# Z-Transformation

## INTRODUCTION

Z-transform plays an important role in **discrete analysis**. Its role in discrete analysis is the same as that of Laplace and Fourier transforms in continuous system. Communication is one of the **field** whose development is based on discrete analysis. Difference equations are also based on discrete system and their solutions and analysis are carried out by Z- transform .

The Z transform is a powerful mathematical tool used in **digital signal processing** and **control systems analysis**. It allows us to **transform signals from the time domain to the frequency domain**, simplifying the analysis and design of digital systems

## SEQUENCE

Sequence **{ *f* (*k*) }** is an ordered list of real or complex numbers.

## REPRESENTATION OF A SEQUENCE

### FIRST METHOD

The elementary way is to list all the members of the sequence such as :

$$\{ f(k) \} = \{\underset{\uparrow}{15}, 10, 7, 4, 1, -1, 0, 3, 6\}$$

The symbol ↑ is used to denote the term in zero position i.e., k = 0,

**k** is an index of position of a term in the sequence.

$$\{g(k)\} = \{15, 10, 7, \underset{\uparrow}{4}, 1, -1, 0, 3, 6\}$$

Two sequences $\{f(k)\}$ and $\{g(k)\}$ have the same terms but these sequences are not identically the same as the zeros term of those sequences are different.

In case the symbol ↑ is not given then left hand end term is considered as the term corresponding to **$K = 0$ .**

### SECOND METHOD

**The second way of specifying the sequence is to define the general term of the sequence $\{f(k)\}$ as function of $k$..**

For example IF, $f(k) = \frac{1}{3^k}$

This sequence represents $\left\{\ldots \frac{1}{3^{-3}}, \frac{1}{3^{-2}}, \frac{1}{3^{-1}}, \underset{\uparrow}{1}, \frac{1}{3^1}, \frac{1}{3^2} \ldots \ldots\right\}$

K=0

## BASIC OPERATIONS ON SEQUENCES

Let $\{f(k)\}$ and $\{g(k)\}$ be two sequences having same number of terms.

**1-Addition**. $\{f(k)\} + \{g(k)\} = \{f(k) + g(k)\}$

**2-Multiplication**. Let $a$ be a scalar, then $a\,\{f(k)\} = \{a\, f(k)\}$

**3-Linearity**. $a\,\{f(k)\} + b\,\{g(k)\} = \{a\, f(k) + b\, g(k)\}$

## EXERCISE

1. Write down the term corresponding to k = 2

$\{6, 7, 5, 1, 0, 4, 6, 8, 10\}$ answer (8)

2 .Write down the term corresponding k = –3

$\{20, 16, 14, 13, 12, 10, 5, 1, 0\}$ answer (14)

3. Write down the sequence $f(k)$ where $\{f(k)= \frac{1}{2^k}\}$

Answer[$f(k)= \frac{1}{2^{-3}}$ , $f(k)= \frac{1}{2^{-2}}$ , $f(k)= \frac{1}{2^{-1}}$ ,1, $f(k)= \frac{1}{2^{1}}$ , $f(k)= \frac{1}{2^{2}}$

$\frac{1}{2^{-3}}$ , $\frac{1}{2^{-2}}$ , $\frac{1}{2^{-1}}$ , 1, $\frac{1}{2^{1}}$ , $\frac{1}{2^{2}}$

4.Write down the sequence {f (k)} where $f(k)= \frac{1}{4^k}$ $\{ -3< k > 4\}$

Answer

$\{ f(k)= \frac{1}{4^{-3}}$ , $f(k)= \frac{1}{4^{-2}}$ , $f(k)= \frac{1}{4^{-1}}$ ,1, $f(k)= \frac{1}{4^{1}}$ , $f(k)= \frac{1}{4^{2}}$ , $f(k)= \frac{1}{4^{3}}\}$

## Arithmetic sequence

**defined by** $\mathbf{y_n = y_1 + (n-1)d}$

**Example/Calculate the $4^{th}$ term of the arithmetic sequence defined by $y_{n+1} - y_n$ = 2,, $y_1$ = 9. Write out the first 4 terms of this sequence explicitly. Suggest why an arithmetic sequence is also known as a linear sequence.**

**Answer We have, using (2),**

**yn = 9 + (n − 1)2 or**

**yn = 2n + 7 so**

**y1 = 9 (as given), y2 = 11, y3 = 13, y4 = 15, . . .**

**Example/Calculate the $12^{th}$ term of the arithmetic sequence defined by 5, 11. 17 , 23 ,29?**

Sol/the different between 5 and 11 is=6,and between 11 and 17 is=6 ….d=6

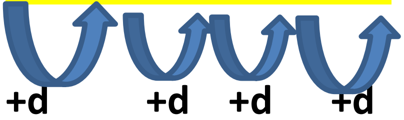
$$a_n = a_1 + (n - 1)d$$

$$a_{12} = 5 + (12 - 1)6 \quad =71$$

Example/Calculate the $3^{th}$ term and the $7^{th}$ term of the arithmetic sequence defined by $a_{3=17} and a_{7=45}$ what is the value of the $14^{th}$ term of the arithmetic sequence

Sol/ $a_3$=17 ,, $a_7$=45 ,, $a_{14}$ =?

$$a3, a4, a5, a6, a7$$



$a_{7=}a_3$+4d ……45=17+4d ……..d=7

$a_3$=a1+2d………17=a1+2d =a1+14 …….a1=3

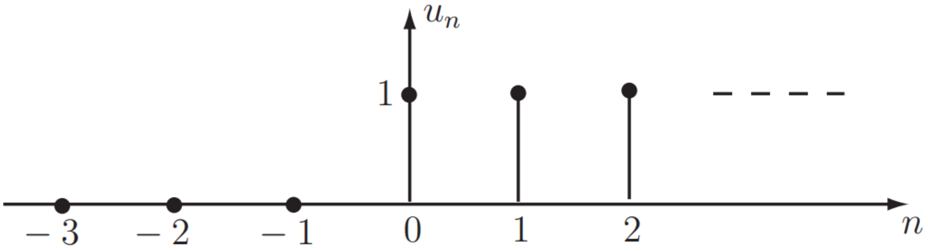
$$a_n = a_1 + (n - 1)d$$

$$a_{14} = 3 + (14 - 1)3 \quad =94$$

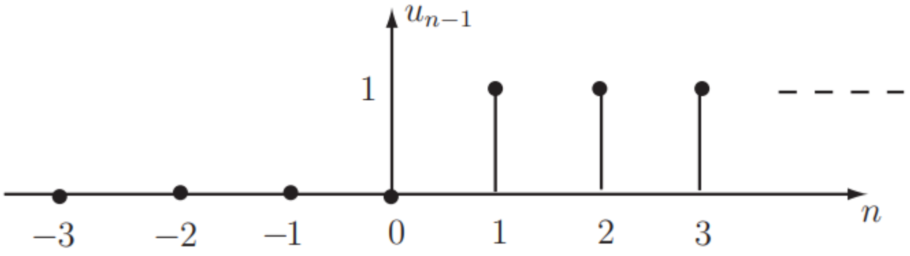
A sequence which is zero for negative integers n is sometimes called a causal sequence. For example the sequence, denoted by {un},

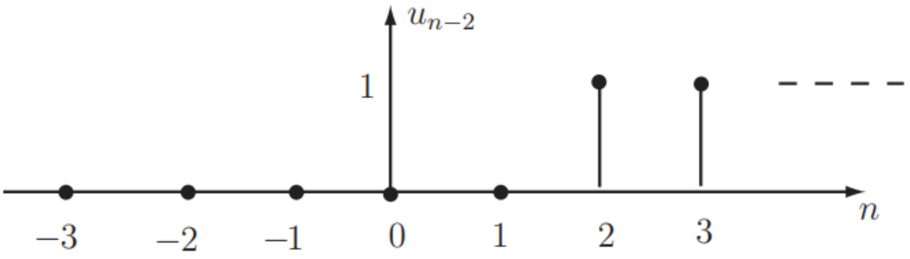
$$u_n = \begin{cases} 0 & n = -1, -2, -3, \\ 1 & n = 0, 1, 2, 3, \dots \end{cases}$$

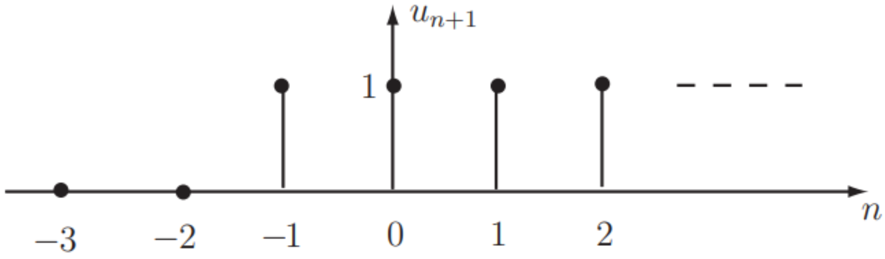
is causal. Figure 4 makes it clear why {un} is called the unit step sequence.



**EXAMPLE/Draw graphs of the sequences {u(n−1)}, {u(n−2)}, {u(n+1)} where {u(n)} is the unit step sequence.**







For example the sequence $\{y_n\} = \{n^2\} \quad n = 0, \pm 1, \pm 2, \ldots$ could be written

$$\{\ldots 9, 4, 1, 0, 1, 4, 9, \ldots\}$$

## Z-TRANSFORM

Definition.The Z- transform of a sequence {f (k)} is denoted as **Z [{f (k)}] .** It is defined as

$$Z\ [\{f(k)\}] = \ F(z) = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k} = \sum_{k=-\infty}^{\infty} \frac{f(k)}{z^k}$$

Where

1. Z is a complex number.

2. Z is an operator of Z-transform

3. F (z) is the Z transform of {f (k)}.

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$z = Ae^{j\phi} = A \cdot (\cos\phi + j\sin\phi)$$

where $A$ is the magnitude of $z$, $j$ is the imaginary unit, and $\phi$ is the *complex argument* (also referred to as *angle* or *phase*) in radians.

# Z – Transform of some time sequences

**1) Right side sequences** As an example, let us find the z-transform and ROC of the right sided sequence **ROC** **Region of Convergence**

$$x(n) = (\underset{\uparrow}{1}, 2, 2, 1)$$

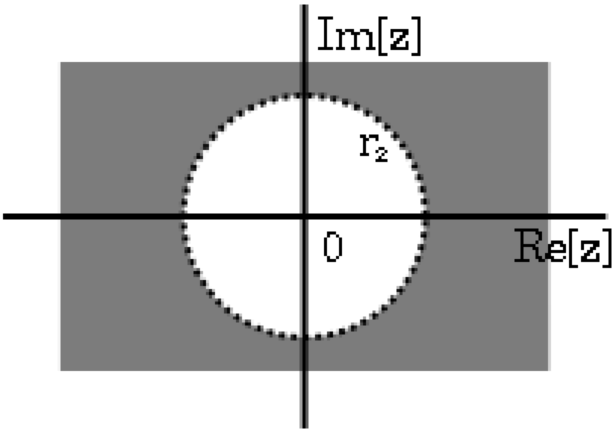
$$X(z) = Z\{x(n)\} = \sum_{n=0}^{3} x(n)z^{-n} = x(0)z^{-0} + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3}$$

$$= 1z^{-0} + 2z^{-1} + 2z^{-2} + 1z^{-3}$$

$$= 1 + 2z^{-1} + 2z^{-2} + z^{-3}$$

We see that X(z) becomes infinity at z =0. Except at z = 0, X(z) is finite for all values of z. Therefore we can say that the ROC of this z transform is the entire z-plane except z = 0. **ie.,**

**ROC : |z| > 0.**



**The ROC of a right-sided sequence**

## 2) Left sided sequences:

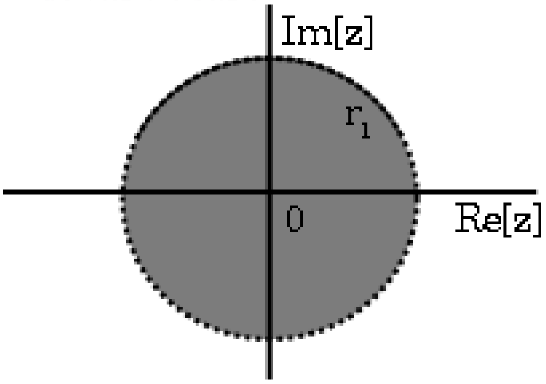
**Let us find the z-transform and ROC of the left sided sequence**

$$x(n) = (1\,,1,\;2\,,\underset{\uparrow}{2})$$

$$X(z) = Z\{x(n)\} = \sum_{n=-3}^{0} x(n)z^{-n} = x(-3)z^{3} + x(-2)z^{2} + x(-1)z^{1} + x(0)z^{0}$$

$$= z^{3} + z^{2} + 2z + 2$$

**We see that X(z) becomes infinity at z = ∞.. Except at z = ∞, X(z) is finite for all values of z. Therefore we can say that the ROC of this z transform is the entire z-plane except z = ∞ ie., ROC : |z| < ∞**



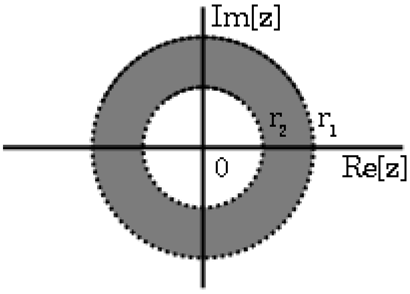
**The ROC of a left-sided sequence.**

## 3 ) Double sided sequences:

**A sequence x(n) is said to be double sided if x(n) has both right and left sides. For example, x(n) = ( 2, 1, 1, 2 ) is a double sided sequence because x(n) exists in the range -2 ≤ n ≤ 1. Z transform of this sequence is given by**

$$X(z) = Z\{x(n)\} = \sum_{n=-2}^{1} x(n)z^{-n} = x(-2)z^{2} + x(-1)z^{1} + x(0)z^{0} + x(1)z^{-1}$$

$$= 2z^{2} + 1z^{1} + 1z^{0} + 2z^{-1}$$



**The ROC of a two-sided sequence.**

Example $\quad If\ g(k) = \{15,\ 10,\ 7,\ 4,\ 1,\ -1,\ 0,\ \underset{\uparrow}{3},\ 6\}$

$$Z[\{g(k)\}] = F(z) = 15z^{7} + 10z^{6} + 7z^{5} + 4z^{4} + z^{3} - z^{2} + 0 + 3 + \frac{6}{z}.$$

Example If $f(k) = \{15,\ 10,\ 7,\ \underset{\uparrow}{4},\ 1,\ -1,\ 0,\ 3,\ 6\}$ then

$$Z[\{f(k)\}] = F(z) = 15z^3 + 10z^2 + 7z + 4 + \frac{1}{z} - \frac{1}{z^2} + 0 + \frac{3}{z^4} + \frac{6}{z^5}$$

Example If $f(k) = \frac{1}{3^k}$, $-4 \le k \le 3$, then

$$Z[\{f(k)\} = 81z^4 + 27z^3 + 9z^2 + 3z + 1 + \frac{1}{3z} + \frac{1}{9z^2} + \frac{1}{27z^3}$$

Example Find Z-transform of the sequence $\left\{\frac{1}{2^k}\right\}$, $-4 \le k \le 4$.

Solution. $F(z) = \sum_{k=-4}^{4} \frac{1}{2^k} z^{-k} = 16z^4 + 8z^3 + 4z^2 + 2z + 1 + \frac{1}{2z} + \frac{1}{4z^2} + \frac{1}{8z^3} + \frac{1}{16z^4}$

Example . Find Z-transform of the sequence $\{a^k\}$, $k \ge 0$.

Solution. $F(z) = \sum_{k=0}^{\infty} a^k z^{-k} = 1 + \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} + \ ....$

This is a Geometrical series whose sum $= \frac{a}{1-r}$

$$= \frac{1}{1 - \frac{a}{z}} = \frac{z}{z-a}$$

**Example** **Find the *z* transform of the finite sequence 1, 0, 0.5, 3.**
***Solution*** **We multiply the terms in the sequence by $z^{-n}$, where *n* =0, 1, 2,. and then sum the terms, giving**

$$F(z) = 1 + 0Z^{-1} + 0.5Z^{-2} + 3Z^{-3}$$

$$= 1 + \frac{0.5}{Z^2} + \frac{3}{Z^3} \quad \text{ans}$$

**Example**: **Write the z-transform for a finite sequence given below.**
**x = {-2, -1, 1, 2, 3, 4, 5}**

**Solution:**

**Given sequence of sample numbers x[n]= is x = {−2, −1, 1, 2, 3, 4, 5}**
**z-transform of x[n] can be written as:**

$$X(z) = -2z^{0} - z^{-1} + z^{-2} + 2z^{-3} + 3z^{-4} + 4z^{-5} + 5z^{-6}$$

**This can be further simplified as below.**

$$X(z) = -2 - z^{-1} + z^{-2} + 2z^{-3} + 3z^{-4} + 4z^{-5} + 5z^{-6}$$

# PROPERTIES OF Z-TRANSFORMS

## Linearity

**Theorem 1: If {f (k)} and {g (k)} are such that they can be added and a and b are constants,then**

**Z {a f (k) + b g (k)} = a Z [{f (k)}] + b Z [{g (k)}]**

**Example:** Write the z-transform of the following power series

| | |
|---|---|
| $f(x) = \begin{cases} a^k, & k \geq 0 \\ 0, & k < 0 \end{cases}$ | It can be expressed using z-transform as:<br>$F(z) = \sum_{k=0}^{\infty} a^k z^{-k}$<br>$= \sum_{k=0}^{\infty} (az^{-1})^k$<br>$= \frac{1}{1-az^{-1}}$<br>$= \frac{z}{z-a}$ |

## Example . Find the Z transform of {f (k)} where

$$f(k) = \begin{cases} 5^k , & k < 0 \\ 3^k , & k \geq 0 \end{cases}$$

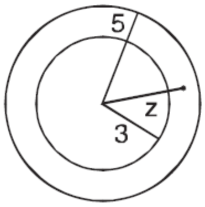
**Solution.** $Z[\{f(k)\}] = \sum_{k=-\infty}^{-1} 5^k z^{-k} + \sum_{k=0}^{\infty} 3^k z^{-k}$

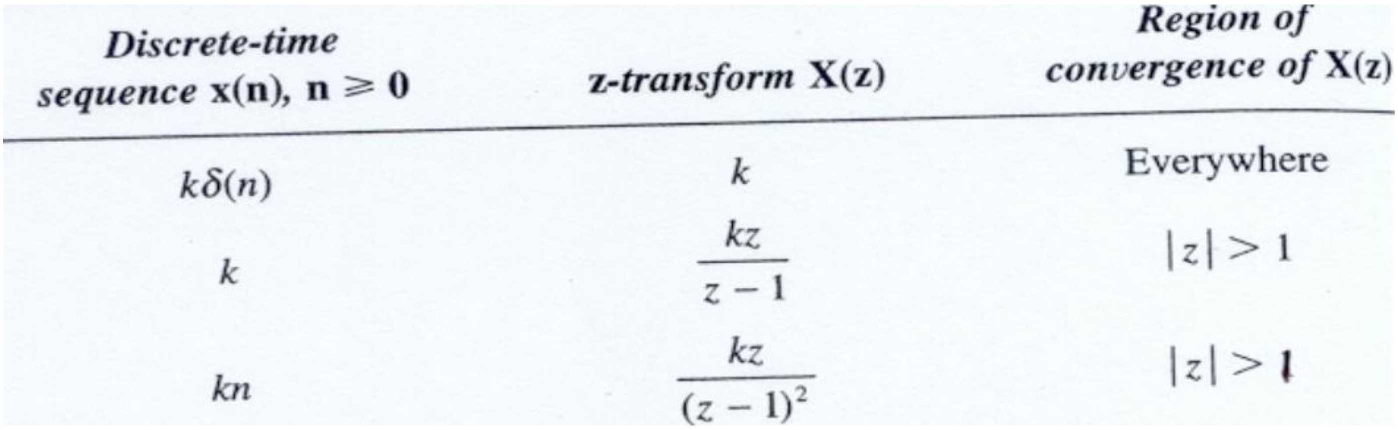
$$= [\,.\,.\,.\,. + 5^{-3} z^3 + 5^{-2} z^2 + 5^{-1} z^1] + \left[1 + \frac{3}{z^{-1}} + \frac{9}{z^{-2}} + \frac{27}{z^{-3}} + .\,.\,.\,.\right] \quad [\text{G.P.}]$$

$$= \frac{5^{-1} z}{1 - 5^{-1} z} + \frac{1}{1 - \frac{3}{z^{-1}}} = \frac{z}{5 - z} + \frac{z}{z - 3} \qquad [\text{G.P.}] \qquad \left[S = \frac{a}{1 - r}\right]$$

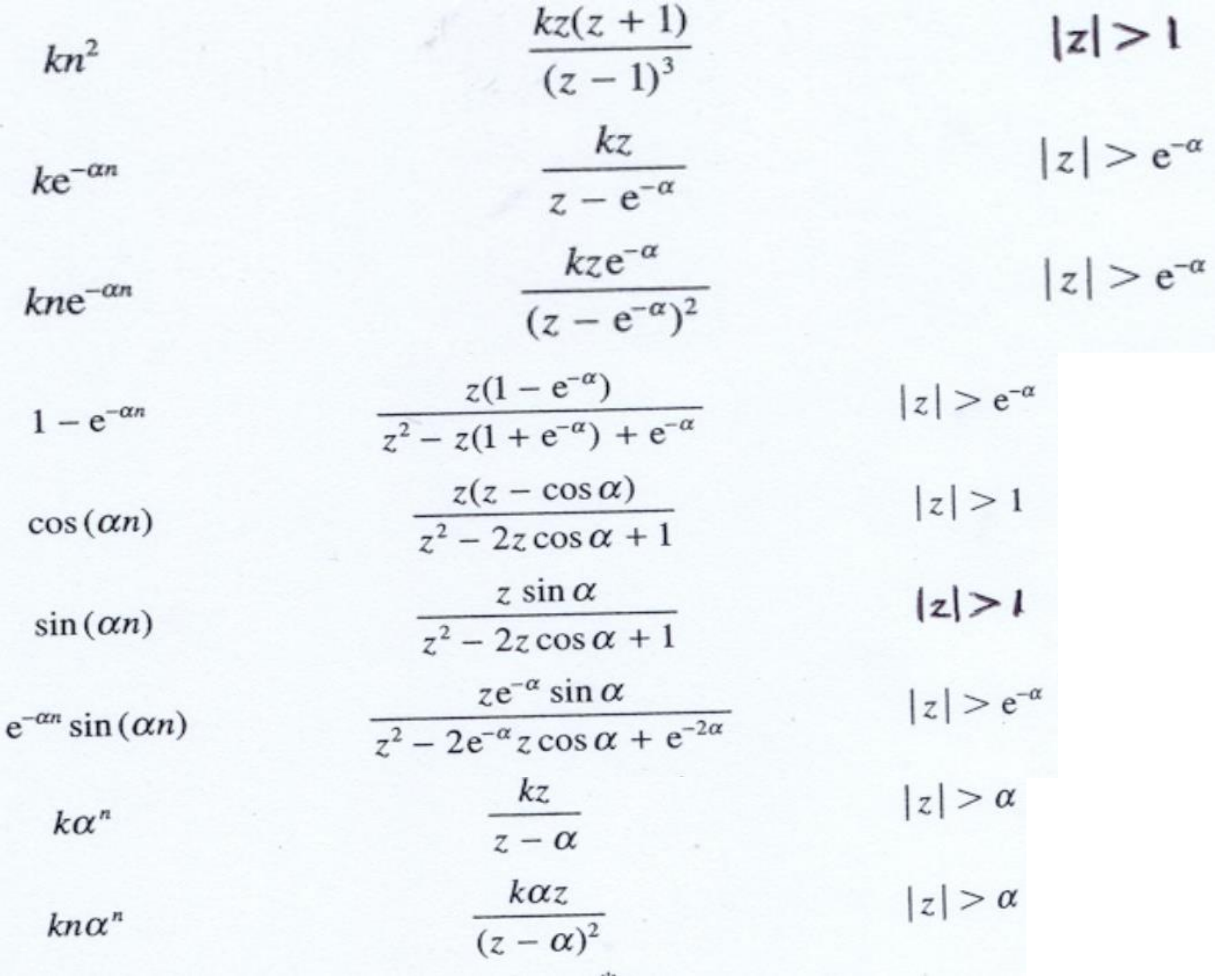
$$= \frac{z^2 - 3z + 5z - z^2}{(5 - z)(z - 3)} = \frac{-2z}{z^2 - 8z + 15} \qquad \left|\frac{z}{5}\right| < 1 , \ \left|\frac{3}{z}\right| < 1$$

Two series are convergent in annulus. Here $3 < |z|$ and $|z| < 5$. **Ans.**

| | |
|---|---|
|  | Reign Of Convergence (**ROC**) |



| *Discrete-time sequence* x(n), n ≥ 0 | *z-transform* X(z) | *Region of convergence of* X(z) |
|---|---|---|
| $k\delta(n)$ | $k$ | Everywhere |
| $k$ | $\frac{kz}{z-1}$ | $\lvert z\rvert > 1$ |
| $kn$ | $\frac{kz}{(z-1)^2}$ | $\lvert z\rvert > 1$ |



| | | |
|---|---|---|
| $kn^2$ | $\frac{kz(z+1)}{(z-1)^3}$ | $\lvert z\rvert > 1$ |
| $ke^{-\alpha n}$ | $\frac{kz}{z-e^{-\alpha}}$ | $\lvert z\rvert > e^{-\alpha}$ |
| $kne^{-\alpha n}$ | $\frac{kze^{-\alpha}}{(z-e^{-\alpha})^2}$ | $\lvert z\rvert > e^{-\alpha}$ |
| $1-e^{-\alpha n}$ | $\frac{z(1-e^{-\alpha})}{z^2-z(1+e^{-\alpha})+e^{-\alpha}}$ | $\lvert z\rvert > e^{-\alpha}$ |
| $\cos(\alpha n)$ | $\frac{z(z-\cos\alpha)}{z^2-2z\cos\alpha+1}$ | $\lvert z\rvert > 1$ |
| $\sin(\alpha n)$ | $\frac{z\sin\alpha}{z^2-2z\cos\alpha+1}$ | $\lvert z\rvert > 1$ |
| $e^{-\alpha n}\sin(\alpha n)$ | $\frac{ze^{-\alpha}\sin\alpha}{z^2-2e^{-\alpha}z\cos\alpha+e^{-2\alpha}}$ | $\lvert z\rvert > e^{-\alpha}$ |
| $k\alpha^n$ | $\frac{kz}{z-\alpha}$ | $\lvert z\rvert > \alpha$ |
| $kn\alpha^n$ | $\frac{k\alpha z}{(z-\alpha)^2}$ | $\lvert z\rvert > \alpha$ |

**Example** (*Linearity*) Find the **z** transform of $3n + 2 \times 3n$.

*Solution* From the linearity property

$$\mathcal{Z}\{3n + 2 \times 3^n\} = 3\mathcal{Z}\{n\} + 2\mathcal{Z}\{3^n\}$$

and from the Table

$$\mathcal{Z}\{n\} = \frac{z}{(z-1)^2} \quad \text{and} \quad \mathcal{Z}\{3^n\} = \frac{z}{z-3}$$

($r^n$ with $r = 3$). Therefore

$$\mathcal{Z}\{3n + 2 \times 3^n\} = \frac{3z}{(z-1)^2} + \frac{2z}{z-3}$$

**Example.** *Find the Z-transform of* **,** $[\frac{1}{2}]^{|k|}$

**Solution.** $$Z\left[\left\{\left(\frac{1}{2}\right)^{|k|}\right\}\right] = \sum \left(\frac{1}{2}\right)^{|k|} z^{-k} = \sum_{k=-\infty}^{-1} \left(\frac{1}{2}\right)^{-k} z^{-k} + \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{-k} z^{-k}$$

$$= \left( \ldots\ldots + \frac{z^4}{16} + \frac{z^3}{8} + \frac{z^2}{4} + \frac{z}{2} \right) + \left( 1 + \frac{1}{2z} + \frac{1}{4z^2} + \frac{1}{8z} + \ldots \right)$$

These infinite series are G.P, and sum of a G.P. $= \dfrac{a}{1-r}$

$$= \frac{\frac{z}{2}}{1 - \frac{z}{2}} + \frac{1}{1 - \frac{1}{2z}} \qquad = \frac{z}{2-z} + \frac{2\,z}{2\,z-1}$$

$$= \frac{2z^2 - z + 4z - 2z^2}{(2-z)\;(2z-1)} \qquad = \frac{3\,z}{(2-z)\;(2z-1)}$$

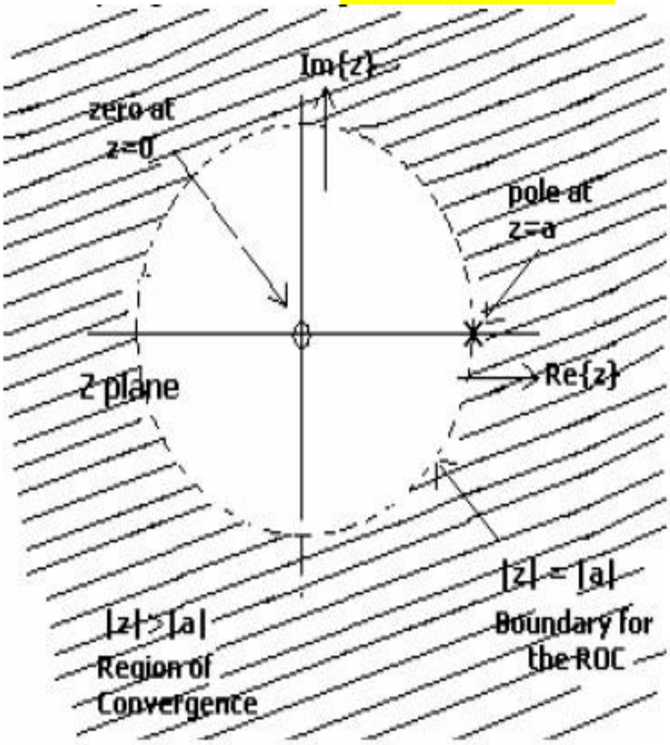
# Poles and zeros of the Z transform

Values of z for which X(z) = 0 are called the **zeros** of X(z). A zero is indicated by a ‘**O**’ in the z plane. Values of z for which X(z) = ∞ are called the **poles** of X(z). A pole is indicated by a ‘**X**’ in the z plane.

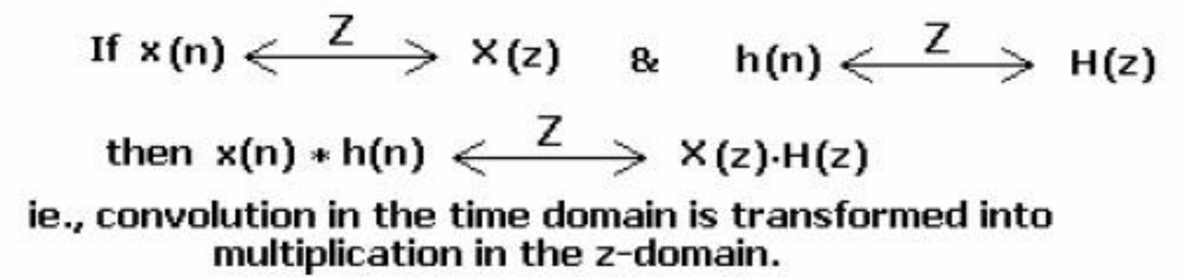
For example, in the previous example, where

**$X(z) = z / (z-a)$**

**The z transform has one zero at z=0 and one pole at z=a.**



ROC for the Z transform of $a^n u(n)$



If $x(n) \xleftrightarrow{Z} X(z)$ & $h(n) \xleftrightarrow{Z} H(z)$

then $x(n) * h(n) \xleftrightarrow{Z} X(z) \cdot H(z)$

ie., convolution in the time domain is transformed into multiplication in the z-domain.

(where * denotes convolution)

## Z- TRANSFORMS OF SOME USEFUL SEQUENCES:

**1) A) Unit impulse $\delta(n)$:**

$$x(n) = \delta(n)$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = \sum_{n=-\infty}^{\infty} \delta(n)z^{-n} = \delta(0)z^{-0} = 1$$ with ROC : the entire z-plane.

**ie., $\delta(n) \xleftrightarrow{z} 1$ with ROC : the entire z-plane**

**B) $x(n) = \delta(n-n_0)$, where $n_0$ is positive.**

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = \sum_{n=-\infty}^{\infty} \delta(n-n_0)z^{-n} = 1.z^{-n}{}_0 = z^{-n}{}_0$$ (because $\delta(n-n_0) = 1$ at $n = n_0$)

with ROC : the entire z-plane except z =0. ie., ROC : $|z| > 0$

**ie., $\delta(n-n_0) \xleftrightarrow{z} z^{-n}{}_0$ with ROC : $|z| > 0$**

**C) $x(n) = \delta(n+n_0)$, where $n_0$ is positive.**

$$X(z) = \sum_{n=-\propto}^{\propto} x(n)z^{-n} = \sum_{n=-\propto}^{\propto} \delta(n+n_0)z^{-n} = 1.z^{n}_{0} = z^{n}_{0} \text{ (because } \delta(n+n_0) = 1 \text{ at } n = -n_0)$$

with ROC : the entire z-plane except z =. ∝ ie., ROC : $|z| < \propto$

$$\textbf{ie., } \delta(n+n_0) \overset{Z}{<\!\!-\!\!-\!\!-\!\!-\!\!>} z^{n}_{0} \quad \textbf{with ROC : } |z| < \propto$$

**Exercise**: Repeat (B) and (C) using the appropriate property of the Z Transform

**2)** $x(n) = a^{n}\, u(n)$

**Example.** *Find the Z-transform of* **UNIT IMPULSE**

$$\delta(k) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

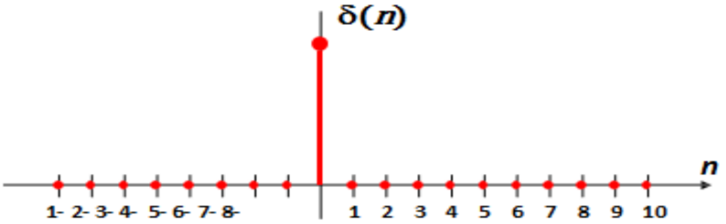
SOLUTION

$$Z[\{f(k)\}] = \sum_{k=-\infty}^{\infty} \delta(k)\, z^{-k}$$

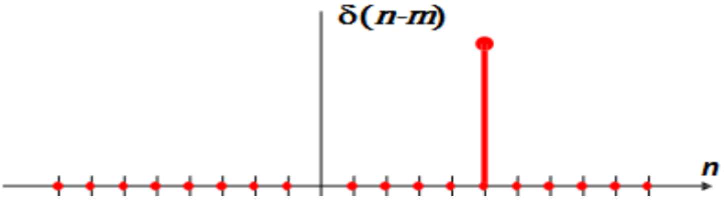
$$= [\ldots + 0 + 0 + 0 + 1 + 0 + 0 + \ldots\ldots]$$

❑ **delta function or unit-impulse (sample) sequence $\delta(n)$**

$$\boxed{\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}}$$



$$\boxed{\delta(n\text{-}m) = \begin{cases} 1 & n = \mathrm{m} \\ 0 & n \neq \mathrm{m} \end{cases}}$$



## Example . Find the Z-transform of discrete UNIT STEP

$$U(k) = \begin{cases} 0 & k < 0 \\ 1 & k \geq 0 \end{cases}$$

SOLUTION

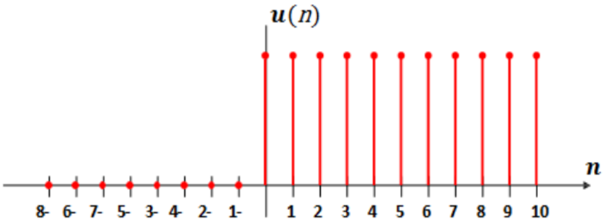
$$Z\,[\{\,U(k)\,\}] = \sum_{k=0}^{\infty} U(k)\, z^{-k} = [\,1 + z^{-1} + z^{-2} + z^{-3} + \ldots\,]$$

G.P. its sum is $\frac{a}{1-r}$.

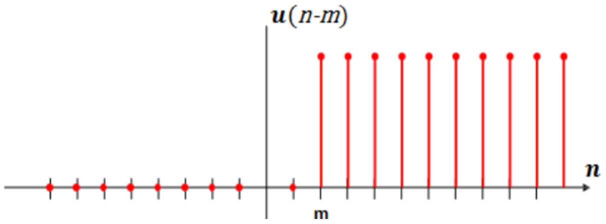
$$= \frac{1}{1 - z^{-1}} = \frac{1}{1 - \frac{1}{z}} = \frac{z}{z-1}$$

- **unit-step sequence** ***U(n)***

$$\boxed{u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}}$$



$$\boxed{u(n\text{-}m) = \begin{cases} 1 & n \geq \mathrm{m} \\ 0 & n < \mathrm{m} \end{cases}}$$

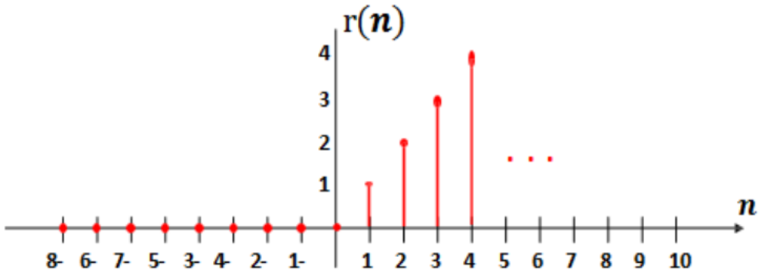


$$u(n) = \sum_{m=0}^{\infty} \delta(n-m)$$

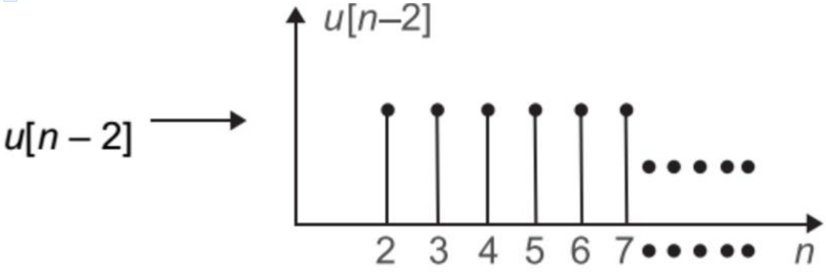
$$\delta(n) = u(n) - u(n-1)$$

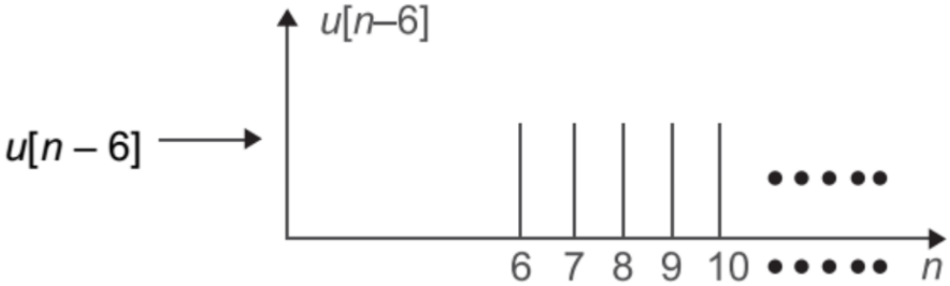
❑ **unit-ramp sequence *r(n)***

$$r(n) = n\,u(n) = \begin{cases} n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

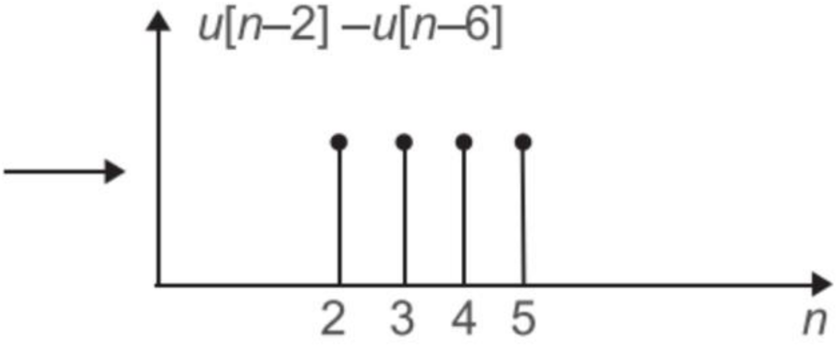


**Example** ***Find***





u[n – 2] – u[n – 6]



**Example. Find the Z-transform of** $\frac{a^k}{k!}$

SOLUTION

$$Z\left[\left\{\frac{a^k}{k\,!}\right\}\right] = \sum_{k=0}^{\infty} \frac{a^k}{k\,!}\, z^{-k}$$

$$= \sum_{k=0}^{\infty} \frac{(a\,z^{-1})^k}{k\,!} = 1 + \frac{a\,z^{-1}}{1!} + \frac{(a\,z^{-2})^2}{2!} + \frac{(a\,z^{-1})^3}{3!} + \ldots$$

## CHANGE OF SCALE

**Theorem.** If $\quad Z[\{f(k)\}] = F(z) \quad$ then $\quad Z\left[\{a^k\, f(k)\}\right] = F\left(\frac{z}{a}\right)$

**Example**. *Find the Z-transform of* $(a^k)\ k > 0$.

**Solution. We know that**

$$Z[\{1\}] = \frac{z}{z-1}$$

For the given sequence, by the scale change formula the Z-transform

$$Z\left[\{a^k \,.\, 1\}\right] = \frac{\frac{z}{a}}{\frac{z}{a} - 1} = \frac{z}{z-a}$$

## SHIFTING PROPERTY

**Theorem.** If $\quad Z[\{f(k)\}] = F(z)\,,$

$$Z[\{f(k \pm n)\}] = z^{\pm n}\, F(z)$$

## CLASSIFICATION OF SYSTEM CAUSALAND NON CAUSAL

**1/ IF THE OUTPUT ≥ INPUT → CAUSAL**

**2/ IF THE OUTPUT < INPUT → NONCAUSAL**

**3/ IN ANY STEP OF THE SOLUTION INPUT > OUTPUT**

**STOP THE SOLUTION AND THE SYSTEM IS NON CAUSAL**

| | |
|---|---|
| EX-y(t)=x(t)<br>y(0)=x(0)<br>y(1)=x(1)<br>y(-1)=x(-1) causal | EX-y(t)=x(2t)<br>y(0)=x(0)<br>y(1)=x(2)<br>y(-1)=x(-2) NONcausal |

| | |
|---|---|
| EX-y(t)=x(t)+x(t-2)<br>y(0)=x(0)+x(-2)<br>y(1)=x(1)+x(-1)<br>y(-1)=x(-1)+x (-3) causal | EX-y(t)=x(t-4) (t+4)<br>y(0)=x(-4) (4)<br>y(1)=x(3) (5)<br>y(-1)=x(-5) (3) noncausal |

For casual sequence

$$Z[\{f(k-1)\}] = z^{-1} F(z) \text{ as } f(-1) = 0$$

$$Z[\{f(k+1)\}] = z F(z) - z f(0)$$

$$Z[\{f(k+2)\}] = z^2 F(z) - z^2 f(0) - z f(1)$$

## Difference Equations

$$Dy_n = y_{n+1} - y_n,$$

$$D^2y_n = y_{n+2} - 2y_{n+1} + y_n.$$

$$D^3y_n = y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n \text{ and so on}$$

**Example** **Form the difference equation for the Fibonacci sequence .**

**The integers 0,1,1,2,3,5,8,13,21, . . . are said to form a Fibonacci sequence.**

**If $y_n$ be the nth term of this sequence, then**

0+1=1+1=2+1=3+2=5+3=8+5=13+8=21…….

DAWAH **Fibonacci sequence**

$$y_n = y_{n-1} + y_{n-2} \text{ for } n > 2$$

$$\text{or } \; y_{n+2} - y_{n+1} - y_n = 0 \text{ for } n > 0$$

**Fibonacci numbers, commonly denoted $F_n$. The sequence commonly starts from 0 and 1, although some authors start the sequence from 1 and 1 or sometimes (as did Fibonacci) from 1 and 2. Starting from 0 and 1, the sequence begins0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144**

## THEOREM

**If $\{f(k)\} = F(z)$, $\{g(k)\} = G(z)$, and $a$ and $b$ are constant,**

**Example/ Find the Z–transform of**

1. **n(n-1)**
2. **$n^2 + 7n + 4$**
3. **(1/2)( n+1)(n+2)**

$$\text{(i) } Z\{\, n(n-1)\} = Z\{n^2\} - Z\{n\}$$

$$= \frac{z(z+1)}{(z-1)^3} - \frac{z}{(z-1)^2}$$

$$= \frac{z(z+1) - z(z-1)}{(z-1)^3}$$

$$= \frac{2z}{(z-1)^3}$$

([redacted]) $Z\{n^2+7n+4\} = Z\{n^2\} + 7\,Z\{n\} + 4\,Z\{1\}$

$$= \frac{z(z+1)}{(z-1)^3} + 7\,\frac{z}{(z-1)^2} + 4\,\frac{z}{z-1}$$

$$= \frac{z\{(z+1)+7(z-1)+4(z-1)^2\}}{(z-1)^3}$$

$$= \frac{2z(z^2-2)}{(z-1)^3}$$

(iii) $Z\left\{\frac{(n+1)(n+2)}{2}\right\} = \frac{1}{2}\{Z\{n^2\} + 3Z\{n\} + 2Z\{1\}\}$

$$= \frac{1}{2}\left\{\frac{z(z+1)}{(z-1)^3} + \frac{3z}{(z-1)^2} + \frac{2z}{(z-1)}\right\} \text{ if } |z| > 1$$

$$= \frac{z^3}{(z-1)^3}$$

## Example

**Show that $Z\{1/n!\} = e^{1/z}$ and hence find $Z\{1/(n+1)!\}$ and $Z\{1/(n+2)!\}$**

$$Z\left\{\frac{1}{n!}\right\} = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{(z^{-1})^n}{n!}$$

$$= 1 + \frac{z^{-1}}{1!} + \frac{(z^{-1})^2}{2!} + \ldots$$

$$= e^{z^{-1}} = e^{1/z}$$

To find $Z\left\{\frac{1}{(n+1)!}\right\}$

**We know that $Z\{f_{n+1}\} = z\{F(z) - f_0\}$**

**Therefore,**

$$Z\left\{\frac{1}{(n+1)!}\right\} = z\left\{Z\left\{\frac{1}{n!}\right\} - 1\right\}$$

$$= z\{e^{1/z} - 1\}$$

Similarly,

$$Z\left\{\frac{1}{(n+2)!}\right\} = z^2\{e^{1/z} - 1 - (1/z)\}.$$

## INVERSE Z-TRANSFORM

Finding the sequence $\{f(k)\}$ from $F(z)$ is defined as inverse $Z$-transform. It is denoted as

$Z^{-1}F(z) = \{f(k)\}$ $Z^{-1}$ is the inverse Z-transform.

## (LINEARITY AND THE INVERSE TRANSFORM)

- There are three methods for finding the inverse Z-transform ie., finding the time sequence x[n] given its Z-transform:
- (1) Long Division Method (Power Series expansion method)
- (2) Partial Fraction Expansion Method
- (3) Complex inversion integral method
- We will study the first two methods only

## (1) Long Division Method (Power Series expansion method)

The Z transform of a sequence x(n) is given by

$$X(z)=\sum_{n=-\infty}^{\infty} x(n)\,z^{-n}=\ldots\ldots+x(-2)z^{2}+x(-1)z+x(0)z^{0}+x(1)z^{-1}+x(2)z^{-2}+\ldots\ldots\ldots$$

Therefore if X(z) can be expanded as a power series, the coefficients represent the inverse sequence values. However, the solution is not obtained **as a closed** form expression. So this method is used when we require the first few numerical values of the inverse z transform of X(z).

For right sided sequences, X(z) will have only negative exponents, and for left sided sequences, X(z) will have only positive exponents.

This method is illustrated by the following

**Example: Find the inverse z transform by division :**

$$X(z)=\frac{z}{3z^{2}-4z+1} \quad \text{for ROCs} \quad \text{(a)} \ |z|>1 \quad \text{(b)} \ |z|<1/3$$

a- |z|> 1 indicates a right sided sequence. So we must divide the numerator by the denominator by long division, to get negative powers of z in the quotient;

$$
\begin{array}{r|l}
 & \frac{1}{3}z^{-1}+\frac{4}{9}z^{-2}+\frac{13}{27}z^{-3}+\ldots\ldots\ldots\ldots \\
\hline
3z^{2}-4z+1 & z \\
 & \underline{z-\frac{4}{3}+\frac{1}{3}z^{-1}} \\
 & \quad \frac{4}{3}-\frac{1}{3}z^{-1} \\
 & \quad \underline{\frac{4}{3}-\frac{16}{9}z^{-1}+\frac{4}{9}z^{-2}} \\
 & \qquad \frac{13}{9}z^{-1}-\frac{4}{9}z^{-2} \\
 & \qquad \ldots\ldots\ldots\ldots\ldots \\
 & \qquad\qquad \ldots\ldots\ldots\ldots\ldots
\end{array}
$$

ie.,

$$X(z)=\frac{1}{3}z^{-1}+\frac{4}{9}z^{-2}+.\frac{13}{27}z^{-3}+\ldots\ldots\ldots$$

$$\therefore x(n)=[\underset{\uparrow}{0},\frac{1}{3},\frac{4}{9},\frac{13}{27},\ldots\ldots\ldots\ldots]$$

**EXAMPLE**: find the **I.Z.T** using long division ?

$$
\begin{array}{r|l}
 & 1+Z^{-1}-Z^{-3} \\
\hline
Z^{2}-Z+1 & Z^{2} \\
 & \underline{Z^{2}-Z+1} \\
 & Z-1 \\
 & \underline{Z-1+Z^{-1}} \\
 & -Z^{-1} \\
 & \underline{-Z^{-1}+Z^{-2}-Z^{-3}} \\
 & -Z^{-2}+Z^{-3}
\end{array}
$$

**EXAMPLE**: find the **I.Z.T** using long division ?

$$
\begin{array}{r|l}
 & \underline{z+4z^{2}+13z^{3}+\ldots\ldots\ldots\ldots} \\
1-4z+3z^{2} & z \\
 & \underline{z-4z^{2}+3z^{3}} \\
 & \quad 4z^{2}-3z^{3} \\
 & \quad \underline{4z^{2}-16z^{3}+12z^{4}} \\
 & \qquad\qquad 13z^{3}-12z^{4} \\
 & \qquad\qquad \ldots\ldots\ldots\ldots\ldots \\
 & \qquad\qquad\qquad \ldots\ldots\ldots\ldots\ldots
\end{array}
$$

ie.,

$$X(z)=z+4z^{2}+13z^{3}+\ldots\ldots\ldots$$

$$\therefore x(n)=[\ldots\ldots\ldots 13,4,1,\underset{\uparrow}{0}]$$

**EXAMPLE**: find the **I.Z.T** using long division ?

$$F(z)=\frac{z}{z-0.5}$$

$$
\begin{array}{r|l}
 & \underline{1+0.5z^{-1}+0.25z^{-2}+\cdots} \\
z-0.5 & z \\
 & \underline{z-0.5} \\
 & \quad 0.5 \\
 & \quad \underline{0.5-0.25z^{-1}} \\
 & \qquad 0.25z^{-1} \\
 & \qquad 0.25z^{-1}-0.125z^{-2}
\end{array}
$$