1}{z-2} = -\frac{1}{2} \frac{|1|}{1-\frac{z}{2}} = \frac{-1}{2} \left(1-\frac{z}{2}\right)^{-1} = -\frac{1}{2} \left[1+\frac{z}{2}+\frac{z^{2}}{2^{2}}+\dots\right]$
 $\{f(k)\} = \{-2^{k-1}\}, k \le 0$
Note The inverse Z-transform can only be settled when region of convergence

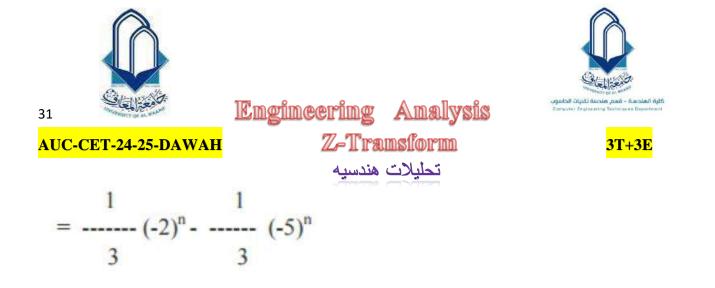
Note. The inverse Z-transform can only be settled when region of convergence (ROC) is given.

Example //

Find the inverse Z – transform of USING Partial Fraction Expansion Method



INVERTING WE GET



CONVOLUTION THEOREM to Find **Inverse Z-Transform**

The inverse Z-transform can be calculated using the convolution theorem. In the convolution integration method, the given Z-transform X(z) is first split into X1(z) and X2(z) such that X(z)=X1(z)X2(z)

The signals x1(n) and x2(n) are then obtained by taking the inverse **Z-transform of X1(z) and X2(z)** respectively. Finally, the function x(n) is obtained by performing the convolution of x1(n) and x2(n) in the time domain

As from the definition of Z-transform of convolution of two signals, we have

$$Z[x_1(n) * x_2(n)] = X_1(z)X_2(z) = X(z)$$

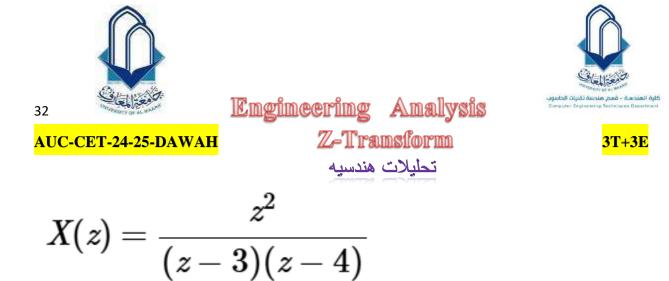
Therefore, the inverse Z-transform is obtained as,

$$x(n) = Z^{-1}[X(z)] = Z^{-1}[Z\{x_1(n) * x_2(n)\}]$$

$$\therefore Z^{-1}[X(z)] = x(n) = x_1(n) * x_2(n) = \sum_{k=0}^n x_1(k) x_2(n-k)$$

<u>Numerical Example</u>

Using the convolution method, find the Z-transform of



$$X(z) = X_1(z)X_2(z) = rac{z}{(z-3)}\;rac{z}{(z-4)}$$

Taking the inverse Z-transform of X1(z) and X2(z) respectively as -

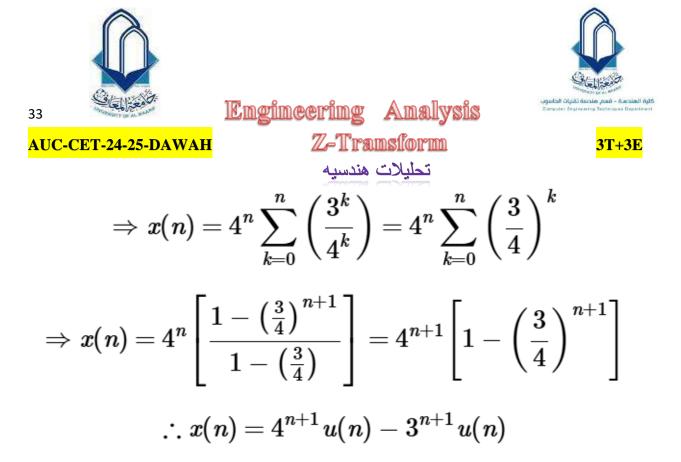
$$Z^{-1}[X_1(z)] = x_1(n) = Z^{-1}igg[rac{z}{(z-3)}igg] = 3^n u(n)$$

Similarly

$$Z^{-1}[X_2(z)] = x_2(n) = Z^{-1}\left[rac{z}{(z-4)}
ight] = 4^n u(n)$$

Now, using the convolution method for finding inverse Ztransform, we have

$$egin{aligned} &Z^{-1}[X(z)] = x(n) = x_1(n) st x_2(n) = \sum_{k=0}^n x_1(k) x_2(n-k) \ &. \ x(n) = \sum_{k=0}^n 3^k u(k) 4^{n-k} u(n-k) = \sum_{k=0}^n 3^k u(k) \left(rac{4^n}{4^k}
ight) u(n-k) \end{aligned}$$



Example/ Using the convolution property of Z-transform, find the Ztransform of the following signal.

$$x(n) = \left(rac{1}{3}
ight)^n u(n) * \left(rac{1}{5}
ight)^n u(n)$$

Solution Let signal is

$$x(n) = x_1(n) \ast x_2(n)$$

$$\therefore x_1(n) = \left(rac{1}{3}
ight)^n u(n)$$

Taking Z-transform, we get,

$$Z[x_1(n)]=X_1(z)=Zigg[igg(rac{1}{3}igg)^nu(n)igg]$$





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$$X_1(z)=rac{z}{\left(z-rac{1}{3}
ight)}; ext{ ROC}
ightarrow |z|>rac{1}{3}$$

Similarly

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$$Z[x_2(n)]=X_2(z)=Zigg[igg(rac{1}{5}igg)^nu(n)igg]$$

$$X_2(z)=rac{z}{\left(z-rac{1}{5}
ight)}; ext{ ROC}
ightarrow |z|>rac{1}{5}$$

Now, using the convolution property of Z-transform

$$\begin{bmatrix} \text{i.e.}, x_1(n) * x_2(n) \stackrel{ZT}{\leftrightarrow} X_1(z) X_2(z) \end{bmatrix}, \text{we get}, \\ Z[x(n)] = X_1(z) X_2(z) \\ \therefore Z\left[\left(\frac{1}{3}\right)^n u(n) * \left(\frac{1}{5}\right)^n u(n)\right] = \frac{z}{\left(z - \frac{1}{3}\right)} \frac{z}{\left(z - \frac{1}{5}\right)}$$

The ROC of the Z-transform of the given sequence is

$$\operatorname{ROC} \to \left\lfloor |z| > \frac{1}{3} \right\rfloor \cap \left\lfloor |z| > \frac{1}{5} \right\rfloor = |z| > \frac{1}{3}$$
ⁿ
(1)ⁿ
₂
₇
₇
₇
₂

$$\therefore \left(rac{1}{3}
ight)^n u(n) st \left(rac{1}{5}
ight)^n u(n) \stackrel{ZT}{\leftrightarrow} rac{z^2}{ig(z-rac{1}{3}ig)ig(z-rac{1}{5}ig)}; ext{ ROC}
ightarrow |z| > rac{1}{3}$$



Example (*Convolution*) Find the inverse **Z**- transform of.

$$\frac{z}{z-1}\frac{z}{z-4}$$

Solution Note that

$$\mathcal{Z}^{-1}\left\{\frac{z}{z-1}\right\} = u_n \quad \text{and} \quad \mathcal{L}^{-1}\left\{\frac{z}{z-4}\right\} = 4^n$$

Hence, using convolution

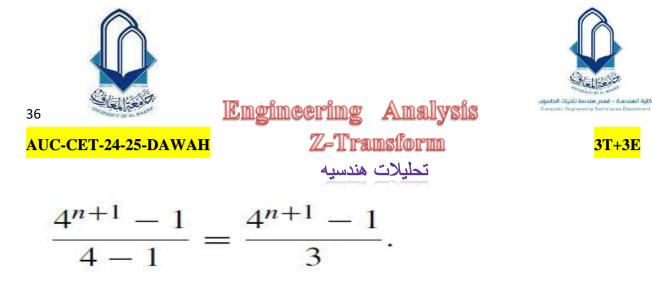
$$\mathcal{Z}^{-1}\left\{\frac{z}{z-1}\frac{z}{z-4}\right\} = u_n * 4^n = \sum_{k=0}^n u_k 4^{n-k}$$

Writing out this sequence for n = 0, 1, 2, 3, ...

1,
$$(1+4)$$
, $1+4+16$, $1+4+16+64$,...
 $(n=0)$ $(n=1)$ $(n=2)$ $(n=3)$

We see that the *n*th term is a geometric series with n + 1 terms and first term 1 and common ratio 4 From the formula for the sum for *n* terms of a geometric progression,

 $Sn = \frac{a(r^n - 1)}{r - 1}$ where *a* is the first term, *r* is the common ratio and *n* is the number of terms. Therefore, for the *n* th term of the above sequence, we get:



So we have found

$$\mathcal{Z}^{-1}\left\{\frac{z}{z-1}\frac{z}{z-4}\right\} = \frac{4^{n+1}-1}{3}.$$

SOLUTION OF DIFFERENCE EQUATIONS

Solution of first order linear constant coefficient difference equations. To solve a difference equation, we have to take the Z - transform of both sides of the difference equation using the property

 $Z\{f_{n+k}\} = z^k\{F(z) - f_0 - (f_1 / z) - \dots - (f_{k-1} / z^{k-1})\} (k > 0)$ Using the initial conditions, we get an algebraic equation of the form $F(z) = \phi(z)$.

By taking the inverse Z-transform, we get the required solution f_n of the given difference equation

Using the given condition, it reduces to



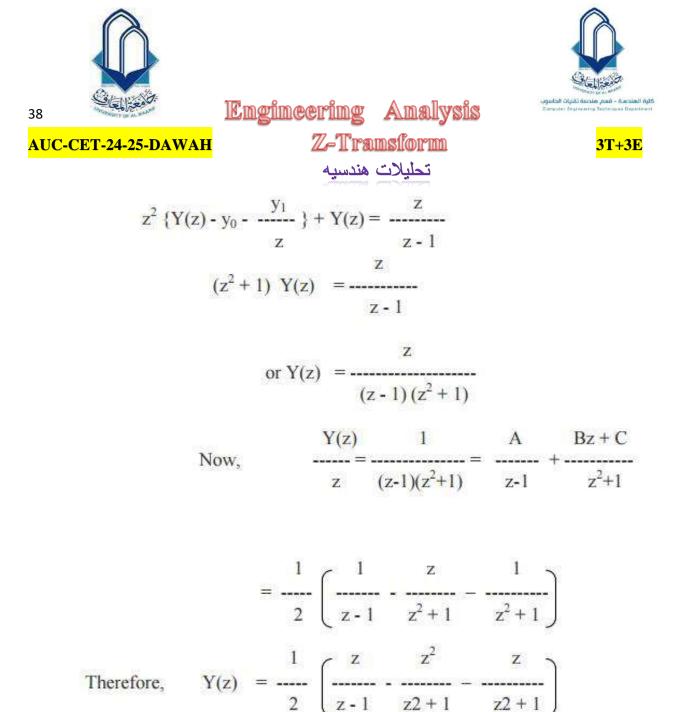
$$(z+1) Y(z) = \frac{z}{z-1}$$

i.e, Y(z) =
$$\frac{z}{(z-1)(z+1)}$$

or Y(z) = $\frac{1}{2} \left\{ \begin{array}{cc} z & z \\ z-1 & z+1 \end{array} \right\}$

On taking inverse Z-transforms, we obtain

 $y_n = (1/2)\{1 - (-1)^n\}$ Example

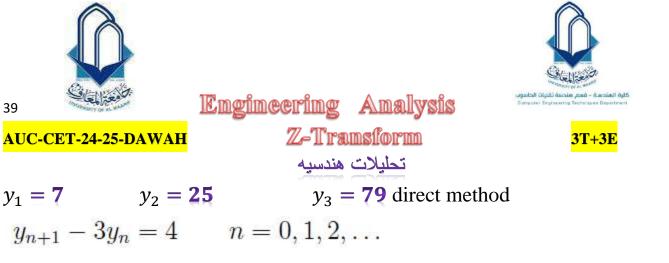


Using Inverse Z-transform, we get $y_n = (\frac{1}{2})\{1 - \cos(\frac{n\pi}{2}) - \sin(\frac{n\pi}{2})\}$ Example: Consider the first order difference equation

 $y_{n+1} - 3y_n = 4$ $n = 0, 1, 2, \dots$

The equation could be solved in a step-by-step or recursive manner, provided that y_0 is known because

 $y_1 = 4 + 3y_0$ $y_2 = 4 + 3y_1$ $y_3 = 4 + 3y_2$ IF WE NOW THAT $y_0 = 1$ THEN



with initial condition $y_0 = 1$

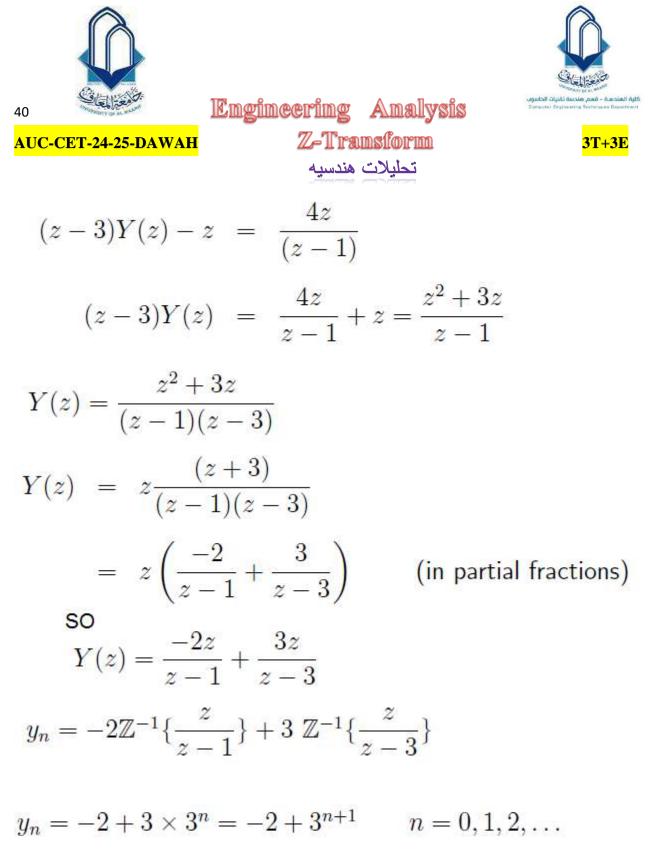
We multiply both sides of (1) by Z^{-n} and sum each side over all positive integer values of n and zero. We obtain

$$\sum_{n=0}^{\infty} (y_{n+1} - 3y_n) z^{-n} = \sum_{n=0}^{\infty} 4z^{-n}$$
OR
$$\sum_{n=0}^{\infty} y_{n+1} z^{-n} - 3 \sum_{n=0}^{\infty} y_n z^{-n} = 4 \sum_{n=0}^{\infty} z^{-n}$$
(2)

The three terms in (2) are clearly recognisable as z-transforms The right-hand side is the z-transform of the constant sequence {4, 4, . . .} which is

$$\frac{4z}{z-1}$$

If $Y(z) = \sum_{n=0}^{\infty} y_n z^{-n}$
$$\sum_{n=0}^{\infty} y_{n+1} z^{-n} = z Y(z) - z y_0 \text{ (by the left shift theorem).}$$
$$z Y(z) - z y_0 - 3 Y(z) = \frac{4z}{z-1}$$



Checking the solution From this solution

$$y_0 = -2 + 3 = 1$$
 (as given)
 $y_1 = -2 + 3^2 = 7$

 $y_{2=25}$, $y_{3=79}$





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Example. Solve the difference equation

$$y_{k+1} - 2y_{k-1} = 0, \quad k \ge 1, \quad y_{(0)} = 1$$

Z-Transform

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Solution.

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 $y_{k+1} - 2y_{k-1} = 0 \qquad \dots \dots (1)$

Engineering Analysis

Taking the Z-transform of both sides of (1), we get

$$Z [y_{k+1} - 2y_{k-1}] = 0$$

$$Z [y_{k+1}] - 2Z [y_{k-1}] = 0$$

$$z Y (z) - y_0 z - 2Y (z) = 0$$
 (y0 = 1

$$(z-2) Y(z) - z = 0$$

$$Y(z) = \frac{z}{z-2}$$

$$\{y_{(k)}\} = Z^{-1} \left[\frac{z}{z-2}\right] = Z^{-1} \left[\frac{1}{1-2z^{-1}}\right]$$

$$= Z^{-1} \left[1-2z^{-1}\right]^{-1} = 1+2z^{-1}+(2z^{-1})^{2}+\dots$$

$$= \{2^{k}\}, \qquad k \ge 0$$

ANS($y_{k=2^k}$) Example Solve the difference equation

$$y_n + 2y_{n-1} = 2u_n$$

for $n \ge 0$ given $y_{-1} = 1$.

Solution We take the z transform of both sides of the difference equation

$$y_n + 2y_{n-1} = 2u_n$$

and using the right shift property to find





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$$\mathcal{Z}\{y_{n-1}\} = z^{-1}Y(z) + y_{-1}$$

$$Y(z) + 2\left(z^{-1}Y(z) + y_{-1}\right) = \frac{2z}{z-1}$$

As $y_{-1} = 1$,

$$Y(z)\left(1+2z^{-1}\right) = \frac{2z}{z-1} - 2$$
$$Y(z) = \frac{2z^2}{(z-1)(z+2)} - \frac{2z}{(z+2)}$$

To take the inverse transform we need to express the first terms using partial fractions. Using the 'cover up' rule we get

$$Y(z) = \frac{2z}{3(z-1)} + \frac{4z}{3(z+2)} - \frac{2z}{(z+2)} = \frac{2z}{3(z-1)} - \frac{2z}{3(z+2)}$$
$$\frac{2z}{(z-1)(z+2)} = \frac{2}{3(z-1)} + \frac{4}{3(z+2)}$$

Taking inverse transforms we find

$$y_n = \frac{2}{3}u_n - \frac{2}{3}(-2)^n$$



Check: To check that we have the correct solution we can substitute in a couple of values for n and see that we get the same value from the difference equation as from the explicit formula found From the explicit formula and using $u_0 = 1$ (by definition of the unit step function),

n = 0 gives

$$y_0 = \frac{2}{3}u_0 - \frac{2}{3}(-2)^0 = \frac{2}{3} - \frac{2}{3} = 0$$

From the difference equation,

$$y_n + 2y_{n-1} = 2u_n$$
, where

 $y_{-1} = 1, n = 0$ gives

 $y_0 + 2y - 1 = 2u_0$

Substituting $y_{-1} = 1$ gives $y_0 = 0$ as before. From the explicit formula,

n = 1 gives

$$y_1 = \frac{2}{3}u_1 - \frac{2}{3}(-2)^1 = \frac{2}{3} + \frac{4}{3} = 2$$

From the difference equation, n = 1 gives

$$y_1 + 2y_0 = 2u_1$$



substituting y0 = 0 gives y1 = 2, confirming the result of the explicit formula.

Example/ Using partial fractions to find the inverse transform

$$Z^{-1}\left\{\frac{z^2}{(z-1)(z-0.5)}\right\}.$$

Solution Notice that most of the values of the transform in Table have a factor of Z in the numerator. We write

$$\frac{z^2}{(z-1)(z-0.5)} = z\left(\frac{z}{(z-1)(z-0.5)}\right)$$

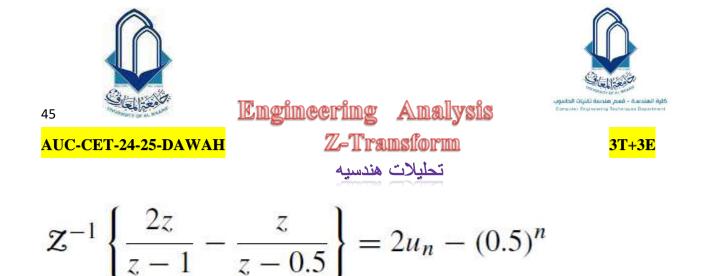
We use the 'cover up' rule to write

$$\frac{z}{(z-1)(z-0.5)} = \frac{1}{0.5(z-1)} - \frac{1}{z-0.5}$$
$$= \frac{2}{z-1} - \frac{1}{z-0.5}$$

So

$$\frac{z^2}{(z-1)(z-0.5)} = \frac{2z}{z-1} - \frac{z}{z-0.5}$$

and using Table we find



The transfer function and impulse response function

$$ay_n + by_{n-1} + cy_{n-2} = f_n.$$

$$Y(z) = \frac{z^2}{az^2 + bz + c}$$

$$H(z) = \frac{z^2}{(az^2 + bz + c)}$$

and $\mathbb{Z}^{-1}{H(z)} = h_n$, where h_n is the impulse response function.

Example Find the transfer function and impulse response of the

system described by the following difference equation:

$$3y_n + 4y_{n-1} = f_n$$

Solution To find the transfer function replace f_n by δn and take the z transform of the resulting equation assuming zero initial conditions: Department of Computer Engineering Techniques, College of Engineering, University of Al Maarif K. DAWAH .ABBAS-2024-2025



$$4y_n + 3y_{n-1} = \delta_n.$$

Taking the z transform of both sides of the equation we get

$$4Y + 3(z^{-1}Y + y_{-1}) = 1.$$

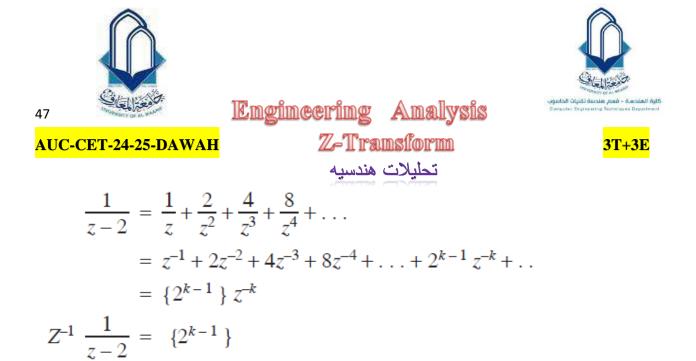
As $y_{-1} = 0$,
$$Y = \frac{z}{4z+3} = H(z) = \frac{z/4}{z+(3/4)}$$

To find the **impulse response sequence** we take the inverse transform of the transfer function to find

$$h_n = \mathbb{Z}^{-1} \left\{ \frac{z/4}{z + \frac{3}{4}} \right\} = \frac{1}{4} \left(-\frac{3}{4} \right)^n$$

INVERSE OF Z-TRANSFORM BY DIVISION

Example Find
$$Z^{-1} \frac{1}{z-2}$$



Example: Find I.Z.T,h[n] using the long division

$$H(z) = \frac{1 + 2z^{-1} - 5z^{-2} + 6z^{-3}}{1 - 3z^{-1} + 2z^{-2}}, \ |z| > 2$$

Solution:

$$\frac{2+3z^{-1}}{1-3z^{-1}+2z^{-2}/1+2z^{-1}-5z^{-2}+6z^{-3}}$$

$$3z^{-1}-9z^{-2}+6z^{-3}$$

$$1-z^{-1}+4z^{-2}$$

$$2-6z^{-1}+4z^{-2}$$

$$-1+5z^{-1}$$

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