



THOMAS' Twelfth Edition
CALCULUS
Early Transcendentals

THOMAS' CALCULUS EARLY TRANSCENDENTALS

Twelfth Edition

Based on the original work by

George B. Thomas, Jr.

Massachusetts Institute of Technology

as revised by

Maurice D. Weir

Naval Postgraduate School

Joel Hass

University of California, Davis

Addison-Wesley

Boston Columbus Indianapolis New York San Francisco Upper Saddle River
Amsterdam Cape Town Dubai London Madrid Milan Munich Paris Montréal Toronto
Delhi Mexico City São Paulo Sydney Hong Kong Seoul Singapore Taipei Tokyo

Editor-in-Chief: Deirdre Lynch
Senior Acquisitions Editor: William Hoffman
Senior Project Editor: Rachel S. Reeve
Associate Editor: Caroline Celano
Associate Project Editor: Leah Goldberg
Senior Managing Editor: Karen Wernholm
Senior Production Project Manager: Sheila Spinney
Senior Design Supervisor: Andrea Nix
Digital Assets Manager: Marianne Groth
Media Producer: Lin Mahoney
Software Development: Mary Durnwald and Bob Carroll
Executive Marketing Manager: Jeff Weidenaar
Marketing Assistant: Kendra Bassi
Senior Author Support/Technology Specialist: Joe Vetere
Senior Prepress Supervisor: Caroline Fell
Manufacturing Manager: Evelyn Beaton
Production Coordinator: Kathy Diamond
Composition: Nesbitt Graphics, Inc.
Illustrations: Karen Heyt, IllustraTech
Cover Design: Rokusek Design

Cover image: Forest Edge, Hokuto, Hokkaido, Japan 2004 © Michael Kenna

About the cover: The cover image of a tree line on a snow-swept landscape, by the photographer Michael Kenna, was taken in Hokkaido, Japan. The artist was not thinking of calculus when he composed the image, but rather, of a visual haiku consisting of a few elements that would spark the viewer's imagination. Similarly, the minimal design of this text allows the central ideas of calculus developed in this book to unfold to ignite the learner's imagination.

For permission to use copyrighted material, grateful acknowledgment is made to the copyright holders on page C-1, which is hereby made part of this copyright page.

Many of the designations used by manufacturers and sellers to distinguish their products are claimed as trademarks. Where those designations appear in this book, and Addison-Wesley was aware of a trademark claim, the designations have been printed in initial caps or all caps.

Library of Congress Cataloging-in-Publication Data

Weir, Maurice D.

Thomas' calculus : early transcendentals / Maurice D. Weir, Joel Hass, George B. Thomas.—12th ed.
p. cm

Includes index.

ISBN 978-0-321-58876-0

1. Calculus—Textbooks. 2. Geometry, Analytic—Textbooks. I. Hass, Joel. II. Thomas, George B. (George Brinton), 1914–2006. III. Title IV. Title: Calculus.

QA303.2.W45 2009

515—dc22

2009023070

Copyright © 2010, 2006, 2001 Pearson Education, Inc. All rights reserved. No part of this publication may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photocopying, recording, or otherwise, without the prior written permission of the publisher. Printed in the United States of America. For information on obtaining permission for use of material in this work, please submit a written request to Pearson Education, Inc., Rights and Contracts Department, 501 Boylston Street, Suite 900, Boston, MA 02116, fax your request to 617-848-7047, or e-mail at <http://www.pearsoned.com/legal/permissions.htm>.

1 2 3 4 5 6 7 8 9 10—CRK—12 11 10 09

Addison-Wesley
is an imprint of



www.pearsoned.com

ISBN-10: 0-321-58876-2

ISBN-13: 978-0-321-58876-0

CONTENTS

Preface

ix

1

Functions

1

- 1.1 Functions and Their Graphs 1
- 1.2 Combining Functions; Shifting and Scaling Graphs 14
- 1.3 Trigonometric Functions 22
- 1.4 Graphing with Calculators and Computers 30
- 1.5 Exponential Functions 34
- 1.6 Inverse Functions and Logarithms 40
- QUESTIONS TO GUIDE YOUR REVIEW 52
- PRACTICE EXERCISES 53
- ADDITIONAL AND ADVANCED EXERCISES 55

2

Limits and Continuity

58

- 2.1 Rates of Change and Tangents to Curves 58
- 2.2 Limit of a Function and Limit Laws 65
- 2.3 The Precise Definition of a Limit 76
- 2.4 One-Sided Limits 85
- 2.5 Continuity 92
- 2.6 Limits Involving Infinity; Asymptotes of Graphs 103
- QUESTIONS TO GUIDE YOUR REVIEW 116
- PRACTICE EXERCISES 117
- ADDITIONAL AND ADVANCED EXERCISES 119

3

Differentiation

122

- 3.1 Tangents and the Derivative at a Point 122
- 3.2 The Derivative as a Function 126

3.3	Differentiation Rules	135
3.4	The Derivative as a Rate of Change	145
3.5	Derivatives of Trigonometric Functions	155
3.6	The Chain Rule	162
3.7	Implicit Differentiation	170
3.8	Derivatives of Inverse Functions and Logarithms	176
3.9	Inverse Trigonometric Functions	186
3.10	Related Rates	192
3.11	Linearization and Differentials	201
	QUESTIONS TO GUIDE YOUR REVIEW	212
	PRACTICE EXERCISES	213
	ADDITIONAL AND ADVANCED EXERCISES	218

4**Applications of Derivatives****222**

4.1	Extreme Values of Functions	222
4.2	The Mean Value Theorem	230
4.3	Monotonic Functions and the First Derivative Test	238
4.4	Concavity and Curve Sketching	243
4.5	Indeterminate Forms and L'Hôpital's Rule	254
4.6	Applied Optimization	263
4.7	Newton's Method	274
4.8	Antiderivatives	279
	QUESTIONS TO GUIDE YOUR REVIEW	289
	PRACTICE EXERCISES	289
	ADDITIONAL AND ADVANCED EXERCISES	293

5**Integration****297**

5.1	Area and Estimating with Finite Sums	297
5.2	Sigma Notation and Limits of Finite Sums	307
5.3	The Definite Integral	313
5.4	The Fundamental Theorem of Calculus	325
5.5	Indefinite Integrals and the Substitution Method	336
5.6	Substitution and Area Between Curves	344
	QUESTIONS TO GUIDE YOUR REVIEW	354
	PRACTICE EXERCISES	354
	ADDITIONAL AND ADVANCED EXERCISES	358

6**Applications of Definite Integrals****363**

6.1	Volumes Using Cross-Sections	363
6.2	Volumes Using Cylindrical Shells	374
6.3	Arc Length	382

6.4	Areas of Surfaces of Revolution	388
6.5	Work and Fluid Forces	393
6.6	Moments and Centers of Mass	402
	QUESTIONS TO GUIDE YOUR REVIEW	413
	PRACTICE EXERCISES	413
	ADDITIONAL AND ADVANCED EXERCISES	415

7

Integrals and Transcendental Functions**417**

7.1	The Logarithm Defined as an Integral	417
7.2	Exponential Change and Separable Differential Equations	427
7.3	Hyperbolic Functions	436
7.4	Relative Rates of Growth	444
	QUESTIONS TO GUIDE YOUR REVIEW	450
	PRACTICE EXERCISES	450
	ADDITIONAL AND ADVANCED EXERCISES	451

8

Techniques of Integration**453**

8.1	Integration by Parts	454
8.2	Trigonometric Integrals	462
8.3	Trigonometric Substitutions	467
8.4	Integration of Rational Functions by Partial Fractions	471
8.5	Integral Tables and Computer Algebra Systems	481
8.6	Numerical Integration	486
8.7	Improper Integrals	496
	QUESTIONS TO GUIDE YOUR REVIEW	507
	PRACTICE EXERCISES	507
	ADDITIONAL AND ADVANCED EXERCISES	509

9

First-Order Differential Equations**514**

9.1	Solutions, Slope Fields, and Euler's Method	514
9.2	First-Order Linear Equations	522
9.3	Applications	528
9.4	Graphical Solutions of Autonomous Equations	534
9.5	Systems of Equations and Phase Planes	541
	QUESTIONS TO GUIDE YOUR REVIEW	547
	PRACTICE EXERCISES	547
	ADDITIONAL AND ADVANCED EXERCISES	548

10

Infinite Sequences and Series**550**

10.1	Sequences	550
10.2	Infinite Series	562

10.3	The Integral Test	571
10.4	Comparison Tests	576
10.5	The Ratio and Root Tests	581
10.6	Alternating Series, Absolute and Conditional Convergence	586
10.7	Power Series	593
10.8	Taylor and Maclaurin Series	602
10.9	Convergence of Taylor Series	607
10.10	The Binomial Series and Applications of Taylor Series	614
	QUESTIONS TO GUIDE YOUR REVIEW	623
	PRACTICE EXERCISES	623
	ADDITIONAL AND ADVANCED EXERCISES	625

11

Parametric Equations and Polar Coordinates

628

11.1	Parametrizations of Plane Curves	628
11.2	Calculus with Parametric Curves	636
11.3	Polar Coordinates	645
11.4	Graphing in Polar Coordinates	649
11.5	Areas and Lengths in Polar Coordinates	653
11.6	Conic Sections	657
11.7	Conics in Polar Coordinates	666
	QUESTIONS TO GUIDE YOUR REVIEW	672
	PRACTICE EXERCISES	673
	ADDITIONAL AND ADVANCED EXERCISES	675

12

Vectors and the Geometry of Space

678

12.1	Three-Dimensional Coordinate Systems	678
12.2	Vectors	683
12.3	The Dot Product	692
12.4	The Cross Product	700
12.5	Lines and Planes in Space	706
12.6	Cylinders and Quadric Surfaces	714
	QUESTIONS TO GUIDE YOUR REVIEW	719
	PRACTICE EXERCISES	720
	ADDITIONAL AND ADVANCED EXERCISES	722

13

Vector-Valued Functions and Motion in Space

725

13.1	Curves in Space and Their Tangents	725
13.2	Integrals of Vector Functions; Projectile Motion	733
13.3	Arc Length in Space	742
13.4	Curvature and Normal Vectors of a Curve	746
13.5	Tangential and Normal Components of Acceleration	752
13.6	Velocity and Acceleration in Polar Coordinates	757

QUESTIONS TO GUIDE YOUR REVIEW	760
PRACTICE EXERCISES	761
ADDITIONAL AND ADVANCED EXERCISES	763

14

Partial Derivatives

765

14.1	Functions of Several Variables	765
14.2	Limits and Continuity in Higher Dimensions	773
14.3	Partial Derivatives	782
14.4	The Chain Rule	793
14.5	Directional Derivatives and Gradient Vectors	802
14.6	Tangent Planes and Differentials	809
14.7	Extreme Values and Saddle Points	820
14.8	Lagrange Multipliers	829
14.9	Taylor's Formula for Two Variables	838
14.10	Partial Derivatives with Constrained Variables	842
	QUESTIONS TO GUIDE YOUR REVIEW	847
	PRACTICE EXERCISES	847
	ADDITIONAL AND ADVANCED EXERCISES	851

15

Multiple Integrals

854

15.1	Double and Iterated Integrals over Rectangles	854
15.2	Double Integrals over General Regions	859
15.3	Area by Double Integration	868
15.4	Double Integrals in Polar Form	871
15.5	Triple Integrals in Rectangular Coordinates	877
15.6	Moments and Centers of Mass	886
15.7	Triple Integrals in Cylindrical and Spherical Coordinates	893
15.8	Substitutions in Multiple Integrals	905
	QUESTIONS TO GUIDE YOUR REVIEW	914
	PRACTICE EXERCISES	914
	ADDITIONAL AND ADVANCED EXERCISES	916

16

Integration in Vector Fields

919

16.1	Line Integrals	919
16.2	Vector Fields and Line Integrals: Work, Circulation, and Flux	925
16.3	Path Independence, Conservative Fields, and Potential Functions	938
16.4	Green's Theorem in the Plane	949
16.5	Surfaces and Area	961
16.6	Surface Integrals	971

16.7	Stokes' Theorem	980
16.8	The Divergence Theorem and a Unified Theory	990
	QUESTIONS TO GUIDE YOUR REVIEW	1001
	PRACTICE EXERCISES	1001
	ADDITIONAL AND ADVANCED EXERCISES	1004

17**Second-Order Differential Equations****online**

17.1	Second-Order Linear Equations
17.2	Nonhomogeneous Linear Equations
17.3	Applications
17.4	Euler Equations
17.5	Power Series Solutions

Appendices**AP-1**

A.1	Real Numbers and the Real Line	AP-1
A.2	Mathematical Induction	AP-6
A.3	Lines, Circles, and Parabolas	AP-10
A.4	Proofs of Limit Theorems	AP-18
A.5	Commonly Occurring Limits	AP-21
A.6	Theory of the Real Numbers	AP-23
A.7	Complex Numbers	AP-25
A.8	The Distributive Law for Vector Cross Products	AP-35
A.9	The Mixed Derivative Theorem and the Increment Theorem	AP-36

Answers to Odd-Numbered Exercises**A-1****Index****I-1****Credits****C-1****A Brief Table of Integrals****T-1**



PREFACE

We have significantly revised this edition of *Thomas' Calculus: Early Transcendentals* to meet the changing needs of today's instructors and students. The result is a book with more examples, more mid-level exercises, more figures, better conceptual flow, and increased clarity and precision. As with previous editions, this new edition provides a modern introduction to calculus that supports conceptual understanding but retains the essential elements of a traditional course. These enhancements are closely tied to an expanded version of MyMathLab[®] for this text (discussed further on), providing additional support for students and flexibility for instructors.

In this twelfth edition early transcendentals version, we introduce the basic transcendental functions in Chapter 1. After reviewing the basic trigonometric functions, we present the family of exponential functions using an algebraic and graphical approach, with the natural exponential described as a particular member of this family. Logarithms are then defined as the inverse functions of the exponentials, and we also discuss briefly the inverse trigonometric functions. We fully incorporate these functions throughout our developments of limits, derivatives, and integrals in the next five chapters of the book, including the examples and exercises. This approach gives students the opportunity to work early with exponential and logarithmic functions in combinations with polynomials, rational and algebraic functions, and trigonometric functions as they learn the concepts, operations, and applications of single-variable calculus. Later, in Chapter 7, we revisit the definition of transcendental functions, now giving a more rigorous presentation. Here we define the natural logarithm function as an integral with the natural exponential as its inverse.

Many of our students were exposed to the terminology and computational aspects of calculus during high school. Despite this familiarity, students' algebra and trigonometry skills often hinder their success in the college calculus sequence. With this text, we have sought to balance the students' prior experience with calculus with the algebraic skill development they may still need, all without undermining or derailing their confidence. We have taken care to provide enough review material, fully stepped-out solutions, and exercises to support complete understanding for students of all levels.

We encourage students to think beyond memorizing formulas and to generalize concepts as they are introduced. Our hope is that after taking calculus, students will be confident in their problem-solving and reasoning abilities. Mastering a beautiful subject with practical applications to the world is its own reward, but the real gift is the ability to think and generalize. We intend this book to provide support and encouragement for both.

Changes for the Twelfth Edition

CONTENT In preparing this edition we have maintained the basic structure of the Table of Contents from the eleventh edition, yet we have paid attention to requests by current users and reviewers to postpone the introduction of parametric equations until we present polar coordinates. We have made numerous revisions to most of the chapters, detailed as follows:

- **Functions** We condensed this chapter to focus on reviewing function concepts and introducing the transcendental functions. Prerequisite material covering real numbers, intervals, increments, straight lines, distances, circles, and parabolas is presented in Appendices 1–3.
- **Limits** To improve the flow of this chapter, we combined the ideas of limits involving infinity and their associations with asymptotes to the graphs of functions, placing them together in the final section of Chapter 3.
- **Differentiation** While we use rates of change and tangents to curves as motivation for studying the limit concept, we now merge the derivative concept into a single chapter. We reorganized and increased the number of related rates examples, and we added new examples and exercises on graphing rational functions. L'Hôpital's Rule is presented as an application section, consistent with our early coverage of the transcendental functions.
- **Antiderivatives and Integration** We maintain the organization of the eleventh edition in placing antiderivatives as the final topic of Chapter 4, covering applications of derivatives. Our focus is on “recovering a function from its derivative” as the solution to the simplest type of first-order differential equation. Integrals, as “limits of Riemann sums,” motivated primarily by the problem of finding the areas of general regions with curved boundaries, are a new topic forming the substance of Chapter 5. After carefully developing the integral *concept*, we turn our attention to its evaluation and connection to antiderivatives captured in the Fundamental Theorem of Calculus. The ensuing applications then *define* the various geometric ideas of area, volume, lengths of paths, and centroids, all as limits of Riemann sums giving definite integrals, which can be evaluated by finding an antiderivative of the integrand. We return later to the topic of solving more complicated first-order differential equations.
- **Differential Equations** Some universities prefer that this subject be treated in a course separate from calculus. Although we do cover solutions to separable differential equations when treating exponential growth and decay applications in Chapter 7 on integrals and transcendental functions, we organize the bulk of our material into two chapters (which may be omitted for the calculus sequence). We give an introductory treatment of first-order differential equations in Chapter 9, including a new section on systems and phase planes, with applications to the competitive-hunter and predator-prey models. We present an introduction to second-order differential equations in Chapter 17, which is included in MyMathLab as well as the *Thomas' Calculus: Early Transcendentals* Web site, www.pearsonhighered.com/thomas.
- **Series** We retain the organizational structure and content of the eleventh edition for the topics of sequences and series. We have added several new figures and exercises to the various sections, and we revised some of the proofs related to convergence of power series in order to improve the accessibility of the material for students. The request stated by one of our users as, “anything you can do to make this material easier for students will be welcomed by our faculty,” drove our thinking for revisions to this chapter.
- **Parametric Equations** Several users requested that we move this topic into Chapter 11, where we also cover polar coordinates and conic sections. We have done this, realizing that many departments choose to cover these topics at the beginning of Calculus III, in preparation for their coverage of vectors and multivariable calculus.
- **Vector-Valued Functions** We streamlined the topics in this chapter to place more emphasis on the conceptual ideas supporting the later material on partial derivatives, the gradient vector, and line integrals. We condensed the discussions of the Frenet frame and Kepler's three laws of planetary motion.
- **Multivariable Calculus** We have further enhanced the art in these chapters, and we have added many new figures, examples, and exercises. We reorganized the opening material on double integrals, and we combined the applications of double and triple integrals to masses and moments into a single section covering both two- and three-dimensional cases. This reorganization allows for better flow of the key mathematical concepts, together with their properties and computational aspects. As with the

eleventh edition, we continue to make the connections of multivariable ideas with their single-variable analogues studied earlier in the book.

- **Vector Fields** We devoted considerable effort to improving the clarity and mathematical precision of our treatment of vector integral calculus, including many additional examples, figures, and exercises. Important theorems and results are stated more clearly and completely, together with enhanced explanations of their hypotheses and mathematical consequences. The area of a surface is now organized into a single section, and surfaces defined implicitly or explicitly are treated as special cases of the more general parametric representation. Surface integrals and their applications then follow as a separate section. Stokes' Theorem and the Divergence Theorem are still presented as generalizations of Green's Theorem to three dimensions.

EXERCISES AND EXAMPLES We know that the exercises and examples are critical components in learning calculus. Because of this importance, we have updated, improved, and increased the number of exercises in nearly every section of the book. There are over 700 new exercises in this edition. We continue our organization and grouping of exercises by topic as in earlier editions, progressing from computational problems to applied and theoretical problems. Exercises requiring the use of computer software systems (such as *Maple*[®] or *Mathematica*[®]) are placed at the end of each exercise section, labeled **Computer Explorations**. Most of the applied exercises have a subheading to indicate the kind of application addressed in the problem.

Many sections include new examples to clarify or deepen the meaning of the topic being discussed and to help students understand its mathematical consequences or applications to science and engineering. At the same time, we have removed examples that were a repetition of material already presented.

ART Because of their importance to learning calculus, we have continued to improve existing figures in *Thomas' Calculus: Early Transcendentals*, and we have created a significant number of new ones. We continue to use color consistently and pedagogically to enhance the conceptual idea that is being illustrated. We have also taken a fresh look at all of the figure captions, paying considerable attention to clarity and precision in short statements.

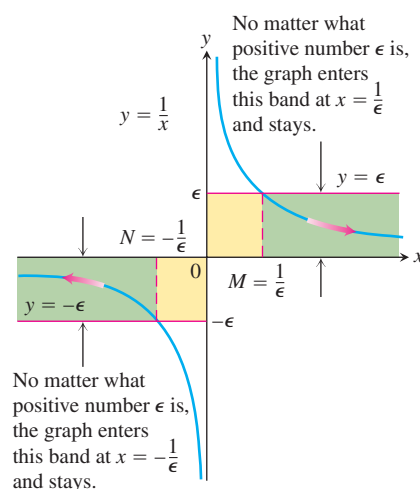


FIGURE 2.50, page 104 The geometric explanation of a finite limit as $x \rightarrow \pm\infty$.

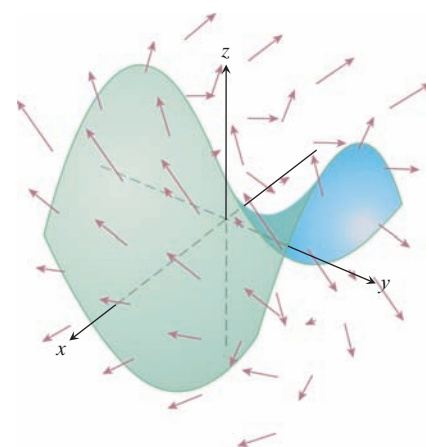


FIGURE 16.9, page 926 A surface in a space occupied by a moving fluid.

MYMATHLAB AND MATHXL The increasing use of and demand for online homework systems has driven the changes to MyMathLab and MathXL[®] for *Thomas' Calculus*:

Early Transcendentals. The **MyMathLab** course now includes significantly more exercises of all types.

Continuing Features

RIGOR The level of rigor is consistent with that of earlier editions. We continue to distinguish between formal and informal discussions and to point out their differences. We think starting with a more intuitive, less formal, approach helps students understand a new or difficult concept so they can then appreciate its full mathematical precision and outcomes. We pay attention to defining ideas carefully and to proving theorems appropriate for calculus students, while mentioning deeper or subtler issues they would study in a more advanced course. Our organization and distinctions between informal and formal discussions give the instructor a degree of flexibility in the amount and depth of coverage of the various topics. For example, while we do not prove the Intermediate Value Theorem or the Extreme Value Theorem for continuous functions on $a \leq x \leq b$, we do state these theorems precisely, illustrate their meanings in numerous examples, and use them to prove other important results. Furthermore, for those instructors who desire greater depth of coverage, in Appendix 6 we discuss the reliance of the validity of these theorems on the completeness of the real numbers.

WRITING EXERCISES Writing exercises placed throughout the text ask students to explore and explain a variety of calculus concepts and applications. In addition, the end of each chapter contains a list of questions for students to review and summarize what they have learned. Many of these exercises make good writing assignments.

END-OF-CHAPTER REVIEWS AND PROJECTS In addition to problems appearing after each section, each chapter culminates with review questions, practice exercises covering the entire chapter, and a series of Additional and Advanced Exercises serving to include more challenging or synthesizing problems. Most chapters also include descriptions of several **Technology Application Projects** that can be worked by individual students or groups of students over a longer period of time. These projects require the use of a computer running *Mathematica* or *Maple* and additional material that is available over the Internet at www.pearsonhighered.com/thomas and in MyMathLab.

WRITING AND APPLICATIONS As always, this text continues to be easy to read, conversational, and mathematically rich. Each new topic is motivated by clear, easy-to-understand examples and is then reinforced by its application to real-world problems of immediate interest to students. A hallmark of this book has been the application of calculus to science and engineering. These applied problems have been updated, improved, and extended continually over the last several editions.

TECHNOLOGY In a course using the text, technology can be incorporated according to the taste of the instructor. Each section contains exercises requiring the use of technology; these are marked with a **T** if suitable for calculator or computer use, or they are labeled **Computer Explorations** if a computer algebra system (CAS, such as *Maple* or *Mathematica*) is required.

Text Versions

THOMAS' CALCULUS: EARLY TRANSCENDENTALS, Twelfth Edition

Complete (Chapters 1–16), ISBN 0-321-58876-2 | 978-0-321-58876-0

Single Variable Calculus (Chapters 1–11), 0-321-62883-7 | 978-0-321-62883-1

Multivariable Calculus (Chapters 10–16), ISBN 0-321-64369-0 | 978-0-321-64369-8

The early transcendentals version of *Thomas' Calculus* introduces and integrates transcendental functions (such as inverse trigonometric, exponential, and logarithmic functions) into the exposition, examples, and exercises of the early chapters alongside the algebraic functions. The Multivariable book for *Thomas' Calculus: Early Transcendentals* is the same text as *Thomas' Calculus, Multivariable*.

THOMAS' CALCULUS, Twelfth Edition

Complete (Chapters 1–16), ISBN 0-321-58799-5 | 978-0-321-58799-2

Single Variable Calculus (Chapters 1–11), ISBN 0-321-63742-9 | 978-0-321-63742-0

Multivariable Calculus (Chapters 10–16), ISBN 0-321-64369-0 | 978-0-321-64369-8

Instructor's Editions

Thomas' Calculus: Early Transcendentals, ISBN 0-321-62718-0 | 978-0-321-62718-6

Thomas' Calculus, ISBN 0-321-60075-4 | 978-0-321-60075-2

In addition to including all of the answers present in the student editions, the *Instructor's Editions* include even-numbered answers for Chapters 1–6.

University Calculus (Early Transcendentals)

University Calculus: Alternate Edition (Late Transcendentals)

University Calculus: Elements with Early Transcendentals

The *University Calculus* texts are based on *Thomas' Calculus* and feature a streamlined presentation of the contents of the calculus course. For more information about these titles, visit www.pearsonhighered.com.

Print Supplements

INSTRUCTOR'S SOLUTIONS MANUAL

Single Variable Calculus (Chapters 1–11), ISBN 0-321-62717-2 | 978-0-321-62717-9

Multivariable Calculus (Chapters 10–16), ISBN 0-321-60072-X | 978-0-321-60072-1

The *Instructor's Solutions Manual* by William Ardis, Collin County Community College, contains complete worked-out solutions to all of the exercises in *Thomas' Calculus: Early Transcendentals*.

STUDENT'S SOLUTIONS MANUAL

Single Variable Calculus (Chapters 1–11), ISBN 0-321-65692-X | 978-0-321-65692-6

Multivariable Calculus (Chapters 10–16), ISBN 0-321-60071-1 | 978-0-321-60071-4

The *Student's Solutions Manual* by William Ardis, Collin County Community College, is designed for the student and contains carefully worked-out solutions to all the odd-numbered exercises in *Thomas' Calculus: Early Transcendentals*.

JUST-IN-TIME ALGEBRA AND TRIGONOMETRY FOR EARLY TRANSCENDENTALS CALCULUS, Third Edition

ISBN 0-321-32050-6 | 978-0-321-32050-6

Sharp algebra and trigonometry skills are critical to mastering calculus, and *Just-in-Time Algebra and Trigonometry for Early Transcendentals Calculus* by Guntram Mueller and Ronald I. Brent is designed to bolster these skills while students study calculus. As students make their way through calculus, this text is with them every step of the way, showing them the necessary algebra or trigonometry topics and pointing out potential problem spots. The easy-to-use table of contents has algebra and trigonometry topics arranged in the order in which students will need them as they study calculus.

CALCULUS REVIEW CARDS

The Calculus Review Cards (one for Single Variable and another for Multivariable) are a student resource containing important formulas, functions, definitions, and theorems that correspond precisely to the *Thomas' Calculus* series. These cards can work as a reference for completing homework assignments or as an aid in studying, and are available bundled with a new text. Contact your Pearson sales representative for more information.

Media and Online Supplements

TECHNOLOGY RESOURCE MANUALS

Maple Manual by James Stapleton, North Carolina State University

Mathematica Manual by Marie Vanisko, Carroll College

TI-Graphing Calculator Manual by Elaine McDonald-Newman, Sonoma State University

These manuals cover *Maple* 13, *Mathematica* 7, and the TI-83 Plus/TI-84 Plus and TI-89, respectively. Each manual provides detailed guidance for integrating a specific software package or graphing calculator throughout the course, including syntax and commands. These manuals are available to qualified instructors through the *Thomas' Calculus: Early Transcendentals* Web site, www.pearsonhighered.com/thomas, and MyMathLab.

WEB SITE www.pearsonhighered.com/thomas

The *Thomas' Calculus: Early Transcendentals* Web site contains the chapter on Second-Order Differential Equations, including odd-numbered answers, and provides the expanded historical biographies and essays referenced in the text. Also available is a collection of *Maple* and *Mathematica* modules, the Technology Resource Manuals, and the **Technology Application Projects**, which can be used as projects by individual students or groups of students.

MyMathLab Online Course (access code required)

MyMathLab is a text-specific, easily customizable online course that integrates interactive multimedia instruction with textbook content. MyMathLab gives you the tools you need to deliver all or a portion of your course online, whether your students are in a lab setting or working from home.

- **Interactive homework exercises**, correlated to your textbook at the objective level, are algorithmically generated for unlimited practice and mastery. Most exercises are free-response and provide guided solutions, sample problems, and learning aids for extra help.
- **“Getting Ready” chapter** includes hundreds of exercises that address prerequisite skills in algebra and trigonometry. Each student can receive remediation for just those skills he or she needs help with.
- **Personalized Study Plan**, generated when students complete a test or quiz, indicates which topics have been mastered and links to tutorial exercises for topics students have not mastered.
- **Multimedia learning aids**, such as video lectures, Java applets, animations, and a complete multimedia textbook, help students independently improve their understanding and performance.
- **Assessment Manager** lets you create online homework, quizzes, and tests that are automatically graded. Select just the right mix of questions from the MyMathLab exercise bank and instructor-created custom exercises.
- **Gradebook**, designed specifically for mathematics and statistics, automatically tracks students' results and gives you control over how to calculate final grades. You can also add offline (paper-and-pencil) grades to the gradebook.
- **MathXL Exercise Builder** allows you to create static and algorithmic exercises for your online assignments. You can use the library of sample exercises as an easy starting point.
- **Pearson Tutor Center (www.pearson tutorservices.com)** access is automatically included with MyMathLab. The Tutor Center is staffed by qualified math instructors who provide textbook-specific tutoring for students via toll-free phone, fax, email, and interactive Web sessions.

MyMathLab is powered by CourseCompass™, Pearson Education's online teaching and learning environment, and by MathXL, our online homework, tutorial, and assessment system. MyMathLab is available to qualified adopters. For more information, visit www.mymathlab.com or contact your Pearson sales representative.

Video Lectures with Optional Captioning

The Video Lectures with Optional Captioning feature an engaging team of mathematics instructors who present comprehensive coverage of topics in the text. The lecturers' presentations include examples and exercises from the text and support an approach that emphasizes visualization and problem solving. Available only through MyMathLab and MathXL.

MathXL Online Course (access code required)

MathXL is an online homework, tutorial, and assessment system that accompanies Pearson's textbooks in mathematics or statistics.

- **Interactive homework exercises**, correlated to your textbook at the objective level, are algorithmically generated for unlimited practice and mastery. Most exercises are free-response and provide guided solutions, sample problems, and learning aids for extra help.
- **“Getting Ready” chapter** includes hundreds of exercises that address prerequisite skills in algebra and trigonometry. Each student can receive remediation for just those skills he or she needs help with.
- **Personalized Study Plan**, generated when students complete a test or quiz, indicates which topics have been mastered and links to tutorial exercises for topics students have not mastered.
- **Multimedia learning aids**, such as video lectures, Java applets, and animations, help students independently improve their understanding and performance.
- **Gradebook**, designed specifically for mathematics and statistics, automatically tracks students' results and gives you control over how to calculate final grades.
- **MathXL Exercise Builder** allows you to create static and algorithmic exercises for your online assignments. You can use the library of sample exercises as an easy starting point.
- **Assessment Manager** lets you create online homework, quizzes, and tests that are automatically graded. Select just the right mix of questions from the MathXL exercise bank, or instructor-created custom exercises.

MathXL is available to qualified adopters. For more information, visit our Web site at www.mathxl.com, or contact your Pearson sales representative.

TestGen®

TestGen (www.pearsonhighered.com/testgen) enables instructors to build, edit, print, and administer tests using a computerized bank of questions developed to cover all the objectives of the text. TestGen is algorithmically based, allowing instructors to create multiple but equivalent versions of the same question or test with the click of a button. Instructors can also modify test bank questions or add new questions. Tests can be printed or administered online. The software and testbank are available for download from Pearson Education's online catalog.

PowerPoint® Lecture Slides

These classroom presentation slides are geared specifically to the sequence and philosophy of the *Thomas' Calculus* series. Key graphics from the book are included to help bring the concepts alive in the classroom. These files are available to qualified instructors through the Pearson Instructor Resource Center, www.pearsonhighered.com/irc, and MyMathLab.

Acknowledgments

We would like to express our thanks to the people who made many valuable contributions to this edition as it developed through its various stages:

Accuracy Checkers

Blaise DeSesa
Paul Lorzak
Kathleen Pellissier
Lauri Semarne
Sarah Streett
Holly Zullo

Reviewers for the Twelfth Edition

Meighan Dillon, *Southern Polytechnic State University*
Anne Dougherty, *University of Colorado*
Said Fariabi, *San Antonio College*
Klaus Fischer, *George Mason University*
Tim Flood, *Pittsburg State University*
Rick Ford, *California State University—Chico*
Robert Gardner, *East Tennessee State University*
Christopher Heil, *Georgia Institute of Technology*
Joshua Brandon Holden, *Rose-Hulman Institute of Technology*
Alexander Hulpke, *Colorado State University*
Jacqueline Jensen, *Sam Houston State University*
Jennifer M. Johnson, *Princeton University*
Hideaki Kaneko, *Old Dominion University*
Przemo Kranz, *University of Mississippi*
Xin Li, *University of Central Florida*
Maura Mast, *University of Massachusetts—Boston*
Val Mohanakumar, *Hillsborough Community College—Dale Mabry Campus*
Aaron Montgomery, *Central Washington University*
Christopher M. Pavone, *California State University at Chico*
Cynthia Piez, *University of Idaho*
Brooke Quinlan, *Hillsborough Community College—Dale Mabry Campus*
Rebecca A. Segal, *Virginia Commonwealth University*
Andrew V. Sills, *Georgia Southern University*
Alex Smith, *University of Wisconsin—Eau Claire*
Mark A. Smith, *Miami University*
Donald Solomon, *University of Wisconsin—Milwaukee*
John Sullivan, *Black Hawk College*
Maria Terrell, *Cornell University*
Blake Thornton, *Washington University in St. Louis*
David Walnut, *George Mason University*
Adrian Wilson, *University of Montevallo*
Bobby Winters, *Pittsburg State University*
Dennis Wortman, *University of Massachusetts—Boston*



1

FUNCTIONS

OVERVIEW Functions are fundamental to the study of calculus. In this chapter we review what functions are and how they are pictured as graphs, how they are combined and transformed, and ways they can be classified. We review the trigonometric functions, and we discuss misrepresentations that can occur when using calculators and computers to obtain a function's graph. We also discuss inverse, exponential, and logarithmic functions. The real number system, Cartesian coordinates, straight lines, parabolas, and circles are reviewed in the Appendices.

1.1

Functions and Their Graphs

Functions are a tool for describing the real world in mathematical terms. A function can be represented by an equation, a graph, a numerical table, or a verbal description; we will use all four representations throughout this book. This section reviews these function ideas.

Functions; Domain and Range

The temperature at which water boils depends on the elevation above sea level (the boiling point drops as you ascend). The interest paid on a cash investment depends on the length of time the investment is held. The area of a circle depends on the radius of the circle. The distance an object travels at constant speed along a straight-line path depends on the elapsed time.

In each case, the value of one variable quantity, say y , depends on the value of another variable quantity, which we might call x . We say that “ y is a function of x ” and write this symbolically as

$$y = f(x) \quad (\text{“}y \text{ equals } f \text{ of } x\text{”}).$$

In this notation, the symbol f represents the function, the letter x is the **independent variable** representing the input value of f , and y is the **dependent variable** or output value of f at x .

DEFINITION A **function** f from a set D to a set Y is a rule that assigns a *unique* (single) element $f(x) \in Y$ to each element $x \in D$.

The set D of all possible input values is called the **domain** of the function. The set of all values of $f(x)$ as x varies throughout D is called the **range** of the function. The range may not include every element in the set Y . The domain and range of a function can be any sets of objects, but often in calculus they are sets of real numbers interpreted as points of a coordinate line. (In Chapters 13–16, we will encounter functions for which the elements of the sets are points in the coordinate plane or in space.)

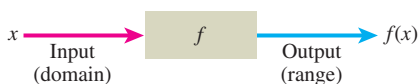


FIGURE 1.1 A diagram showing a function as a kind of machine.

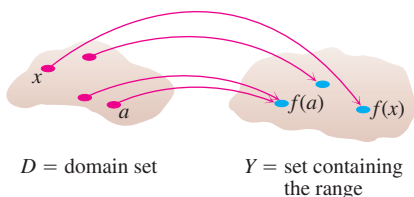


FIGURE 1.2 A function from a set D to a set Y assigns a unique element of Y to each element in D .

Often a function is given by a formula that describes how to calculate the output value from the input variable. For instance, the equation $A = \pi r^2$ is a rule that calculates the area A of a circle from its radius r (so r , interpreted as a length, can only be positive in this formula). When we define a function $y = f(x)$ with a formula and the domain is not stated explicitly or restricted by context, the domain is assumed to be the largest set of real x -values for which the formula gives real y -values, the so-called **natural domain**. If we want to restrict the domain in some way, we must say so. The domain of $y = x^2$ is the entire set of real numbers. To restrict the domain of the function to, say, positive values of x , we would write “ $y = x^2, x > 0$.”

Changing the domain to which we apply a formula usually changes the range as well. The range of $y = x^2$ is $[0, \infty)$. The range of $y = x^2, x \geq 2$, is the set of all numbers obtained by squaring numbers greater than or equal to 2. In set notation (see Appendix 1), the range is $\{x^2 \mid x \geq 2\}$ or $\{y \mid y \geq 4\}$ or $[4, \infty)$.

When the range of a function is a set of real numbers, the function is said to be **real-valued**. The domains and ranges of many real-valued functions of a real variable are intervals or combinations of intervals. The intervals may be open, closed, or half open, and may be finite or infinite. The range of a function is not always easy to find.

A function f is like a machine that produces an output value $f(x)$ in its range whenever we feed it an input value x from its domain (Figure 1.1). The function keys on a calculator give an example of a function as a machine. For instance, the \sqrt{x} key on a calculator gives an output value (the square root) whenever you enter a nonnegative number x and press the \sqrt{x} key.

A function can also be pictured as an **arrow diagram** (Figure 1.2). Each arrow associates an element of the domain D with a unique or single element in the set Y . In Figure 1.2, the arrows indicate that $f(a)$ is associated with a , $f(x)$ is associated with x , and so on. Notice that a function can have the same *value* at two different input elements in the domain (as occurs with $f(a)$ in Figure 1.2), but each input element x is assigned a *single* output value $f(x)$.

EXAMPLE 1 Let’s verify the natural domains and associated ranges of some simple functions. The domains in each case are the values of x for which the formula makes sense.

Function	Domain (x)	Range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

Solution The formula $y = x^2$ gives a real y -value for any real number x , so the domain is $(-\infty, \infty)$. The range of $y = x^2$ is $[0, \infty)$ because the square of any real number is nonnegative and every nonnegative number y is the square of its own square root, $y = (\sqrt{y})^2$ for $y \geq 0$.

The formula $y = 1/x$ gives a real y -value for every x except $x = 0$. For consistency in the rules of arithmetic, we cannot divide any number by zero. The range of $y = 1/x$, the set of reciprocals of all nonzero real numbers, is the set of all nonzero real numbers, since $y = 1/(1/y)$. That is, for $y \neq 0$ the number $x = 1/y$ is the input assigned to the output value y .

The formula $y = \sqrt{x}$ gives a real y -value only if $x \geq 0$. The range of $y = \sqrt{x}$ is $[0, \infty)$ because every nonnegative number is some number’s square root (namely, it is the square root of its own square).

In $y = \sqrt{4 - x}$, the quantity $4 - x$ cannot be negative. That is, $4 - x \geq 0$, or $x \leq 4$. The formula gives real y -values for all $x \leq 4$. The range of $\sqrt{4 - x}$ is $[0, \infty)$, the set of all nonnegative numbers.

The formula $y = \sqrt{1 - x^2}$ gives a real y -value for every x in the closed interval from -1 to 1 . Outside this domain, $1 - x^2$ is negative and its square root is not a real number. The values of $1 - x^2$ vary from 0 to 1 on the given domain, and the square roots of these values do the same. The range of $\sqrt{1 - x^2}$ is $[0, 1]$. ■

Graphs of Functions

If f is a function with domain D , its **graph** consists of the points in the Cartesian plane whose coordinates are the input-output pairs for f . In set notation, the graph is

$$\{(x, f(x)) \mid x \in D\}.$$

The graph of the function $f(x) = x + 2$ is the set of points with coordinates (x, y) for which $y = x + 2$. Its graph is the straight line sketched in Figure 1.3.

The graph of a function f is a useful picture of its behavior. If (x, y) is a point on the graph, then $y = f(x)$ is the height of the graph above the point x . The height may be positive or negative, depending on the sign of $f(x)$ (Figure 1.4).

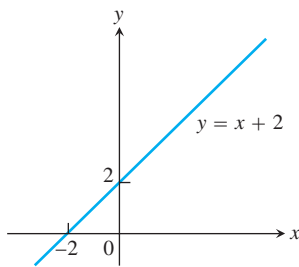


FIGURE 1.3 The graph of $f(x) = x + 2$ is the set of points (x, y) for which y has the value $x + 2$.

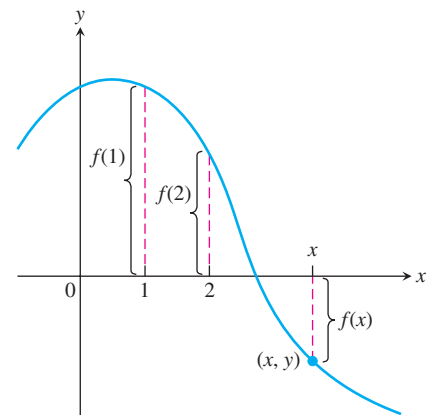


FIGURE 1.4 If (x, y) lies on the graph of f , then the value $y = f(x)$ is the height of the graph above the point x (or below x if $f(x)$ is negative).

x	$y = x^2$
-2	4
-1	1
0	0
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4

EXAMPLE 2 Graph the function $y = x^2$ over the interval $[-2, 2]$.

Solution Make a table of xy -pairs that satisfy the equation $y = x^2$. Plot the points (x, y) whose coordinates appear in the table, and draw a *smooth* curve (labeled with its equation) through the plotted points (see Figure 1.5). ■

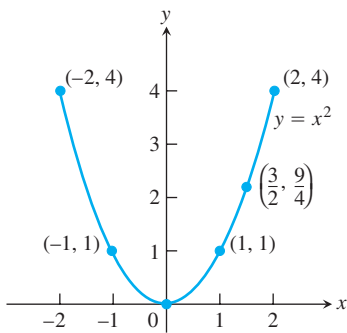
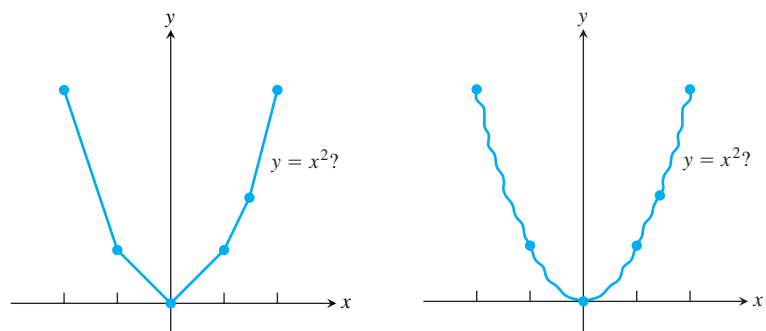


FIGURE 1.5 Graph of the function in Example 2.

How do we know that the graph of $y = x^2$ doesn't look like one of these curves?



To find out, we could plot more points. But how would we then connect *them*? The basic question still remains: How do we know for sure what the graph looks like between the points we plot? Calculus answers this question, as we will see in Chapter 4. Meanwhile we will have to settle for plotting points and connecting them as best we can.

Representing a Function Numerically

We have seen how a function may be represented algebraically by a formula (the area function) and visually by a graph (Example 2). Another way to represent a function is **numerically**, through a table of values. Numerical representations are often used by engineers and scientists. From an appropriate table of values, a graph of the function can be obtained using the method illustrated in Example 2, possibly with the aid of a computer. The graph consisting of only the points in the table is called a **scatterplot**.

EXAMPLE 3 Musical notes are pressure waves in the air. The data in Table 1.1 give recorded pressure displacement versus time in seconds of a musical note produced by a tuning fork. The table provides a representation of the pressure function over time. If we first make a scatterplot and then connect approximately the data points (t, p) from the table, we obtain the graph shown in Figure 1.6.

TABLE 1.1 Tuning fork data

Time	Pressure	Time	Pressure
0.00091	-0.080	0.00362	0.217
0.00108	0.200	0.00379	0.480
0.00125	0.480	0.00398	0.681
0.00144	0.693	0.00416	0.810
0.00162	0.816	0.00435	0.827
0.00180	0.844	0.00453	0.749
0.00198	0.771	0.00471	0.581
0.00216	0.603	0.00489	0.346
0.00234	0.368	0.00507	0.077
0.00253	0.099	0.00525	-0.164
0.00271	-0.141	0.00543	-0.320
0.00289	-0.309	0.00562	-0.354
0.00307	-0.348	0.00579	-0.248
0.00325	-0.248	0.00598	-0.035
0.00344	-0.041		

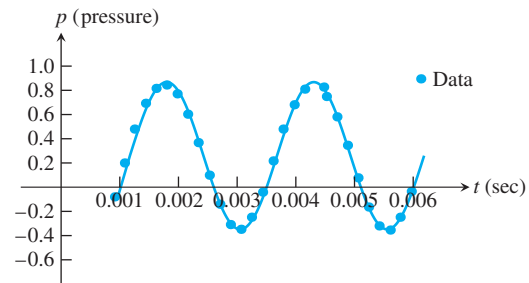


FIGURE 1.6 A smooth curve through the plotted points gives a graph of the pressure function represented by Table 1.1 (Example 3).

The Vertical Line Test for a Function

Not every curve in the coordinate plane can be the graph of a function. A function f can have only one value $f(x)$ for each x in its domain, so *no vertical* line can intersect the graph of a function more than once. If a is in the domain of the function f , then the vertical line $x = a$ will intersect the graph of f at the single point $(a, f(a))$.

A circle cannot be the graph of a function since some vertical lines intersect the circle twice. The circle in Figure 1.7a, however, does contain the graphs of *two* functions of x : the upper semicircle defined by the function $f(x) = \sqrt{1 - x^2}$ and the lower semicircle defined by the function $g(x) = -\sqrt{1 - x^2}$ (Figures 1.7b and 1.7c).

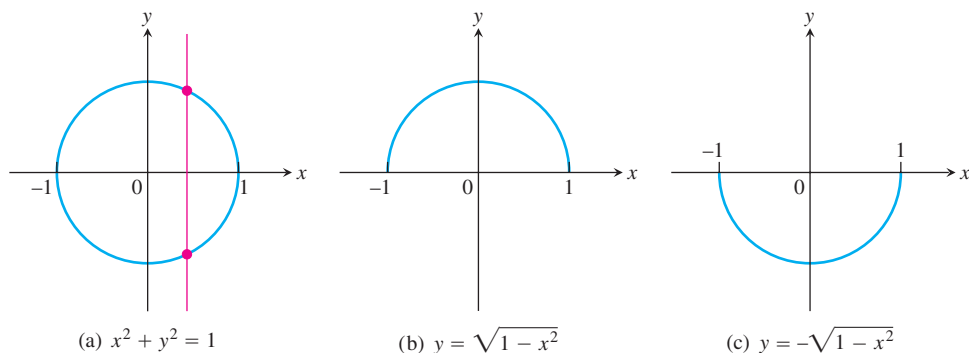


FIGURE 1.7 (a) The circle is not the graph of a function; it fails the vertical line test. (b) The upper semicircle is the graph of a function $f(x) = \sqrt{1 - x^2}$. (c) The lower semicircle is the graph of a function $g(x) = -\sqrt{1 - x^2}$.

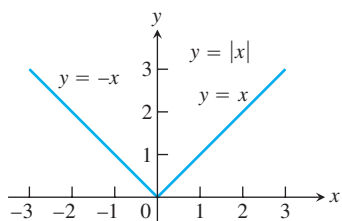


FIGURE 1.8 The absolute value function has domain $(-\infty, \infty)$ and range $[0, \infty)$.

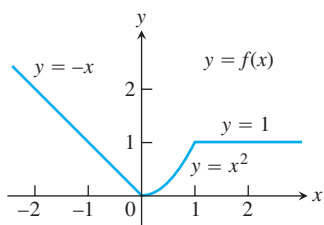


FIGURE 1.9 To graph the function $y = f(x)$ shown here, we apply different formulas to different parts of its domain (Example 4).

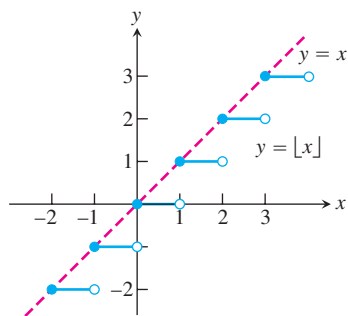


FIGURE 1.10 The graph of the greatest integer function $y = \lfloor x \rfloor$ lies on or below the line $y = x$, so it provides an integer floor for x (Example 5).

Piecewise-Defined Functions

Sometimes a function is described by using different formulas on different parts of its domain. One example is the **absolute value function**

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0, \end{cases}$$

whose graph is given in Figure 1.8. The right-hand side of the equation means that the function equals x if $x \geq 0$, and equals $-x$ if $x < 0$. Here are some other examples.

EXAMPLE 4 The function

$$f(x) = \begin{cases} -x, & x < 0 \\ x^2, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

is defined on the entire real line but has values given by different formulas depending on the position of x . The values of f are given by $y = -x$ when $x < 0$, $y = x^2$ when $0 \leq x \leq 1$, and $y = 1$ when $x > 1$. The function, however, is *just one function* whose domain is the entire set of real numbers (Figure 1.9). ■

EXAMPLE 5 The function whose value at any number x is the *greatest integer less than or equal to* x is called the **greatest integer function** or the **integer floor function**. It is denoted $\lfloor x \rfloor$. Figure 1.10 shows the graph. Observe that

$$\begin{aligned} \lfloor 2.4 \rfloor &= 2, & \lfloor 1.9 \rfloor &= 1, & \lfloor 0 \rfloor &= 0, & \lfloor -1.2 \rfloor &= -2, \\ \lfloor 2 \rfloor &= 2, & \lfloor 0.2 \rfloor &= 0, & \lfloor -0.3 \rfloor &= -1, & \lfloor -2 \rfloor &= -2. \end{aligned}$$

EXAMPLE 6 The function whose value at any number x is the *smallest integer greater than or equal to* x is called the **least integer function** or the **integer ceiling function**. It is denoted $\lceil x \rceil$. Figure 1.11 shows the graph. For positive values of x , this function might represent, for example, the cost of parking x hours in a parking lot which charges \$1 for each hour or part of an hour. ■

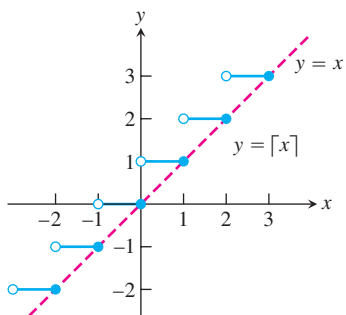


FIGURE 1.11 The graph of the least integer function $y = [x]$ lies on or above the line $y = x$, so it provides an integer ceiling for x (Example 6).

Increasing and Decreasing Functions

If the graph of a function *climbs* or *rises* as you move from left to right, we say that the function is *increasing*. If the graph *descends* or *falls* as you move from left to right, the function is *decreasing*.

DEFINITIONS Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I .

1. If $f(x_2) > f(x_1)$ whenever $x_1 < x_2$, then f is said to be **increasing** on I .
2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then f is said to be **decreasing** on I .

It is important to realize that the definitions of increasing and decreasing functions must be satisfied for *every* pair of points x_1 and x_2 in I with $x_1 < x_2$. Because we use the inequality $<$ to compare the function values, instead of \leq , it is sometimes said that f is *strictly* increasing or decreasing on I . The interval I may be finite (also called bounded) or infinite (unbounded) and by definition never consists of a single point (Appendix 1).

EXAMPLE 7 The function graphed in Figure 1.9 is decreasing on $(-\infty, 0]$ and increasing on $[0, 1]$. The function is neither increasing nor decreasing on the interval $[1, \infty)$ because of the strict inequalities used to compare the function values in the definitions. ■

Even Functions and Odd Functions: Symmetry

The graphs of *even* and *odd* functions have characteristic symmetry properties.

DEFINITIONS A function $y = f(x)$ is an

even function of x if $f(-x) = f(x)$,

odd function of x if $f(-x) = -f(x)$,

for every x in the function's domain.

The names *even* and *odd* come from powers of x . If y is an even power of x , as in $y = x^2$ or $y = x^4$, it is an even function of x because $(-x)^2 = x^2$ and $(-x)^4 = x^4$. If y is an odd power of x , as in $y = x$ or $y = x^3$, it is an odd function of x because $(-x)^1 = -x$ and $(-x)^3 = -x^3$.

The graph of an even function is **symmetric about the y-axis**. Since $f(-x) = f(x)$, a point (x, y) lies on the graph if and only if the point $(-x, y)$ lies on the graph (Figure 1.12a). A reflection across the y -axis leaves the graph unchanged.

The graph of an odd function is **symmetric about the origin**. Since $f(-x) = -f(x)$, a point (x, y) lies on the graph if and only if the point $(-x, -y)$ lies on the graph (Figure 1.12b). Equivalently, a graph is symmetric about the origin if a rotation of 180° about the origin leaves the graph unchanged. Notice that the definitions imply that both x and $-x$ must be in the domain of f .

EXAMPLE 8

- | | |
|------------------|---|
| $f(x) = x^2$ | Even function: $(-x)^2 = x^2$ for all x ; symmetry about y -axis. |
| $f(x) = x^2 + 1$ | Even function: $(-x)^2 + 1 = x^2 + 1$ for all x ; symmetry about y -axis (Figure 1.13a). |
| $f(x) = x$ | Odd function: $(-x) = -x$ for all x ; symmetry about the origin. |
| $f(x) = x + 1$ | Not odd: $f(-x) = -x + 1$, but $-f(x) = -x - 1$. The two are not equal.
Not even: $(-x) + 1 \neq x + 1$ for all $x \neq 0$ (Figure 1.13b). ■ |

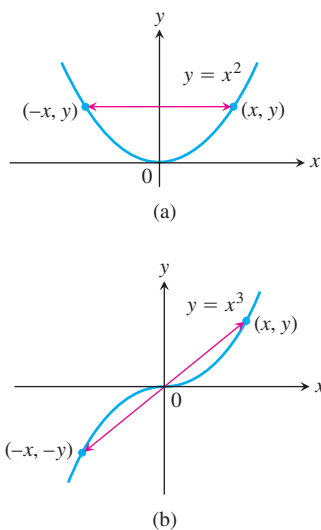


FIGURE 1.12 (a) The graph of $y = x^2$ (an even function) is symmetric about the y -axis. (b) The graph of $y = x^3$ (an odd function) is symmetric about the origin.

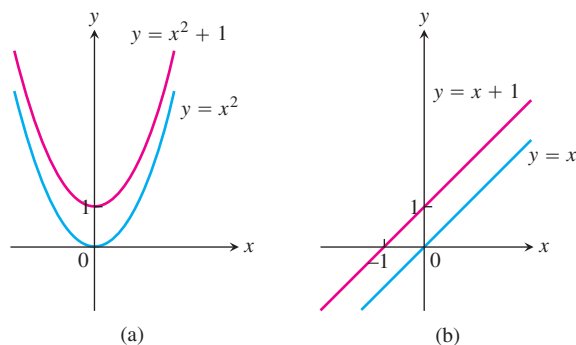


FIGURE 1.13 (a) When we add the constant term 1 to the function $y = x^2$, the resulting function $y = x^2 + 1$ is still even and its graph is still symmetric about the y -axis. (b) When we add the constant term 1 to the function $y = x$, the resulting function $y = x + 1$ is no longer odd. The symmetry about the origin is lost (Example 8).

Common Functions

A variety of important types of functions are frequently encountered in calculus. We identify and briefly describe them here.

Linear Functions A function of the form $f(x) = mx + b$, for constants m and b , is called a **linear function**. Figure 1.14a shows an array of lines $f(x) = mx$ where $b = 0$, so these lines pass through the origin. The function $f(x) = x$ where $m = 1$ and $b = 0$ is called the **identity function**. Constant functions result when the slope $m = 0$ (Figure 1.14b). A linear function with positive slope whose graph passes through the origin is called a *proportionality* relationship.

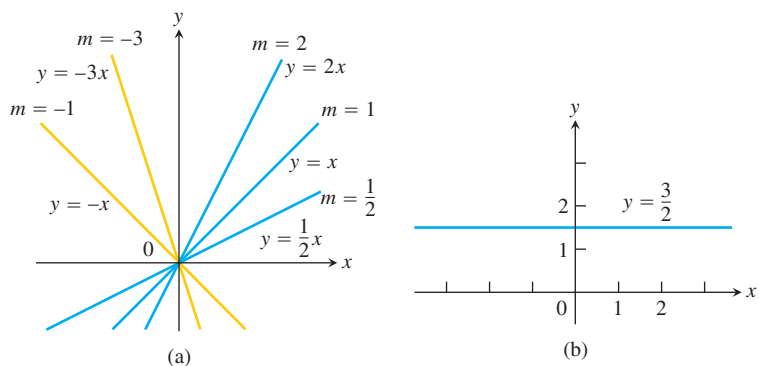


FIGURE 1.14 (a) Lines through the origin with slope m . (b) A constant function with slope $m = 0$.

DEFINITION Two variables y and x are **proportional** (to one another) if one is always a constant multiple of the other; that is, if $y = kx$ for some nonzero constant k .

If the variable y is proportional to the reciprocal $1/x$, then sometimes it is said that y is **inversely proportional** to x (because $1/x$ is the multiplicative inverse of x).

Power Functions A function $f(x) = x^a$, where a is a constant, is called a **power function**. There are several important cases to consider.

(a) $a = n$, a positive integer.

The graphs of $f(x) = x^n$, for $n = 1, 2, 3, 4, 5$, are displayed in Figure 1.15. These functions are defined for all real values of x . Notice that as the power n gets larger, the curves tend to flatten toward the x -axis on the interval $(-1, 1)$, and also rise more steeply for $|x| > 1$. Each curve passes through the point $(1, 1)$ and through the origin. The graphs of functions with even powers are symmetric about the y -axis; those with odd powers are symmetric about the origin. The even-powered functions are decreasing on the interval $(-\infty, 0]$ and increasing on $[0, \infty)$; the odd-powered functions are increasing over the entire real line $(-\infty, \infty)$.

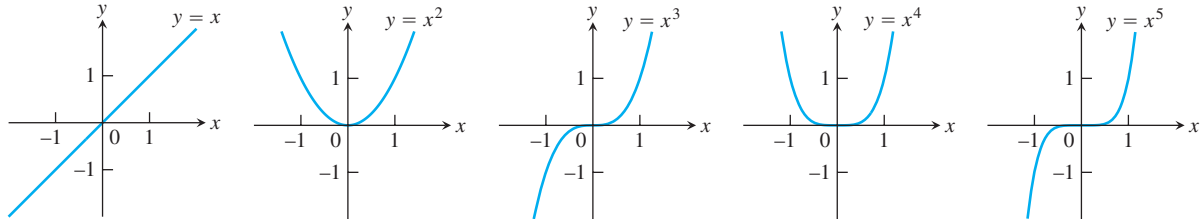


FIGURE 1.15 Graphs of $f(x) = x^n$, $n = 1, 2, 3, 4, 5$, defined for $-\infty < x < \infty$.

(b) $a = -1$ or $a = -2$.

The graphs of the functions $f(x) = x^{-1} = 1/x$ and $g(x) = x^{-2} = 1/x^2$ are shown in Figure 1.16. Both functions are defined for all $x \neq 0$ (you can never divide by zero). The graph of $y = 1/x$ is the hyperbola $xy = 1$, which approaches the coordinate axes far from the origin. The graph of $y = 1/x^2$ also approaches the coordinate axes. The graph of the function f is symmetric about the origin; f is decreasing on the intervals $(-\infty, 0)$ and $(0, \infty)$. The graph of the function g is symmetric about the y -axis; g is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.

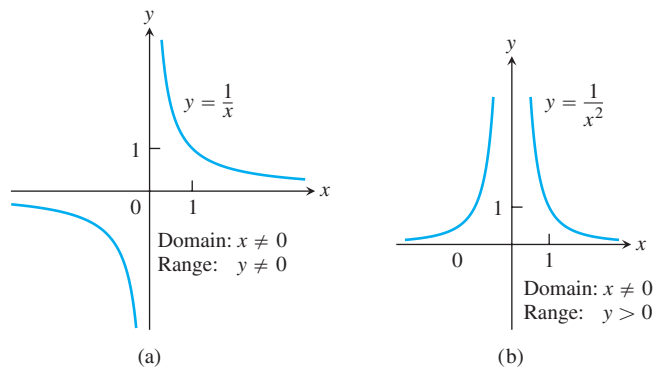


FIGURE 1.16 Graphs of the power functions $f(x) = x^a$ for part (a) $a = -1$ and for part (b) $a = -2$.

(c) $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}$, and $\frac{2}{3}$.

The functions $f(x) = x^{1/2} = \sqrt{x}$ and $g(x) = x^{1/3} = \sqrt[3]{x}$ are the **square root** and **cube root** functions, respectively. The domain of the square root function is $[0, \infty)$, but the cube root function is defined for all real x . Their graphs are displayed in Figure 1.17 along with the graphs of $y = x^{3/2}$ and $y = x^{2/3}$. (Recall that $x^{3/2} = (x^{1/2})^3$ and $x^{2/3} = (x^{1/3})^2$.)

Polynomials A function p is a **polynomial** if

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where n is a nonnegative integer and the numbers $a_0, a_1, a_2, \dots, a_n$ are real constants (called the **coefficients** of the polynomial). All polynomials have domain $(-\infty, \infty)$. If the

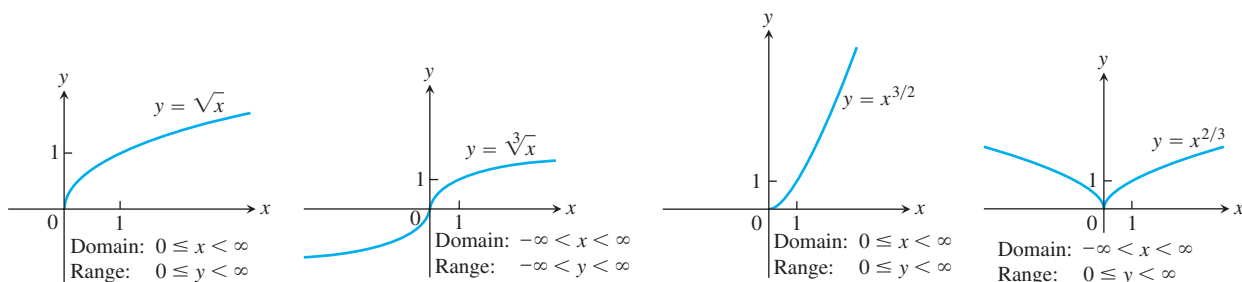


FIGURE 1.17 Graphs of the power functions $f(x) = x^a$ for $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2},$ and $\frac{2}{3}$.

leading coefficient $a_n \neq 0$ and $n > 0$, then n is called the **degree** of the polynomial. Linear functions with $m \neq 0$ are polynomials of degree 1. Polynomials of degree 2, usually written as $p(x) = ax^2 + bx + c$, are called **quadratic functions**. Likewise, **cubic functions** are polynomials $p(x) = ax^3 + bx^2 + cx + d$ of degree 3. Figure 1.18 shows the graphs of three polynomials. Techniques to graph polynomials are studied in Chapter 4.

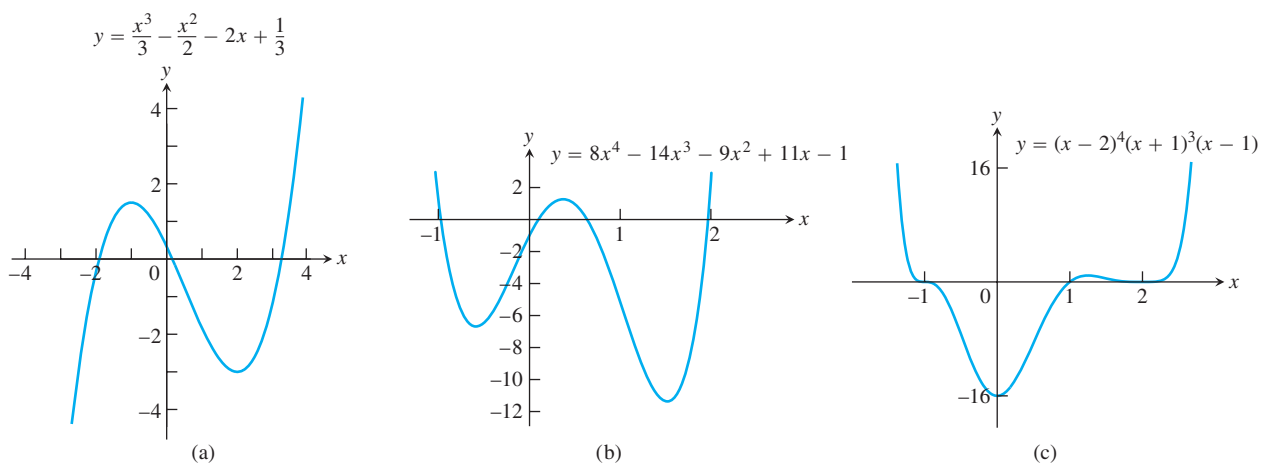


FIGURE 1.18 Graphs of three polynomial functions.

Rational Functions A **rational function** is a quotient or ratio $f(x) = p(x)/q(x)$, where p and q are polynomials. The domain of a rational function is the set of all real x for which $q(x) \neq 0$. The graphs of several rational functions are shown in Figure 1.19.

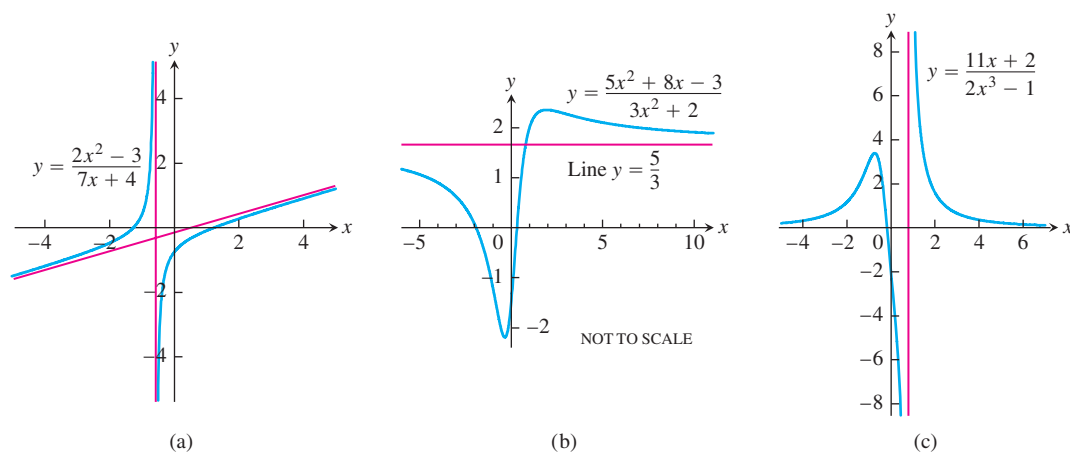


FIGURE 1.19 Graphs of three rational functions. The straight red lines are called *asymptotes* and are not part of the graph.

Algebraic Functions Any function constructed from polynomials using algebraic operations (addition, subtraction, multiplication, division, and taking roots) lies within the class of **algebraic functions**. All rational functions are algebraic, but also included are more complicated functions (such as those satisfying an equation like $y^3 - 9xy + x^3 = 0$, studied in Section 3.7). Figure 1.20 displays the graphs of three algebraic functions.

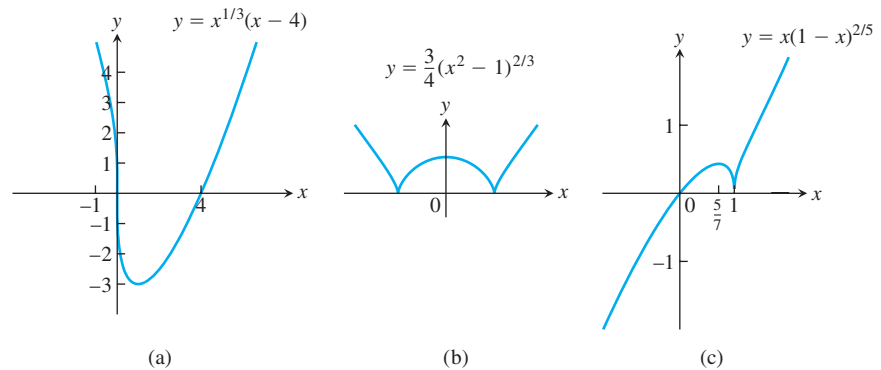


FIGURE 1.20 Graphs of three algebraic functions.

Trigonometric Functions The six basic trigonometric functions are reviewed in Section 1.3. The graphs of the sine and cosine functions are shown in Figure 1.21.

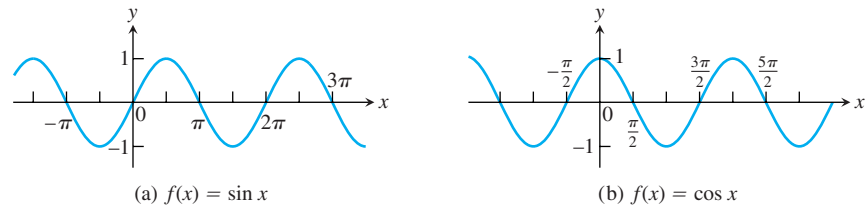


FIGURE 1.21 Graphs of the sine and cosine functions.

Exponential Functions Functions of the form $f(x) = a^x$, where the base $a > 0$ is a positive constant and $a \neq 1$, are called **exponential functions**. All exponential functions have domain $(-\infty, \infty)$ and range $(0, \infty)$, so an exponential function never assumes the value 0. We discuss exponential functions in Section 1.5. The graphs of some exponential functions are shown in Figure 1.22.

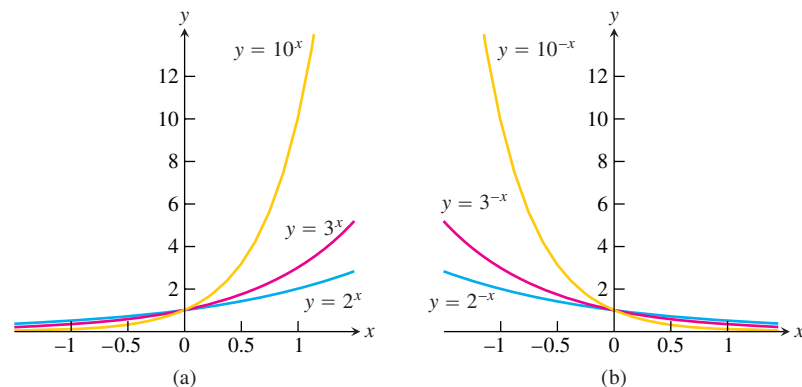


FIGURE 1.22 Graphs of exponential functions.

Logarithmic Functions These are the functions $f(x) = \log_a x$, where the base $a \neq 1$ is a positive constant. They are the *inverse functions* of the exponential functions, and we discuss these functions in Section 1.6. Figure 1.23 shows the graphs of four logarithmic functions with various bases. In each case the domain is $(0, \infty)$ and the range is $(-\infty, \infty)$.

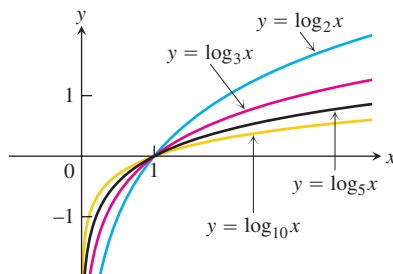


FIGURE 1.23 Graphs of four logarithmic functions.

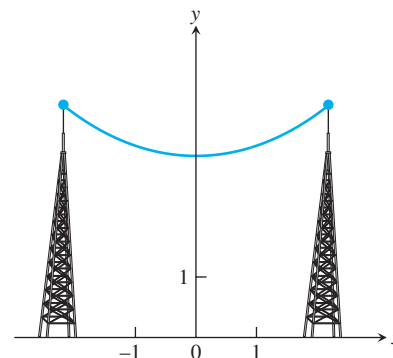


FIGURE 1.24 Graph of a catenary or hanging cable. (The Latin word *catena* means “chain.”)

Transcendental Functions These are functions that are not algebraic. They include the trigonometric, inverse trigonometric, exponential, and logarithmic functions, and many other functions as well. A particular example of a transcendental function is a **catenary**. Its graph has the shape of a cable, like a telephone line or electric cable, strung from one support to another and hanging freely under its own weight (Figure 1.24). The function defining the graph is discussed in Section 7.3.

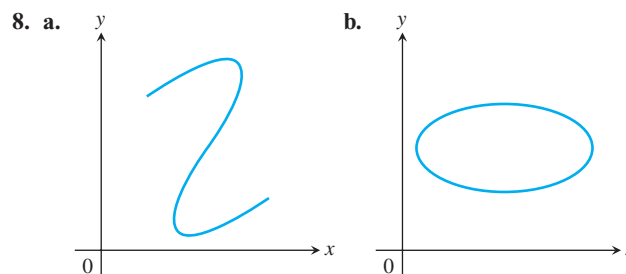
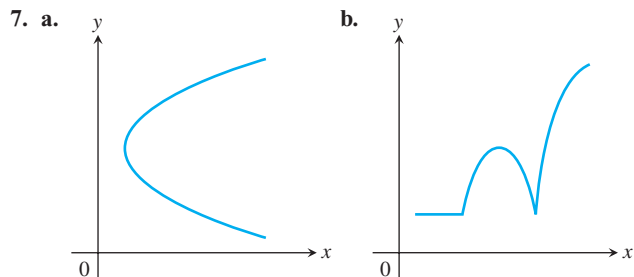
Exercises 1.1

Functions

In Exercises 1–6, find the domain and range of each function.

1. $f(x) = 1 + x^2$
2. $f(x) = 1 - \sqrt{x}$
3. $F(x) = \sqrt{5x + 10}$
4. $g(x) = \sqrt{x^2 - 3x}$
5. $f(t) = \frac{4}{3 - t}$
6. $G(t) = \frac{2}{t^2 - 16}$

In Exercises 7 and 8, which of the graphs are graphs of functions of x , and which are not? Give reasons for your answers.



Finding Formulas for Functions

9. Express the area and perimeter of an equilateral triangle as a function of the triangle's side length x .
10. Express the side length of a square as a function of the length d of the square's diagonal. Then express the area as a function of the diagonal length.
11. Express the edge length of a cube as a function of the cube's diagonal length d . Then express the surface area and volume of the cube as a function of the diagonal length.

12. A point P in the first quadrant lies on the graph of the function $f(x) = \sqrt{x}$. Express the coordinates of P as functions of the slope of the line joining P to the origin.
13. Consider the point (x, y) lying on the graph of the line $2x + 4y = 5$. Let L be the distance from the point (x, y) to the origin $(0, 0)$. Write L as a function of x .
14. Consider the point (x, y) lying on the graph of $y = \sqrt{x - 3}$. Let L be the distance between the points (x, y) and $(4, 0)$. Write L as a function of y .

Functions and Graphs

Find the domain and graph the functions in Exercises 15–20.

15. $f(x) = 5 - 2x$ 16. $f(x) = 1 - 2x - x^2$
 17. $g(x) = \sqrt{|x|}$ 18. $g(x) = \sqrt{-x}$
 19. $F(t) = t/|t|$ 20. $G(t) = 1/|t|$

21. Find the domain of $y = \frac{x + 3}{4 - \sqrt{x^2 - 9}}$.

22. Find the range of $y = 2 + \frac{x^2}{x^2 + 4}$.

23. Graph the following equations and explain why they are not graphs of functions of x .

- a. $|y| = x$ b. $y^2 = x^2$

24. Graph the following equations and explain why they are not graphs of functions of x .

- a. $|x| + |y| = 1$ b. $|x + y| = 1$

Piecewise-Defined Functions

Graph the functions in Exercises 25–28.

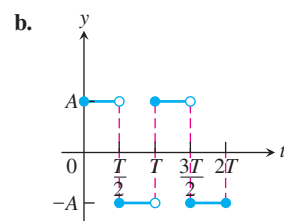
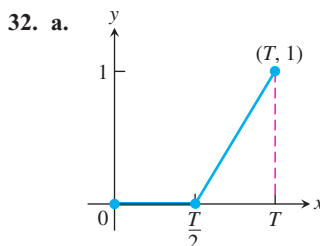
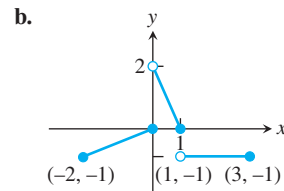
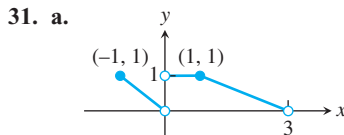
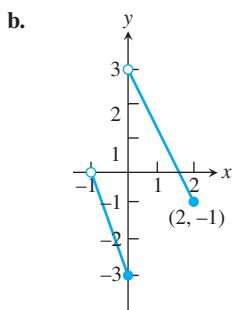
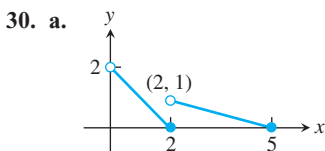
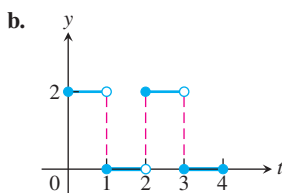
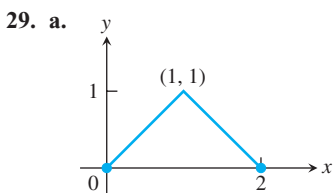
25. $f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 2 - x, & 1 < x \leq 2 \end{cases}$

26. $g(x) = \begin{cases} 1 - x, & 0 \leq x \leq 1 \\ 2 - x, & 1 < x \leq 2 \end{cases}$

27. $F(x) = \begin{cases} 4 - x^2, & x \leq 1 \\ x^2 + 2x, & x > 1 \end{cases}$

28. $G(x) = \begin{cases} 1/x, & x < 0 \\ x, & 0 \leq x \end{cases}$

Find a formula for each function graphed in Exercises 29–32.



The Greatest and Least Integer Functions

33. For what values of x is
 a. $\lfloor x \rfloor = 0$? b. $\lceil x \rceil = 0$?
34. What real numbers x satisfy the equation $\lfloor x \rfloor = \lceil x \rceil$?
35. Does $\lceil -x \rceil = -\lfloor x \rfloor$ for all real x ? Give reasons for your answer.
36. Graph the function

$$f(x) = \begin{cases} \lfloor x \rfloor, & x \geq 0 \\ \lceil x \rceil, & x < 0. \end{cases}$$

Why is $f(x)$ called the *integer part* of x ?

Increasing and Decreasing Functions

Graph the functions in Exercises 37–46. What symmetries, if any, do the graphs have? Specify the intervals over which the function is increasing and the intervals where it is decreasing.

37. $y = -x^3$ 38. $y = -\frac{1}{x^2}$
 39. $y = -\frac{1}{x}$ 40. $y = \frac{1}{|x|}$
 41. $y = \sqrt{|x|}$ 42. $y = \sqrt{-x}$
 43. $y = x^3/8$ 44. $y = -4\sqrt{x}$
 45. $y = -x^{3/2}$ 46. $y = (-x)^{2/3}$

Even and Odd Functions

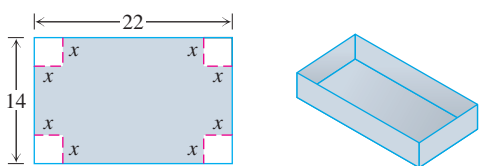
In Exercises 47–58, say whether the function is even, odd, or neither. Give reasons for your answer.

47. $f(x) = 3$ 48. $f(x) = x^{-5}$
 49. $f(x) = x^2 + 1$ 50. $f(x) = x^2 + x$
 51. $g(x) = x^3 + x$ 52. $g(x) = x^4 + 3x^2 - 1$
 53. $g(x) = \frac{1}{x^2 - 1}$ 54. $g(x) = \frac{x}{x^2 - 1}$
 55. $h(t) = \frac{1}{t - 1}$ 56. $h(t) = |t^3|$
 57. $h(t) = 2t + 1$ 58. $h(t) = 2|t| + 1$

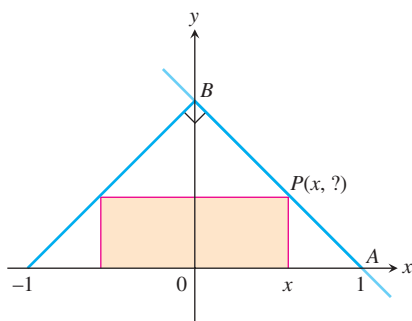
Theory and Examples

59. The variable s is proportional to t , and $s = 25$ when $t = 75$. Determine t when $s = 60$.

60. **Kinetic energy** The kinetic energy K of a mass is proportional to the square of its velocity v . If $K = 12,960$ joules when $v = 18$ m/sec, what is K when $v = 10$ m/sec?
61. The variables r and s are inversely proportional, and $r = 6$ when $s = 4$. Determine s when $r = 10$.
62. **Boyle's Law** Boyle's Law says that the volume V of a gas at constant temperature increases whenever the pressure P decreases, so that V and P are inversely proportional. If $P = 14.7$ lbs/in² when $V = 1000$ in³, then what is V when $P = 23.4$ lbs/in²?
63. A box with an open top is to be constructed from a rectangular piece of cardboard with dimensions 14 in. by 22 in. by cutting out equal squares of side x at each corner and then folding up the sides as in the figure. Express the volume V of the box as a function of x .

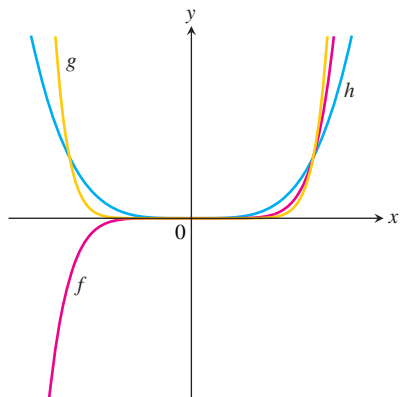


64. The accompanying figure shows a rectangle inscribed in an isosceles right triangle whose hypotenuse is 2 units long.
- Express the y -coordinate of P in terms of x . (You might start by writing an equation for the line AB .)
 - Express the area of the rectangle in terms of x .

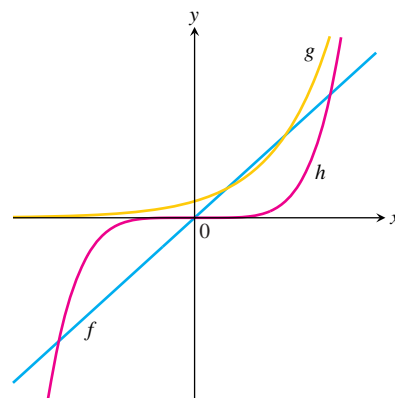


In Exercises 65 and 66, match each equation with its graph. Do not use a graphing device, and give reasons for your answer.

65. a. $y = x^4$ b. $y = x^7$ c. $y = x^{10}$



66. a. $y = 5x$ b. $y = 5^x$ c. $y = x^5$



- T 67. a. Graph the functions $f(x) = x/2$ and $g(x) = 1 + (4/x)$ together to identify the values of x for which

$$\frac{x}{2} > 1 + \frac{4}{x}.$$

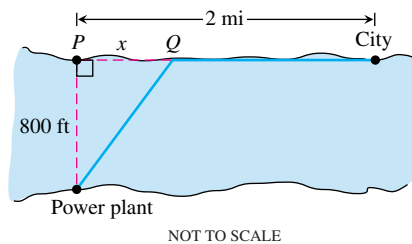
- b. Confirm your findings in part (a) algebraically.

- T 68. a. Graph the functions $f(x) = 3/(x - 1)$ and $g(x) = 2/(x + 1)$ together to identify the values of x for which

$$\frac{3}{x - 1} < \frac{2}{x + 1}.$$

- b. Confirm your findings in part (a) algebraically.

69. For a curve to be *symmetric about the x -axis*, the point (x, y) must lie on the curve if and only if the point $(x, -y)$ lies on the curve. Explain why a curve that is symmetric about the x -axis is not the graph of a function, unless the function is $y = 0$.
70. Three hundred books sell for \$40 each, resulting in a revenue of $(300)(\$40) = \$12,000$. For each \$5 increase in the price, 25 fewer books are sold. Write the revenue R as a function of the number x of \$5 increases.
71. A pen in the shape of an isosceles right triangle with legs of length x ft and hypotenuse of length h ft is to be built. If fencing costs \$5/ft for the legs and \$10/ft for the hypotenuse, write the total cost C of construction as a function of h .
72. **Industrial costs** A power plant sits next to a river where the river is 800 ft wide. To lay a new cable from the plant to a location in the city 2 mi downstream on the opposite side costs \$180 per foot across the river and \$100 per foot along the land.



- Suppose that the cable goes from the plant to a point Q on the opposite side that is x ft from the point P directly opposite the plant. Write a function $C(x)$ that gives the cost of laying the cable in terms of the distance x .
- Generate a table of values to determine if the least expensive location for point Q is less than 2000 ft or greater than 2000 ft from point P .

1.2 Combining Functions; Shifting and Scaling Graphs

In this section we look at the main ways functions are combined or transformed to form new functions.

Sums, Differences, Products, and Quotients

Like numbers, functions can be added, subtracted, multiplied, and divided (except where the denominator is zero) to produce new functions. If f and g are functions, then for every x that belongs to the domains of both f and g (that is, for $x \in D(f) \cap D(g)$), we define functions $f + g$, $f - g$, and fg by the formulas

$$\begin{aligned}(f + g)(x) &= f(x) + g(x). \\ (f - g)(x) &= f(x) - g(x). \\ (fg)(x) &= f(x)g(x).\end{aligned}$$

Notice that the $+$ sign on the left-hand side of the first equation represents the operation of addition of *functions*, whereas the $+$ on the right-hand side of the equation means addition of the real numbers $f(x)$ and $g(x)$.

At any point of $D(f) \cap D(g)$ at which $g(x) \neq 0$, we can also define the function f/g by the formula

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad (\text{where } g(x) \neq 0).$$

Functions can also be multiplied by constants: If c is a real number, then the function cf is defined for all x in the domain of f by

$$(cf)(x) = cf(x).$$

EXAMPLE 1 The functions defined by the formulas

$$f(x) = \sqrt{x} \quad \text{and} \quad g(x) = \sqrt{1-x}$$

have domains $D(f) = [0, \infty)$ and $D(g) = (-\infty, 1]$. The points common to these domains are the points

$$[0, \infty) \cap (-\infty, 1] = [0, 1].$$

The following table summarizes the formulas and domains for the various algebraic combinations of the two functions. We also write $f \cdot g$ for the product function fg .

Function	Formula	Domain
$f + g$	$(f + g)(x) = \sqrt{x} + \sqrt{1-x}$	$[0, 1] = D(f) \cap D(g)$
$f - g$	$(f - g)(x) = \sqrt{x} - \sqrt{1-x}$	$[0, 1]$
$g - f$	$(g - f)(x) = \sqrt{1-x} - \sqrt{x}$	$[0, 1]$
$f \cdot g$	$(f \cdot g)(x) = f(x)g(x) = \sqrt{x(1-x)}$	$[0, 1]$
f/g	$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \sqrt{\frac{x}{1-x}}$	$[0, 1)$ ($x = 1$ excluded)
g/f	$\frac{g}{f}(x) = \frac{g(x)}{f(x)} = \sqrt{\frac{1-x}{x}}$	$(0, 1]$ ($x = 0$ excluded)

The graph of the function $f + g$ is obtained from the graphs of f and g by adding the corresponding y -coordinates $f(x)$ and $g(x)$ at each point $x \in D(f) \cap D(g)$, as in Figure 1.25. The graphs of $f + g$ and $f \cdot g$ from Example 1 are shown in Figure 1.26.

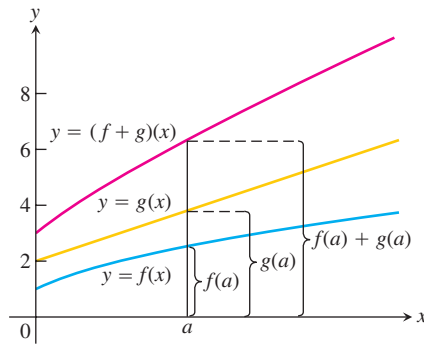


FIGURE 1.25 Graphical addition of two functions.

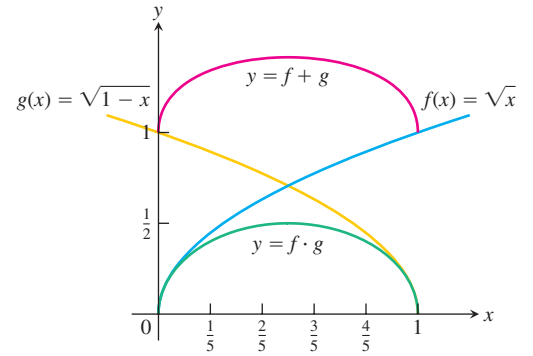


FIGURE 1.26 The domain of the function $f + g$ is the intersection of the domains of f and g , the interval $[0, 1]$ on the x -axis where these domains overlap. This interval is also the domain of the function $f \cdot g$ (Example 1).

Composite Functions

Composition is another method for combining functions.

DEFINITION If f and g are functions, the **composite** function $f \circ g$ (“ f composed with g ”) is defined by

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f .

The definition implies that $f \circ g$ can be formed when the range of g lies in the domain of f . To find $(f \circ g)(x)$, *first* find $g(x)$ and *second* find $f(g(x))$. Figure 1.27 pictures $f \circ g$ as a machine diagram and Figure 1.28 shows the composite as an arrow diagram.

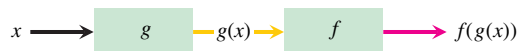


FIGURE 1.27 Two functions can be composed at x whenever the value of one function at x lies in the domain of the other. The composite is denoted by $f \circ g$.

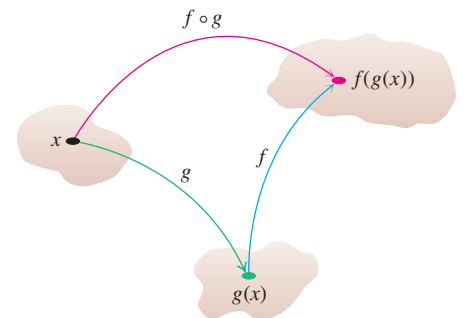


FIGURE 1.28 Arrow diagram for $f \circ g$.

To evaluate the composite function $g \circ f$ (when defined), we find $f(x)$ first and then $g(f(x))$. The domain of $g \circ f$ is the set of numbers x in the domain of f such that $f(x)$ lies in the domain of g .

The functions $f \circ g$ and $g \circ f$ are usually quite different.

EXAMPLE 2 If $f(x) = \sqrt{x}$ and $g(x) = x + 1$, find

- (a) $(f \circ g)(x)$ (b) $(g \circ f)(x)$ (c) $(f \circ f)(x)$ (d) $(g \circ g)(x)$.

Solution

Composite	Domain
(a) $(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x + 1}$	$[-1, \infty)$
(b) $(g \circ f)(x) = g(f(x)) = f(x) + 1 = \sqrt{x} + 1$	$[0, \infty)$
(c) $(f \circ f)(x) = f(f(x)) = \sqrt{f(x)} = \sqrt{\sqrt{x}} = x^{1/4}$	$[0, \infty)$
(d) $(g \circ g)(x) = g(g(x)) = g(x) + 1 = (x + 1) + 1 = x + 2$	$(-\infty, \infty)$

To see why the domain of $f \circ g$ is $[-1, \infty)$, notice that $g(x) = x + 1$ is defined for all real x but belongs to the domain of f only if $x + 1 \geq 0$, that is to say, when $x \geq -1$. ■

Notice that if $f(x) = x^2$ and $g(x) = \sqrt{x}$, then $(f \circ g)(x) = (\sqrt{x})^2 = x$. However, the domain of $f \circ g$ is $[0, \infty)$, not $(-\infty, \infty)$, since \sqrt{x} requires $x \geq 0$.

Shifting a Graph of a Function

A common way to obtain a new function from an existing one is by adding a constant to each output of the existing function, or to its input variable. The graph of the new function is the graph of the original function shifted vertically or horizontally, as follows.

Shift Formulas

Vertical Shifts

$y = f(x) + k$ Shifts the graph of f up k units if $k > 0$
 Shifts it down $|k|$ units if $k < 0$

Horizontal Shifts

$y = f(x + h)$ Shifts the graph of f left h units if $h > 0$
 Shifts it right $|h|$ units if $h < 0$

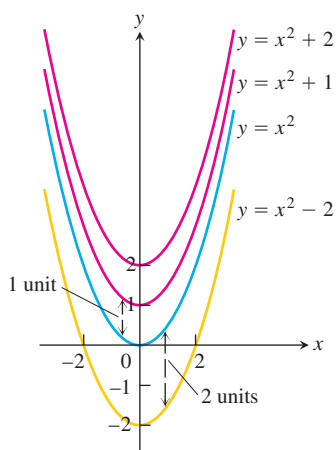


FIGURE 1.29 To shift the graph of $f(x) = x^2$ up (or down), we add positive (or negative) constants to the formula for f (Examples 3a and b).

EXAMPLE 3

- (a) Adding 1 to the right-hand side of the formula $y = x^2$ to get $y = x^2 + 1$ shifts the graph up 1 unit (Figure 1.29).
 (b) Adding -2 to the right-hand side of the formula $y = x^2$ to get $y = x^2 - 2$ shifts the graph down 2 units (Figure 1.29).
 (c) Adding 3 to x in $y = x^2$ to get $y = (x + 3)^2$ shifts the graph 3 units to the left (Figure 1.30).
 (d) Adding -2 to x in $y = |x|$, and then adding -1 to the result, gives $y = |x - 2| - 1$ and shifts the graph 2 units to the right and 1 unit down (Figure 1.31). ■

Scaling and Reflecting a Graph of a Function

To scale the graph of a function $y = f(x)$ is to stretch or compress it, vertically or horizontally. This is accomplished by multiplying the function f , or the independent variable x , by an appropriate constant c . Reflections across the coordinate axes are special cases where $c = -1$.

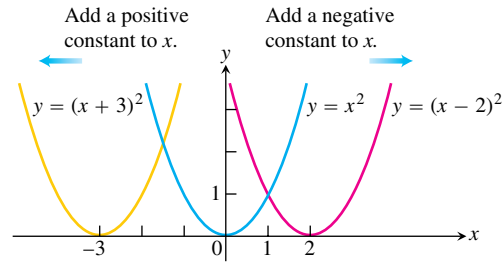


FIGURE 1.30 To shift the graph of $y = x^2$ to the left, we add a positive constant to x (Example 3c). To shift the graph to the right, we add a negative constant to x .

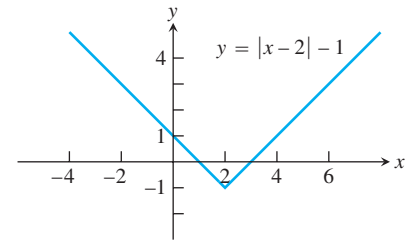


FIGURE 1.31 Shifting the graph of $y = |x|$ 2 units to the right and 1 unit down (Example 3d).

Vertical and Horizontal Scaling and Reflecting Formulas

For $c > 1$, the graph is scaled:

$y = cf(x)$ Stretches the graph of f vertically by a factor of c .

$y = \frac{1}{c}f(x)$ Compresses the graph of f vertically by a factor of c .

$y = f(cx)$ Compresses the graph of f horizontally by a factor of c .

$y = f(x/c)$ Stretches the graph of f horizontally by a factor of c .

For $c = -1$, the graph is reflected:

$y = -f(x)$ Reflects the graph of f across the x -axis.

$y = f(-x)$ Reflects the graph of f across the y -axis.

EXAMPLE 4 Here we scale and reflect the graph of $y = \sqrt{x}$.

- (a) **Vertical:** Multiplying the right-hand side of $y = \sqrt{x}$ by 3 to get $y = 3\sqrt{x}$ stretches the graph vertically by a factor of 3, whereas multiplying by $1/3$ compresses the graph by a factor of 3 (Figure 1.32).
- (b) **Horizontal:** The graph of $y = \sqrt{3x}$ is a horizontal compression of the graph of $y = \sqrt{x}$ by a factor of 3, and $y = \sqrt{x/3}$ is a horizontal stretching by a factor of 3 (Figure 1.33). Note that $y = \sqrt{3x} = \sqrt{3}\sqrt{x}$ so a horizontal compression *may* correspond to a vertical stretching by a different scaling factor. Likewise, a horizontal stretching may correspond to a vertical compression by a different scaling factor.
- (c) **Reflection:** The graph of $y = -\sqrt{x}$ is a reflection of $y = \sqrt{x}$ across the x -axis, and $y = \sqrt{-x}$ is a reflection across the y -axis (Figure 1.34). ■

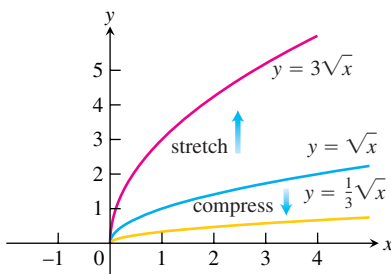


FIGURE 1.32 Vertically stretching and compressing the graph $y = \sqrt{x}$ by a factor of 3 (Example 4a).

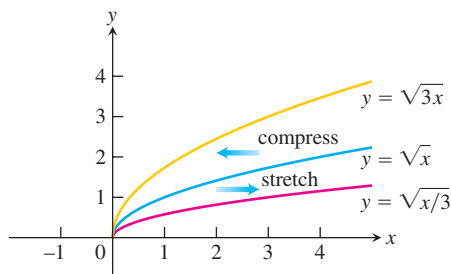


FIGURE 1.33 Horizontally stretching and compressing the graph $y = \sqrt{x}$ by a factor of 3 (Example 4b).

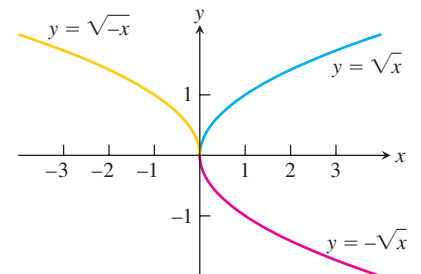


FIGURE 1.34 Reflections of the graph $y = \sqrt{x}$ across the coordinate axes (Example 4c).

EXAMPLE 5 Given the function $f(x) = x^4 - 4x^3 + 10$ (Figure 1.35a), find formulas to

- (a) compress the graph horizontally by a factor of 2 followed by a reflection across the y -axis (Figure 1.35b).
- (b) compress the graph vertically by a factor of 2 followed by a reflection across the x -axis (Figure 1.35c).

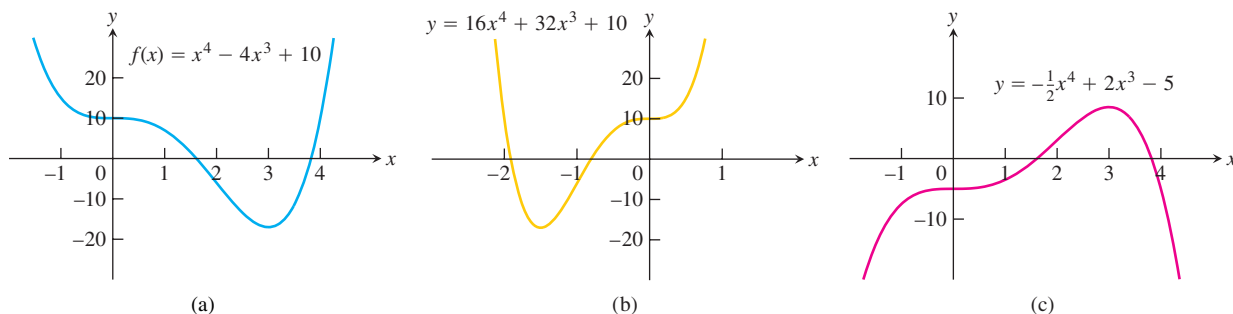


FIGURE 1.35 (a) The original graph of f . (b) The horizontal compression of $y = f(x)$ in part (a) by a factor of 2, followed by a reflection across the y -axis. (c) The vertical compression of $y = f(x)$ in part (a) by a factor of 2, followed by a reflection across the x -axis (Example 5).

Solution

- (a) We multiply x by 2 to get the horizontal compression, and by -1 to give reflection across the y -axis. The formula is obtained by substituting $-2x$ for x in the right-hand side of the equation for f :

$$\begin{aligned}
 y &= f(-2x) = (-2x)^4 - 4(-2x)^3 + 10 \\
 &= 16x^4 + 32x^3 + 10.
 \end{aligned}$$

- (b) The formula is

$$y = -\frac{1}{2}f(x) = -\frac{1}{2}x^4 + 2x^3 - 5. \quad \blacksquare$$

Ellipses

Although they are not the graphs of functions, circles can be stretched horizontally or vertically in the same way as the graphs of functions. The standard equation for a circle of radius r centered at the origin is

$$x^2 + y^2 = r^2.$$

Substituting cx for x in the standard equation for a circle (Figure 1.36a) gives

$$c^2x^2 + y^2 = r^2. \tag{1}$$

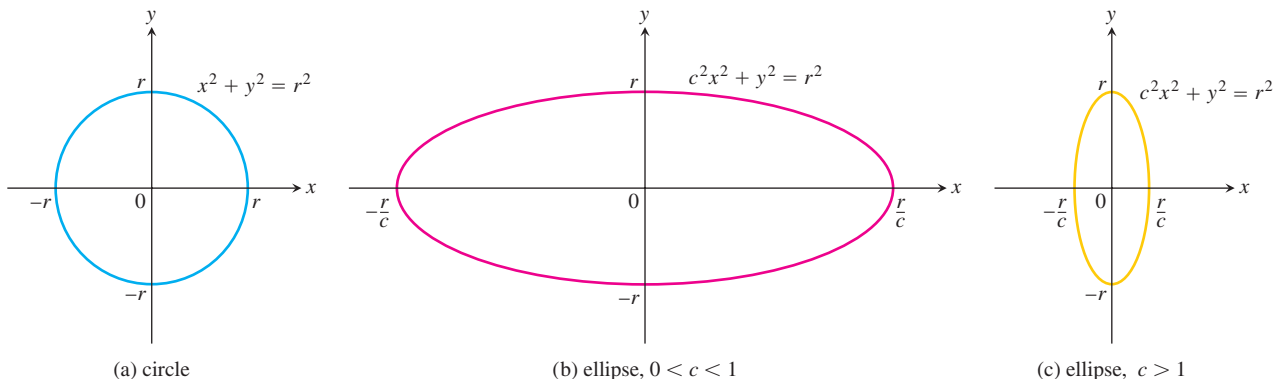


FIGURE 1.36 Horizontal stretching or compression of a circle produces graphs of ellipses.

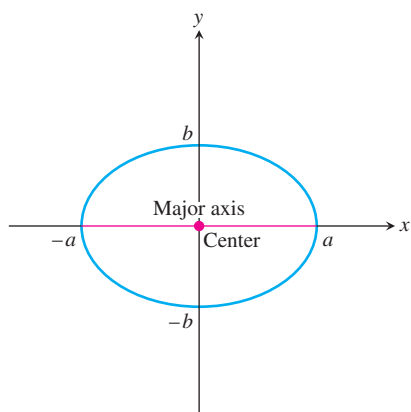


FIGURE 1.37 Graph of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b$, where the major axis is horizontal.

If $0 < c < 1$, the graph of Equation (1) horizontally stretches the circle; if $c > 1$ the circle is compressed horizontally. In either case, the graph of Equation (1) is an ellipse (Figure 1.36). Notice in Figure 1.36 that the y -intercepts of all three graphs are always $-r$ and r . In Figure 1.36b, the line segment joining the points $(\pm r/c, 0)$ is called the **major axis** of the ellipse; the **minor axis** is the line segment joining $(0, \pm r)$. The axes of the ellipse are reversed in Figure 1.36c: The major axis is the line segment joining the points $(0, \pm r)$, and the minor axis is the line segment joining the points $(\pm r/c, 0)$. In both cases, the major axis is the longer line segment.

If we divide both sides of Equation (1) by r^2 , we obtain

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (2)$$

where $a = r/c$ and $b = r$. If $a > b$, the major axis is horizontal; if $a < b$, the major axis is vertical. The **center** of the ellipse given by Equation (2) is the origin (Figure 1.37).

Substituting $x - h$ for x , and $y - k$ for y , in Equation (2) results in

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1. \quad (3)$$

Equation (3) is the **standard equation of an ellipse** with center at (h, k) . The geometric definition and properties of ellipses are reviewed in Section 11.6.

Exercises 1.2

Algebraic Combinations

In Exercises 1 and 2, find the domains and ranges of f , g , $f + g$, and $f \cdot g$.

- $f(x) = x$, $g(x) = \sqrt{x - 1}$
- $f(x) = \sqrt{x + 1}$, $g(x) = \sqrt{x - 1}$

In Exercises 3 and 4, find the domains and ranges of f , g , f/g , and g/f .

- $f(x) = 2$, $g(x) = x^2 + 1$
- $f(x) = 1$, $g(x) = 1 + \sqrt{x}$

Composites of Functions

- If $f(x) = x + 5$ and $g(x) = x^2 - 3$, find the following.
 - $f(g(0))$
 - $g(f(0))$
 - $f(g(x))$
 - $g(f(x))$
 - $f(f(-5))$
 - $g(g(2))$
 - $f(f(x))$
 - $g(g(x))$
- If $f(x) = x - 1$ and $g(x) = 1/(x + 1)$, find the following.
 - $f(g(1/2))$
 - $g(f(1/2))$
 - $f(g(x))$
 - $g(f(x))$
 - $f(f(2))$
 - $g(g(2))$
 - $f(f(x))$
 - $g(g(x))$

In Exercises 7–10, write a formula for $f \circ g \circ h$.

- $f(x) = x + 1$, $g(x) = 3x$, $h(x) = 4 - x$
- $f(x) = 3x + 4$, $g(x) = 2x - 1$, $h(x) = x^2$

$$9. f(x) = \sqrt{x + 1}, \quad g(x) = \frac{1}{x + 4}, \quad h(x) = \frac{1}{x}$$

$$10. f(x) = \frac{x + 2}{3 - x}, \quad g(x) = \frac{x^2}{x^2 + 1}, \quad h(x) = \sqrt{2 - x}$$

Let $f(x) = x - 3$, $g(x) = \sqrt{x}$, $h(x) = x^3$, and $j(x) = 2x$. Express each of the functions in Exercises 11 and 12 as a composite involving one or more of f , g , h , and j .

- $y = \sqrt{x} - 3$
 - $y = 2\sqrt{x}$
 - $y = x^{1/4}$
 - $y = 4x$
 - $y = \sqrt{(x - 3)^3}$
 - $y = (2x - 6)^3$
- $y = 2x - 3$
 - $y = x^{3/2}$
 - $y = x^9$
 - $y = x - 6$
 - $y = 2\sqrt{x - 3}$
 - $y = \sqrt{x^3 - 3}$

13. Copy and complete the following table.

$g(x)$	$f(x)$	$(f \circ g)(x)$
a. $x - 7$	\sqrt{x}	?
b. $x + 2$	$3x$?
c. ?	$\sqrt{x - 5}$	$\sqrt{x^2 - 5}$
d. $\frac{x}{x - 1}$	$\frac{x}{x - 1}$?
e. ?	$1 + \frac{1}{x}$	x
f. $\frac{1}{x}$?	x

14. Copy and complete the following table.

$g(x)$	$f(x)$	$(f \circ g)(x)$
a. $\frac{1}{x-1}$	$ x $?
b. ?	$\frac{x-1}{x}$	$\frac{x}{x+1}$
c. ?	\sqrt{x}	$ x $
d. \sqrt{x}	?	$ x $

15. Evaluate each expression using the given table of values

x	-2	-1	0	1	2
$f(x)$	1	0	-2	1	2
$g(x)$	2	1	0	-1	0

- a. $f(g(-1))$ b. $g(f(0))$ c. $f(f(-1))$
 d. $g(g(2))$ e. $g(f(-2))$ f. $f(g(1))$

16. Evaluate each expression using the functions

$$f(x) = 2 - x, \quad g(x) = \begin{cases} -x, & -2 \leq x < 0 \\ x - 1, & 0 \leq x \leq 2. \end{cases}$$

- a. $f(g(0))$ b. $g(f(3))$ c. $g(g(-1))$
 d. $f(f(2))$ e. $g(f(0))$ f. $f(g(1/2))$

In Exercises 17 and 18, (a) write formulas for $f \circ g$ and $g \circ f$ and find the (b) domain and (c) range of each.

17. $f(x) = \sqrt{x+1}$, $g(x) = \frac{1}{x}$

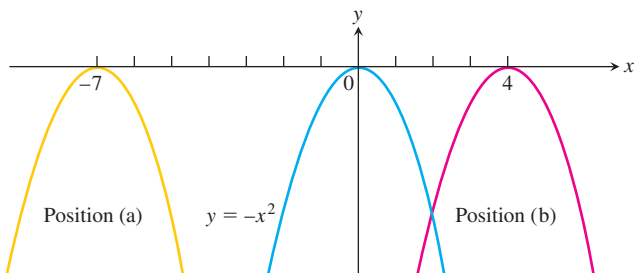
18. $f(x) = x^2$, $g(x) = 1 - \sqrt{x}$

19. Let $f(x) = \frac{x}{x-2}$. Find a function $y = g(x)$ so that $(f \circ g)(x) = x$.

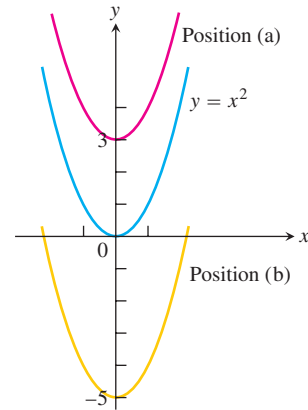
20. Let $f(x) = 2x^3 - 4$. Find a function $y = g(x)$ so that $(f \circ g)(x) = x + 2$.

Shifting Graphs

21. The accompanying figure shows the graph of $y = -x^2$ shifted to two new positions. Write equations for the new graphs.

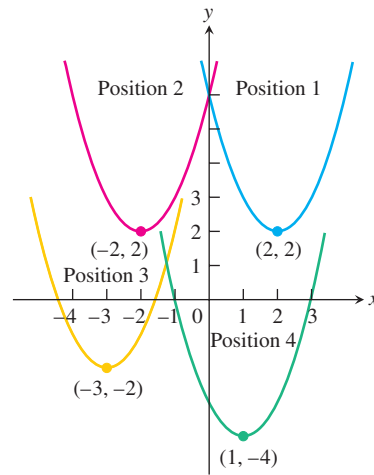


22. The accompanying figure shows the graph of $y = x^2$ shifted to two new positions. Write equations for the new graphs.

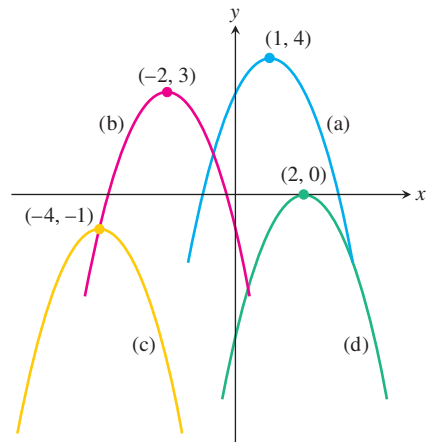


23. Match the equations listed in parts (a)–(d) to the graphs in the accompanying figure.

- a. $y = (x - 1)^2 - 4$ b. $y = (x - 2)^2 + 2$
 c. $y = (x + 2)^2 + 2$ d. $y = (x + 3)^2 - 2$



24. The accompanying figure shows the graph of $y = -x^2$ shifted to four new positions. Write an equation for each new graph.



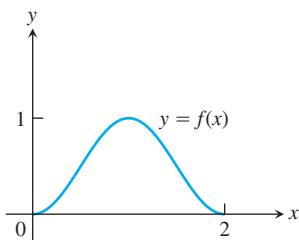
Exercises 25–34 tell how many units and in what directions the graphs of the given equations are to be shifted. Give an equation for the shifted graph. Then sketch the original and shifted graphs together, labeling each graph with its equation.

25. $x^2 + y^2 = 49$ Down 3, left 2
 26. $x^2 + y^2 = 25$ Up 3, left 4
 27. $y = x^3$ Left 1, down 1
 28. $y = x^{2/3}$ Right 1, down 1
 29. $y = \sqrt{x}$ Left 0.81
 30. $y = -\sqrt{x}$ Right 3
 31. $y = 2x - 7$ Up 7
 32. $y = \frac{1}{2}(x + 1) + 5$ Down 5, right 1
 33. $y = 1/x$ Up 1, right 1
 34. $y = 1/x^2$ Left 2, down 1

Graph the functions in Exercises 35–54.

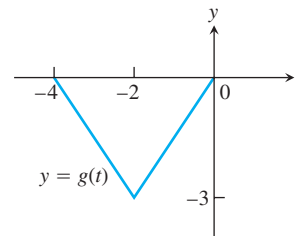
35. $y = \sqrt{x + 4}$ 36. $y = \sqrt{9 - x}$
 37. $y = |x - 2|$ 38. $y = |1 - x| - 1$
 39. $y = 1 + \sqrt{x - 1}$ 40. $y = 1 - \sqrt{x}$
 41. $y = (x + 1)^{2/3}$ 42. $y = (x - 8)^{2/3}$
 43. $y = 1 - x^{2/3}$ 44. $y + 4 = x^{2/3}$
 45. $y = \sqrt[3]{x - 1} - 1$ 46. $y = (x + 2)^{3/2} + 1$
 47. $y = \frac{1}{x - 2}$ 48. $y = \frac{1}{x} - 2$
 49. $y = \frac{1}{x} + 2$ 50. $y = \frac{1}{x + 2}$
 51. $y = \frac{1}{(x - 1)^2}$ 52. $y = \frac{1}{x^2} - 1$
 53. $y = \frac{1}{x^2} + 1$ 54. $y = \frac{1}{(x + 1)^2}$

55. The accompanying figure shows the graph of a function $f(x)$ with domain $[0, 2]$ and range $[0, 1]$. Find the domains and ranges of the following functions, and sketch their graphs.



- a. $f(x) + 2$ b. $f(x) - 1$
 c. $2f(x)$ d. $-f(x)$
 e. $f(x + 2)$ f. $f(x - 1)$
 g. $f(-x)$ h. $-f(x + 1) + 1$

56. The accompanying figure shows the graph of a function $g(t)$ with domain $[-4, 0]$ and range $[-3, 0]$. Find the domains and ranges of the following functions, and sketch their graphs.



- a. $g(-t)$ b. $-g(t)$
 c. $g(t) + 3$ d. $1 - g(t)$
 e. $g(-t + 2)$ f. $g(t - 2)$
 g. $g(1 - t)$ h. $-g(t - 4)$

Vertical and Horizontal Scaling

Exercises 57–66 tell by what factor and direction the graphs of the given functions are to be stretched or compressed. Give an equation for the stretched or compressed graph.

57. $y = x^2 - 1$, stretched vertically by a factor of 3
 58. $y = x^2 - 1$, compressed horizontally by a factor of 2
 59. $y = 1 + \frac{1}{x^2}$, compressed vertically by a factor of 2
 60. $y = 1 + \frac{1}{x^2}$, stretched horizontally by a factor of 3
 61. $y = \sqrt{x + 1}$, compressed horizontally by a factor of 4
 62. $y = \sqrt{x + 1}$, stretched vertically by a factor of 3
 63. $y = \sqrt{4 - x^2}$, stretched horizontally by a factor of 2
 64. $y = \sqrt{4 - x^2}$, compressed vertically by a factor of 3
 65. $y = 1 - x^3$, compressed horizontally by a factor of 3
 66. $y = 1 - x^3$, stretched horizontally by a factor of 2

Graphing

In Exercises 67–74, graph each function, not by plotting points, but by starting with the graph of one of the standard functions presented in Figures 1.14–1.17 and applying an appropriate transformation.

67. $y = -\sqrt{2x + 1}$ 68. $y = \sqrt{1 - \frac{x}{2}}$
 69. $y = (x - 1)^3 + 2$ 70. $y = (1 - x)^3 + 2$
 71. $y = \frac{1}{2x} - 1$ 72. $y = \frac{2}{x^2} + 1$
 73. $y = -\sqrt[3]{x}$ 74. $y = (-2x)^{2/3}$
 75. Graph the function $y = |x^2 - 1|$.
 76. Graph the function $y = \sqrt{|x|}$.

Ellipses

Exercises 77–82 give equations of ellipses. Put each equation in standard form and sketch the ellipse.

77. $9x^2 + 25y^2 = 225$ 78. $16x^2 + 7y^2 = 112$
 79. $3x^2 + (y - 2)^2 = 3$ 80. $(x + 1)^2 + 2y^2 = 4$

81. $3(x - 1)^2 + 2(y + 2)^2 = 6$
82. $6\left(x + \frac{3}{2}\right)^2 + 9\left(y - \frac{1}{2}\right)^2 = 54$
83. Write an equation for the ellipse $(x^2/16) + (y^2/9) = 1$ shifted 4 units to the left and 3 units up. Sketch the ellipse and identify its center and major axis.
84. Write an equation for the ellipse $(x^2/4) + (y^2/25) = 1$ shifted 3 units to the right and 2 units down. Sketch the ellipse and identify its center and major axis.

Combining Functions

85. Assume that f is an even function, g is an odd function, and both f and g are defined on the entire real line \mathbb{R} . Which of the following (where defined) are even? odd?

- a. fg b. f/g c. g/f
 d. $f^2 = ff$ e. $g^2 = gg$ f. $f \circ g$
 g. $g \circ f$ h. $f \circ f$ i. $g \circ g$

86. Can a function be both even and odd? Give reasons for your answer.

T 87. (Continuation of Example 1.) Graph the functions $f(x) = \sqrt{x}$ and $g(x) = \sqrt{1 - x}$ together with their (a) sum, (b) product, (c) two differences, (d) two quotients.

T 88. Let $f(x) = x - 7$ and $g(x) = x^2$. Graph f and g together with $f \circ g$ and $g \circ f$.

1.3 Trigonometric Functions

This section reviews radian measure and the basic trigonometric functions.

Angles

Angles are measured in degrees or radians. The number of **radians** in the central angle $A'CB'$ within a circle of radius r is defined as the number of “radius units” contained in the arc s subtended by that central angle. If we denote this central angle by θ when measured in radians, this means that $\theta = s/r$ (Figure 1.38), or

$$s = r\theta \quad (\theta \text{ in radians}). \tag{1}$$

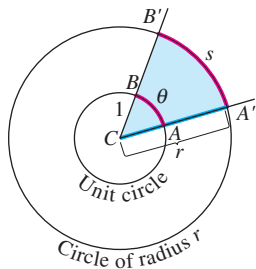


FIGURE 1.38 The radian measure of the central angle $A'CB'$ is the number $\theta = s/r$. For a unit circle of radius $r = 1$, θ is the length of arc AB that central angle ACB cuts from the unit circle.

If the circle is a unit circle having radius $r = 1$, then from Figure 1.38 and Equation (1), we see that the central angle θ measured in radians is just the length of the arc that the angle cuts from the unit circle. Since one complete revolution of the unit circle is 360° or 2π radians, we have

$$\pi \text{ radians} = 180^\circ \tag{2}$$

and

$$1 \text{ radian} = \frac{180}{\pi} (\approx 57.3) \text{ degrees} \quad \text{or} \quad 1 \text{ degree} = \frac{\pi}{180} (\approx 0.017) \text{ radians}.$$

Table 1.2 shows the equivalence between degree and radian measures for some basic angles.

TABLE 1.2 Angles measured in degrees and radians

Degrees	-180	-135	-90	-45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

An angle in the xy -plane is said to be in **standard position** if its vertex lies at the origin and its initial ray lies along the positive x -axis (Figure 1.39). Angles measured counterclockwise from the positive x -axis are assigned positive measures; angles measured clockwise are assigned negative measures.

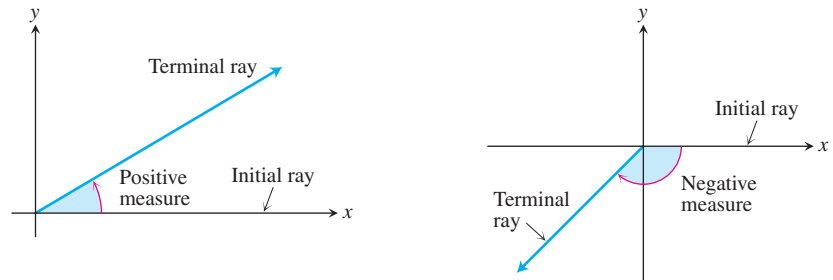


FIGURE 1.39 Angles in standard position in the xy -plane.

Angles describing counterclockwise rotations can go arbitrarily far beyond 2π radians or 360° . Similarly, angles describing clockwise rotations can have negative measures of all sizes (Figure 1.40).

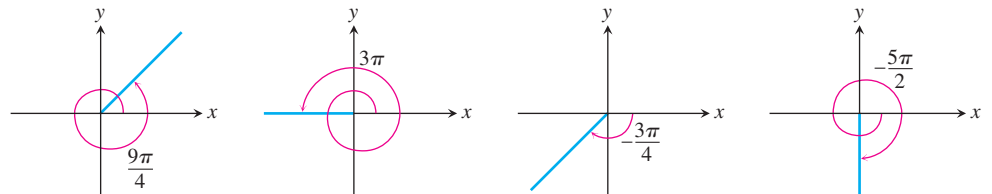
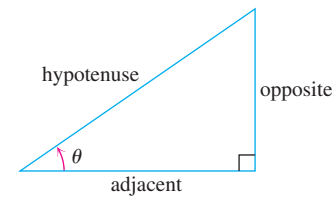


FIGURE 1.40 Nonzero radian measures can be positive or negative and can go beyond 2π .



$$\begin{aligned}\sin \theta &= \frac{\text{opp}}{\text{hyp}} & \csc \theta &= \frac{\text{hyp}}{\text{opp}} \\ \cos \theta &= \frac{\text{adj}}{\text{hyp}} & \sec \theta &= \frac{\text{hyp}}{\text{adj}} \\ \tan \theta &= \frac{\text{opp}}{\text{adj}} & \cot \theta &= \frac{\text{adj}}{\text{opp}}\end{aligned}$$

FIGURE 1.41 Trigonometric ratios of an acute angle.

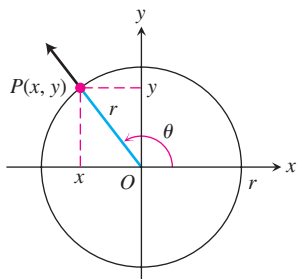


FIGURE 1.42 The trigonometric functions of a general angle θ are defined in terms of x , y , and r .

Angle Convention: Use Radians From now on, in this book it is assumed that all angles are measured in radians unless degrees or some other unit is stated explicitly. When we talk about the angle $\pi/3$, we mean $\pi/3$ radians (which is 60°), not $\pi/3$ degrees. We use radians because it simplifies many of the operations in calculus, and some results we will obtain involving the trigonometric functions are not true when angles are measured in degrees.

The Six Basic Trigonometric Functions

You are probably familiar with defining the trigonometric functions of an acute angle in terms of the sides of a right triangle (Figure 1.41). We extend this definition to obtuse and negative angles by first placing the angle in standard position in a circle of radius r . We then define the trigonometric functions in terms of the coordinates of the point $P(x, y)$ where the angle's terminal ray intersects the circle (Figure 1.42).

$$\begin{aligned}\text{sine: } \sin \theta &= \frac{y}{r} & \text{cosecant: } \csc \theta &= \frac{r}{y} \\ \text{cosine: } \cos \theta &= \frac{x}{r} & \text{secant: } \sec \theta &= \frac{r}{x} \\ \text{tangent: } \tan \theta &= \frac{y}{x} & \text{cotangent: } \cot \theta &= \frac{x}{y}\end{aligned}$$

These extended definitions agree with the right-triangle definitions when the angle is acute. Notice also that whenever the quotients are defined,

$$\begin{aligned}\tan \theta &= \frac{\sin \theta}{\cos \theta} & \cot \theta &= \frac{1}{\tan \theta} \\ \sec \theta &= \frac{1}{\cos \theta} & \csc \theta &= \frac{1}{\sin \theta}\end{aligned}$$

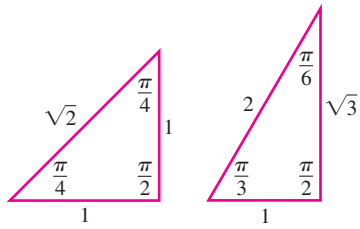


FIGURE 1.43 Radian angles and side lengths of two common triangles.

As you can see, $\tan \theta$ and $\sec \theta$ are not defined if $x = \cos \theta = 0$. This means they are not defined if θ is $\pm\pi/2, \pm3\pi/2, \dots$. Similarly, $\cot \theta$ and $\csc \theta$ are not defined for values of θ for which $y = 0$, namely $\theta = 0, \pm\pi, \pm2\pi, \dots$.

The exact values of these trigonometric ratios for some angles can be read from the triangles in Figure 1.43. For instance,

$$\begin{aligned} \sin \frac{\pi}{4} &= \frac{1}{\sqrt{2}} & \sin \frac{\pi}{6} &= \frac{1}{2} & \sin \frac{\pi}{3} &= \frac{\sqrt{3}}{2} \\ \cos \frac{\pi}{4} &= \frac{1}{\sqrt{2}} & \cos \frac{\pi}{6} &= \frac{\sqrt{3}}{2} & \cos \frac{\pi}{3} &= \frac{1}{2} \\ \tan \frac{\pi}{4} &= 1 & \tan \frac{\pi}{6} &= \frac{1}{\sqrt{3}} & \tan \frac{\pi}{3} &= \sqrt{3} \end{aligned}$$

The CAST rule (Figure 1.44) is useful for remembering when the basic trigonometric functions are positive or negative. For instance, from the triangle in Figure 1.45, we see that

$$\sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2}, \quad \cos \frac{2\pi}{3} = -\frac{1}{2}, \quad \tan \frac{2\pi}{3} = -\sqrt{3}.$$

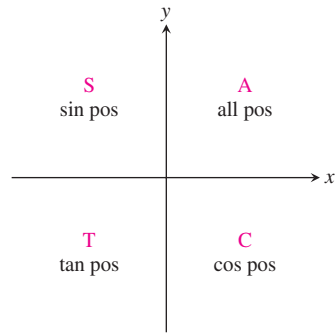


FIGURE 1.44 The CAST rule, remembered by the statement “Calculus Activates Student Thinking,” tells which trigonometric functions are positive in each quadrant.

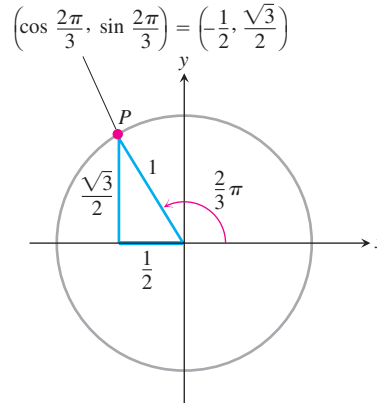


FIGURE 1.45 The triangle for calculating the sine and cosine of $2\pi/3$ radians. The side lengths come from the geometry of right triangles.

Using a similar method we determined the values of $\sin \theta$, $\cos \theta$, and $\tan \theta$ shown in Table 1.3.

Degrees	-180	-135	-90	-45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin \theta$	0	$-\frac{\sqrt{2}}{2}$	-1	$-\frac{\sqrt{2}}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0
$\cos \theta$	-1	$-\frac{\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	0	1
$\tan \theta$	0	1		-1	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$		$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0		0

Periodicity and Graphs of the Trigonometric Functions

When an angle of measure θ and an angle of measure $\theta + 2\pi$ are in standard position, their terminal rays coincide. The two angles therefore have the same trigonometric function values: $\sin(\theta + 2\pi) = \sin \theta$, $\tan(\theta + 2\pi) = \tan \theta$, and so on. Similarly, $\cos(\theta - 2\pi) = \cos \theta$, $\sin(\theta - 2\pi) = \sin \theta$, and so on. We describe this repeating behavior by saying that the six basic trigonometric functions are *periodic*.

Periods of Trigonometric Functions

- Period π :** $\tan(x + \pi) = \tan x$
 $\cot(x + \pi) = \cot x$
- Period 2π :** $\sin(x + 2\pi) = \sin x$
 $\cos(x + 2\pi) = \cos x$
 $\sec(x + 2\pi) = \sec x$
 $\csc(x + 2\pi) = \csc x$

DEFINITION A function $f(x)$ is **periodic** if there is a positive number p such that $f(x + p) = f(x)$ for every value of x . The smallest such value of p is the **period** of f .

When we graph trigonometric functions in the coordinate plane, we usually denote the independent variable by x instead of θ . Figure 1.46 shows that the tangent and cotangent functions have period $p = \pi$, and the other four functions have period 2π . Also, the symmetries in these graphs reveal that the cosine and secant functions are even and the other four functions are odd (although this does not prove those results).

Even

$\cos(-x) = \cos x$
 $\sec(-x) = \sec x$

Odd

$\sin(-x) = -\sin x$
 $\tan(-x) = -\tan x$
 $\csc(-x) = -\csc x$
 $\cot(-x) = -\cot x$

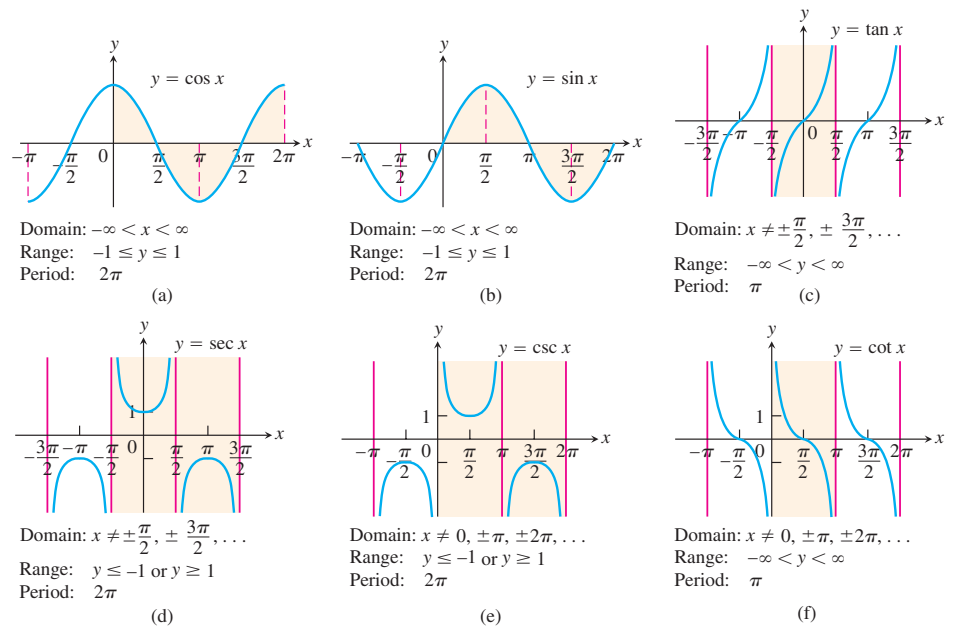


FIGURE 1.46 Graphs of the six basic trigonometric functions using radian measure. The shading for each trigonometric function indicates its periodicity.

Trigonometric Identities

The coordinates of any point $P(x, y)$ in the plane can be expressed in terms of the point's distance r from the origin and the angle θ that ray OP makes with the positive x -axis (Figure 1.42). Since $x/r = \cos \theta$ and $y/r = \sin \theta$, we have

$$x = r \cos \theta, \quad y = r \sin \theta.$$

When $r = 1$ we can apply the Pythagorean theorem to the reference right triangle in Figure 1.47 and obtain the equation

$$\cos^2 \theta + \sin^2 \theta = 1. \tag{3}$$

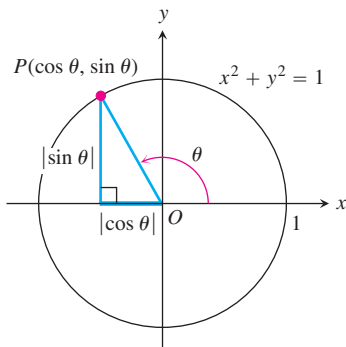


FIGURE 1.47 The reference triangle for a general angle θ .

This equation, true for all values of θ , is the most frequently used identity in trigonometry. Dividing this identity in turn by $\cos^2 \theta$ and $\sin^2 \theta$ gives

$$\begin{aligned} 1 + \tan^2 \theta &= \sec^2 \theta \\ 1 + \cot^2 \theta &= \csc^2 \theta \end{aligned}$$

The following formulas hold for all angles A and B (Exercise 58).

Addition Formulas

$$\begin{aligned} \cos(A + B) &= \cos A \cos B - \sin A \sin B \\ \sin(A + B) &= \sin A \cos B + \cos A \sin B \end{aligned} \quad (4)$$

There are similar formulas for $\cos(A - B)$ and $\sin(A - B)$ (Exercises 35 and 36). All the trigonometric identities needed in this book derive from Equations (3) and (4). For example, substituting θ for both A and B in the addition formulas gives

Double-Angle Formulas

$$\begin{aligned} \cos 2\theta &= \cos^2 \theta - \sin^2 \theta \\ \sin 2\theta &= 2 \sin \theta \cos \theta \end{aligned} \quad (5)$$

Additional formulas come from combining the equations

$$\cos^2 \theta + \sin^2 \theta = 1, \quad \cos^2 \theta - \sin^2 \theta = \cos 2\theta.$$

We add the two equations to get $2 \cos^2 \theta = 1 + \cos 2\theta$ and subtract the second from the first to get $2 \sin^2 \theta = 1 - \cos 2\theta$. This results in the following identities, which are useful in integral calculus.

Half-Angle Formulas

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2} \quad (6)$$

$$\sin^2 \theta = \frac{1 - \cos 2\theta}{2} \quad (7)$$

The Law of Cosines

If a , b , and c are sides of a triangle ABC and if θ is the angle opposite c , then

$$c^2 = a^2 + b^2 - 2ab \cos \theta. \quad (8)$$

This equation is called the **law of cosines**.

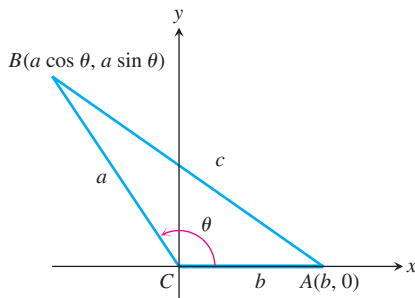


FIGURE 1.48 The square of the distance between A and B gives the law of cosines.

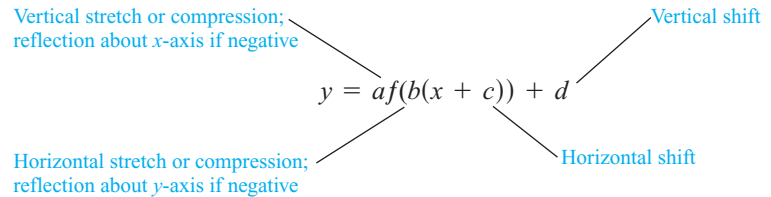
We can see why the law holds if we introduce coordinate axes with the origin at C and the positive x -axis along one side of the triangle, as in Figure 1.48. The coordinates of A are $(b, 0)$; the coordinates of B are $(a \cos \theta, a \sin \theta)$. The square of the distance between A and B is therefore

$$\begin{aligned} c^2 &= (a \cos \theta - b)^2 + (a \sin \theta)^2 \\ &= a^2(\underbrace{\cos^2 \theta + \sin^2 \theta}_1) + b^2 - 2ab \cos \theta \\ &= a^2 + b^2 - 2ab \cos \theta. \end{aligned}$$

The law of cosines generalizes the Pythagorean theorem. If $\theta = \pi/2$, then $\cos \theta = 0$ and $c^2 = a^2 + b^2$.

Transformations of Trigonometric Graphs

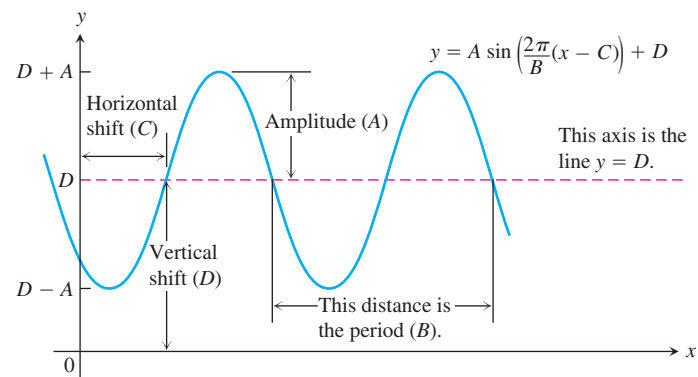
The rules for shifting, stretching, compressing, and reflecting the graph of a function summarized in the following diagram apply to the trigonometric functions we have discussed in this section.



The transformation rules applied to the sine function give the **general sine function** or **sinusoid** formula

$$f(x) = A \sin\left(\frac{2\pi}{B}(x - C)\right) + D,$$

where $|A|$ is the *amplitude*, $|B|$ is the *period*, C is the *horizontal shift*, and D is the *vertical shift*. A graphical interpretation of the various terms is revealing and given below.



Two Special Inequalities

For any angle θ measured in radians,

$$-|\theta| \leq \sin \theta \leq |\theta| \quad \text{and} \quad -|\theta| \leq 1 - \cos \theta \leq |\theta|.$$

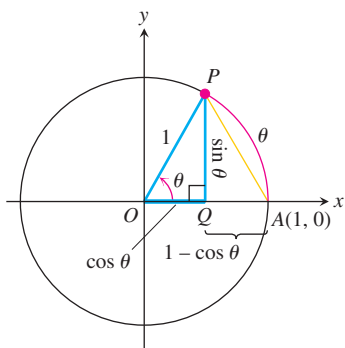


FIGURE 1.49 From the geometry of this figure, drawn for $\theta > 0$, we get the inequality $\sin^2 \theta + (1 - \cos \theta)^2 \leq \theta^2$.

To establish these inequalities, we picture θ as a nonzero angle in standard position (Figure 1.49). The circle in the figure is a unit circle, so $|\theta|$ equals the length of the circular arc AP . The length of line segment AP is therefore less than $|\theta|$.

Triangle APQ is a right triangle with sides of length

$$QP = |\sin \theta|, \quad AQ = 1 - \cos \theta.$$

From the Pythagorean theorem and the fact that $AP < |\theta|$, we get

$$\sin^2 \theta + (1 - \cos \theta)^2 = (AP)^2 \leq \theta^2. \tag{9}$$

The terms on the left-hand side of Equation (9) are both positive, so each is smaller than their sum and hence is less than or equal to θ^2 :

$$\sin^2 \theta \leq \theta^2 \quad \text{and} \quad (1 - \cos \theta)^2 \leq \theta^2.$$

By taking square roots, this is equivalent to saying that

$$|\sin \theta| \leq |\theta| \quad \text{and} \quad |1 - \cos \theta| \leq |\theta|,$$

so

$$-\theta \leq \sin \theta \leq \theta \quad \text{and} \quad -\theta \leq 1 - \cos \theta \leq \theta.$$

These inequalities will be useful in the next chapter.

Exercises 1.3

Radians and Degrees

- On a circle of radius 10 m, how long is an arc that subtends a central angle of (a) $4\pi/5$ radians? (b) 110° ?
- A central angle in a circle of radius 8 is subtended by an arc of length 10π . Find the angle's radian and degree measures.
- You want to make an 80° angle by marking an arc on the perimeter of a 12-in.-diameter disk and drawing lines from the ends of the arc to the disk's center. To the nearest tenth of an inch, how long should the arc be?
- If you roll a 1-m-diameter wheel forward 30 cm over level ground, through what angle will the wheel turn? Answer in radians (to the nearest tenth) and degrees (to the nearest degree).

Evaluating Trigonometric Functions

- Copy and complete the following table of function values. If the function is undefined at a given angle, enter "UND." Do not use a calculator or tables.

θ	$-\pi$	$-2\pi/3$	0	$\pi/2$	$3\pi/4$
----------	--------	-----------	---	---------	----------

$\sin \theta$
 $\cos \theta$
 $\tan \theta$
 $\cot \theta$
 $\sec \theta$
 $\csc \theta$

- Copy and complete the following table of function values. If the function is undefined at a given angle, enter "UND." Do not use a calculator or tables.

θ	$-3\pi/2$	$-\pi/3$	$-\pi/6$	$\pi/4$	$5\pi/6$
$\sin \theta$					
$\cos \theta$					
$\tan \theta$					
$\cot \theta$					
$\sec \theta$					
$\csc \theta$					

In Exercises 7–12, one of $\sin x$, $\cos x$, and $\tan x$ is given. Find the other two if x lies in the specified interval.

- $\sin x = \frac{3}{5}$, $x \in \left[\frac{\pi}{2}, \pi\right]$
- $\tan x = 2$, $x \in \left[0, \frac{\pi}{2}\right]$
- $\cos x = \frac{1}{3}$, $x \in \left[-\frac{\pi}{2}, 0\right]$
- $\cos x = -\frac{5}{13}$, $x \in \left[\frac{\pi}{2}, \pi\right]$
- $\tan x = \frac{1}{2}$, $x \in \left[\pi, \frac{3\pi}{2}\right]$
- $\sin x = -\frac{1}{2}$, $x \in \left[\pi, \frac{3\pi}{2}\right]$

Graphing Trigonometric Functions

Graph the functions in Exercises 13–22. What is the period of each function?

- $\sin 2x$
- $\sin(x/2)$
- $\cos \pi x$
- $\cos \frac{\pi x}{2}$
- $-\sin \frac{\pi x}{3}$
- $-\cos 2\pi x$
- $\cos\left(x - \frac{\pi}{2}\right)$
- $\sin\left(x + \frac{\pi}{6}\right)$

$$21. \sin\left(x - \frac{\pi}{4}\right) + 1 \qquad 22. \cos\left(x + \frac{2\pi}{3}\right) - 2$$

Graph the functions in Exercises 23–26 in the ts -plane (t -axis horizontal, s -axis vertical). What is the period of each function? What symmetries do the graphs have?

$$23. s = \cot 2t \qquad 24. s = -\tan \pi t$$

$$25. s = \sec\left(\frac{\pi t}{2}\right) \qquad 26. s = \csc\left(\frac{t}{2}\right)$$

- T** 27. a. Graph $y = \cos x$ and $y = \sec x$ together for $-3\pi/2 \leq x \leq 3\pi/2$. Comment on the behavior of $\sec x$ in relation to the signs and values of $\cos x$.
- b. Graph $y = \sin x$ and $y = \csc x$ together for $-\pi \leq x \leq 2\pi$. Comment on the behavior of $\csc x$ in relation to the signs and values of $\sin x$.
- T** 28. Graph $y = \tan x$ and $y = \cot x$ together for $-7 \leq x \leq 7$. Comment on the behavior of $\cot x$ in relation to the signs and values of $\tan x$.
29. Graph $y = \sin x$ and $y = \lfloor \sin x \rfloor$ together. What are the domain and range of $\lfloor \sin x \rfloor$?
30. Graph $y = \sin x$ and $y = \lceil \sin x \rceil$ together. What are the domain and range of $\lceil \sin x \rceil$?

Using the Addition Formulas

Use the addition formulas to derive the identities in Exercises 31–36.

$$31. \cos\left(x - \frac{\pi}{2}\right) = \sin x \qquad 32. \cos\left(x + \frac{\pi}{2}\right) = -\sin x$$

$$33. \sin\left(x + \frac{\pi}{2}\right) = \cos x \qquad 34. \sin\left(x - \frac{\pi}{2}\right) = -\cos x$$

35. $\cos(A - B) = \cos A \cos B + \sin A \sin B$ (Exercise 57 provides a different derivation.)
36. $\sin(A - B) = \sin A \cos B - \cos A \sin B$
37. What happens if you take $B = A$ in the trigonometric identity $\cos(A - B) = \cos A \cos B + \sin A \sin B$? Does the result agree with something you already know?
38. What happens if you take $B = 2\pi$ in the addition formulas? Do the results agree with something you already know?

In Exercises 39–42, express the given quantity in terms of $\sin x$ and $\cos x$.

$$39. \cos(\pi + x) \qquad 40. \sin(2\pi - x)$$

$$41. \sin\left(\frac{3\pi}{2} - x\right) \qquad 42. \cos\left(\frac{3\pi}{2} + x\right)$$

$$43. \text{Evaluate } \sin \frac{7\pi}{12} \text{ as } \sin\left(\frac{\pi}{4} + \frac{\pi}{3}\right).$$

$$44. \text{Evaluate } \cos \frac{11\pi}{12} \text{ as } \cos\left(\frac{\pi}{4} + \frac{2\pi}{3}\right).$$

$$45. \text{Evaluate } \cos \frac{\pi}{12}. \qquad 46. \text{Evaluate } \sin \frac{5\pi}{12}.$$

Using the Half-Angle Formulas

Find the function values in Exercises 47–50.

$$47. \cos^2 \frac{\pi}{8} \qquad 48. \cos^2 \frac{5\pi}{12}$$

$$49. \sin^2 \frac{\pi}{12} \qquad 50. \sin^2 \frac{3\pi}{8}$$

Solving Trigonometric Equations

For Exercises 51–54, solve for the angle θ , where $0 \leq \theta \leq 2\pi$.

$$51. \sin^2 \theta = \frac{3}{4}$$

$$52. \sin^2 \theta = \cos^2 \theta$$

$$53. \sin 2\theta - \cos \theta = 0$$

$$54. \cos 2\theta + \cos \theta = 0$$

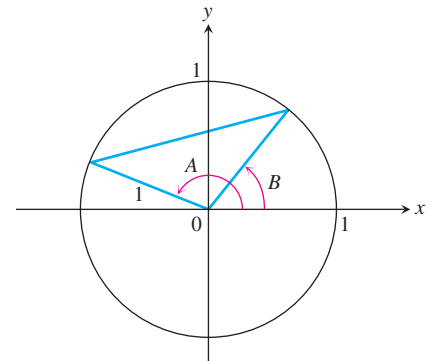
Theory and Examples

55. **The tangent sum formula** The standard formula for the tangent of the sum of two angles is

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

Derive the formula.

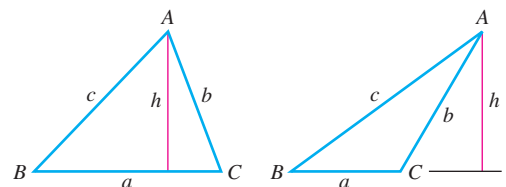
56. (Continuation of Exercise 55.) Derive a formula for $\tan(A - B)$.
57. Apply the law of cosines to the triangle in the accompanying figure to derive the formula for $\cos(A - B)$.



58. a. Apply the formula for $\cos(A - B)$ to the identity $\sin \theta = \cos\left(\frac{\pi}{2} - \theta\right)$ to obtain the addition formula for $\sin(A + B)$.
- b. Derive the formula for $\cos(A + B)$ by substituting $-B$ for B in the formula for $\cos(A - B)$ from Exercise 35.
59. A triangle has sides $a = 2$ and $b = 3$ and angle $C = 60^\circ$. Find the length of side c .
60. A triangle has sides $a = 2$ and $b = 3$ and angle $C = 40^\circ$. Find the length of side c .
61. **The law of sines** The law of sines says that if a , b , and c are the sides opposite the angles A , B , and C in a triangle, then

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

Use the accompanying figures and the identity $\sin(\pi - \theta) = \sin \theta$, if required, to derive the law.



62. A triangle has sides $a = 2$ and $b = 3$ and angle $C = 60^\circ$ (as in Exercise 59). Find the sine of angle B using the law of sines.

63. A triangle has side $c = 2$ and angles $A = \pi/4$ and $B = \pi/3$. Find the length a of the side opposite A .

T 64. **The approximation $\sin x \approx x$** It is often useful to know that, when x is measured in radians, $\sin x \approx x$ for numerically small values of x . In Section 3.11, we will see why the approximation holds. The approximation error is less than 1 in 5000 if $|x| < 0.1$.

- With your grapher in radian mode, graph $y = \sin x$ and $y = x$ together in a viewing window about the origin. What do you see happening as x nears the origin?
- With your grapher in degree mode, graph $y = \sin x$ and $y = x$ together about the origin again. How is the picture different from the one obtained with radian mode?

General Sine Curves

For

$$f(x) = A \sin\left(\frac{2\pi}{B}(x - C)\right) + D,$$

identify A , B , C , and D for the sine functions in Exercises 65–68 and sketch their graphs.

65. $y = 2 \sin(x + \pi) - 1$ 66. $y = \frac{1}{2} \sin(\pi x - \pi) + \frac{1}{2}$

67. $y = -\frac{2}{\pi} \sin\left(\frac{\pi}{2}t\right) + \frac{1}{\pi}$ 68. $y = \frac{L}{2\pi} \sin \frac{2\pi t}{L}, \quad L > 0$

COMPUTER EXPLORATIONS

In Exercises 69–72, you will explore graphically the general sine function

$$f(x) = A \sin\left(\frac{2\pi}{B}(x - C)\right) + D$$

as you change the values of the constants A , B , C , and D . Use a CAS or computer grapher to perform the steps in the exercises.

69. **The period B** Set the constants $A = 3$, $C = D = 0$.

- Plot $f(x)$ for the values $B = 1, 3, 2\pi, 5\pi$ over the interval $-4\pi \leq x \leq 4\pi$. Describe what happens to the graph of the general sine function as the period increases.
- What happens to the graph for negative values of B ? Try it with $B = -3$ and $B = -2\pi$.

70. **The horizontal shift C** Set the constants $A = 3$, $B = 6$, $D = 0$.

- Plot $f(x)$ for the values $C = 0, 1$, and 2 over the interval $-4\pi \leq x \leq 4\pi$. Describe what happens to the graph of the general sine function as C increases through positive values.
- What happens to the graph for negative values of C ?
- What smallest positive value should be assigned to C so the graph exhibits no horizontal shift? Confirm your answer with a plot.

71. **The vertical shift D** Set the constants $A = 3$, $B = 6$, $C = 0$.

- Plot $f(x)$ for the values $D = 0, 1$, and 3 over the interval $-4\pi \leq x \leq 4\pi$. Describe what happens to the graph of the general sine function as D increases through positive values.
- What happens to the graph for negative values of D ?

72. **The amplitude A** Set the constants $B = 6$, $C = D = 0$.

- Describe what happens to the graph of the general sine function as A increases through positive values. Confirm your answer by plotting $f(x)$ for the values $A = 1, 5$, and 9 .
- What happens to the graph for negative values of A ?

1.4

Graphing with Calculators and Computers

A graphing calculator or a computer with graphing software enables us to graph very complicated functions with high precision. Many of these functions could not otherwise be easily graphed. However, care must be taken when using such devices for graphing purposes, and in this section we address some of the issues involved. In Chapter 4 we will see how calculus helps us determine that we are accurately viewing all the important features of a function's graph.

Graphing Windows

When using a graphing calculator or computer as a graphing tool, a portion of the graph is displayed in a rectangular **display** or **viewing window**. Often the default window gives an incomplete or misleading picture of the graph. We use the term *square window* when the units or scales on both axes are the same. This term does not mean that the display window itself is square (usually it is rectangular), but instead it means that the x -unit is the same as the y -unit.

When a graph is displayed in the default window, the x -unit may differ from the y -unit of scaling in order to fit the graph in the window. The viewing window is set by specifying an interval $[a, b]$ for the x -values and an interval $[c, d]$ for the y -values. The machine selects equally spaced x -values in $[a, b]$ and then plots the points $(x, f(x))$. A point is plotted if and

only if x lies in the domain of the function and $f(x)$ lies within the interval $[c, d]$. A short line segment is then drawn between each plotted point and its next neighboring point. We now give illustrative examples of some common problems that may occur with this procedure.

EXAMPLE 1 Graph the function $f(x) = x^3 - 7x^2 + 28$ in each of the following display or viewing windows:

- (a) $[-10, 10]$ by $[-10, 10]$ (b) $[-4, 4]$ by $[-50, 10]$ (c) $[-4, 10]$ by $[-60, 60]$

Solution

- (a) We select $a = -10$, $b = 10$, $c = -10$, and $d = 10$ to specify the interval of x -values and the range of y -values for the window. The resulting graph is shown in Figure 1.50a. It appears that the window is cutting off the bottom part of the graph and that the interval of x -values is too large. Let's try the next window.

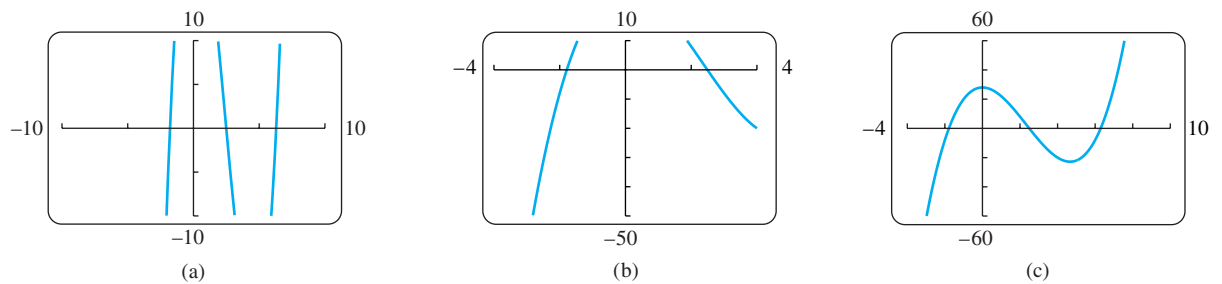


FIGURE 1.50 The graph of $f(x) = x^3 - 7x^2 + 28$ in different viewing windows. Selecting a window that gives a clear picture of a graph is often a trial-and-error process (Example 1).

- (b) Now we see more features of the graph (Figure 1.50b), but the top is missing and we need to view more to the right of $x = 4$ as well. The next window should help.
- (c) Figure 1.50c shows the graph in this new viewing window. Observe that we get a more complete picture of the graph in this window, and it is a reasonable graph of a third-degree polynomial. ■

EXAMPLE 2 When a graph is displayed, the x -unit may differ from the y -unit, as in the graphs shown in Figures 1.50b and 1.50c. The result is distortion in the picture, which may be misleading. The display window can be made square by compressing or stretching the units on one axis to match the scale on the other, giving the true graph. Many systems have built-in functions to make the window “square.” If yours does not, you will have to do some calculations and set the window size manually to get a square window, or bring to your viewing some foreknowledge of the true picture.

Figure 1.51a shows the graphs of the perpendicular lines $y = x$ and $y = -x + 3\sqrt{2}$, together with the semicircle $y = \sqrt{9 - x^2}$, in a nonsquare $[-4, 4]$ by $[-6, 8]$ display window. Notice the distortion. The lines do not appear to be perpendicular, and the semicircle appears to be elliptical in shape.

Figure 1.51b shows the graphs of the same functions in a square window in which the x -units are scaled to be the same as the y -units. Notice that the scaling on the x -axis for Figure 1.51a has been compressed in Figure 1.51b to make the window square. Figure 1.51c gives an enlarged view of Figure 1.51b with a square $[-3, 3]$ by $[0, 4]$ window. ■

If the denominator of a rational function is zero at some x -value within the viewing window, a calculator or graphing computer software may produce a steep near-vertical line segment from the top to the bottom of the window. Here is an example.

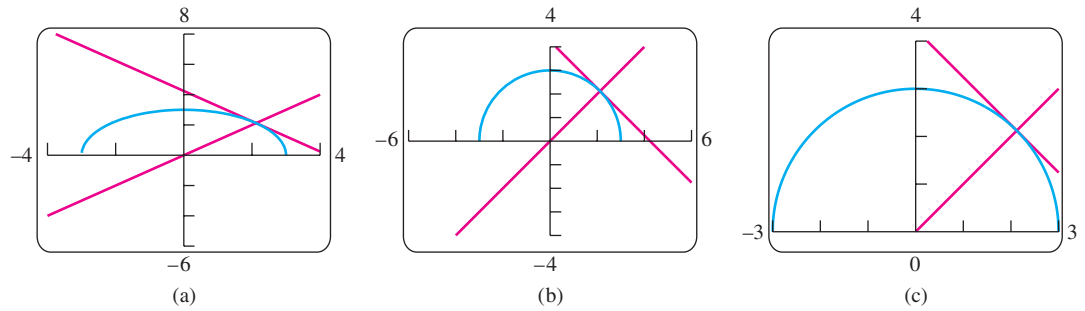


FIGURE 1.51 Graphs of the perpendicular lines $y = x$ and $y = -x + 3\sqrt{2}$, and the semicircle $y = \sqrt{9 - x^2}$ appear distorted (a) in a nonsquare window, but clear (b) and (c) in square windows (Example 2).

EXAMPLE 3 Graph the function $y = \frac{1}{2 - x}$.

Solution Figure 1.52a shows the graph in the $[-10, 10]$ by $[-10, 10]$ default square window with our computer graphing software. Notice the near-vertical line segment at $x = 2$. It is not truly a part of the graph and $x = 2$ does not belong to the domain of the function. By trial and error we can eliminate the line by changing the viewing window to the smaller $[-6, 6]$ by $[-4, 4]$ view, revealing a better graph (Figure 1.52b). ■

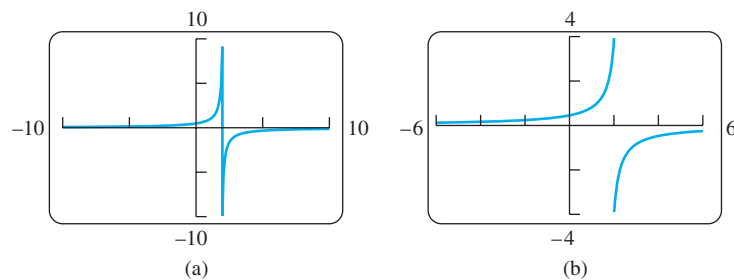


FIGURE 1.52 Graphs of the function $y = \frac{1}{2 - x}$. A vertical line may appear without a careful choice of the viewing window (Example 3).

Sometimes the graph of a trigonometric function oscillates very rapidly. When a calculator or computer software plots the points of the graph and connects them, many of the maximum and minimum points are actually missed. The resulting graph is then very misleading.

EXAMPLE 4 Graph the function $f(x) = \sin 100x$.

Solution Figure 1.53a shows the graph of f in the viewing window $[-12, 12]$ by $[-1, 1]$. We see that the graph looks very strange because the sine curve should oscillate periodically between -1 and 1 . This behavior is not exhibited in Figure 1.53a. We might

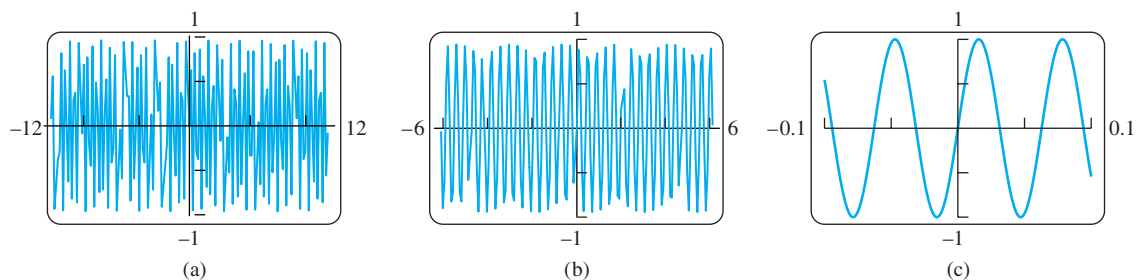


FIGURE 1.53 Graphs of the function $y = \sin 100x$ in three viewing windows. Because the period is $2\pi/100 \approx 0.063$, the smaller window in (c) best displays the true aspects of this rapidly oscillating function (Example 4).

experiment with a smaller viewing window, say $[-6, 6]$ by $[-1, 1]$, but the graph is not better (Figure 1.53b). The difficulty is that the period of the trigonometric function $y = \sin 100x$ is very small ($2\pi/100 \approx 0.063$). If we choose the much smaller viewing window $[-0.1, 0.1]$ by $[-1, 1]$ we get the graph shown in Figure 1.53c. This graph reveals the expected oscillations of a sine curve. ■

EXAMPLE 5 Graph the function $y = \cos x + \frac{1}{50} \sin 50x$.

Solution In the viewing window $[-6, 6]$ by $[-1, 1]$ the graph appears much like the cosine function with some small sharp wiggles on it (Figure 1.54a). We get a better look when we significantly reduce the window to $[-0.6, 0.6]$ by $[0.8, 1.02]$, obtaining the graph in Figure 1.54b. We now see the small but rapid oscillations of the second term, $\frac{1}{50} \sin 50x$, added to the comparatively larger values of the cosine curve. ■

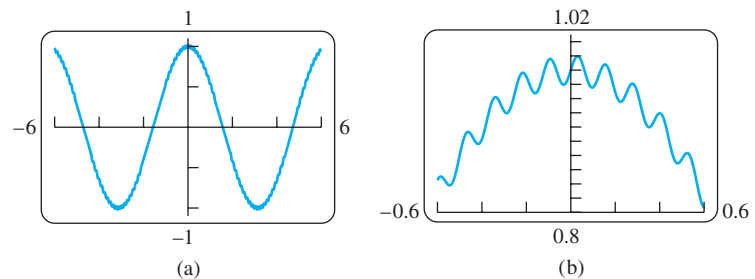


FIGURE 1.54 In (b) we see a close-up view of the function

$y = \cos x + \frac{1}{50} \sin 50x$ graphed in (a). The term $\cos x$ clearly dominates the second term, $\frac{1}{50} \sin 50x$, which produces the rapid oscillations along the cosine curve. Both views are needed for a clear idea of the graph (Example 5).

Obtaining a Complete Graph

Some graphing devices will not display the portion of a graph for $f(x)$ when $x < 0$. Usually that happens because of the procedure the device is using to calculate the function values. Sometimes we can obtain the complete graph by defining the formula for the function in a different way.

EXAMPLE 6 Graph the function $y = x^{1/3}$.

Solution Some graphing devices display the graph shown in Figure 1.55a. When we compare it with the graph of $y = x^{1/3} = \sqrt[3]{x}$ in Figure 1.17, we see that the left branch for

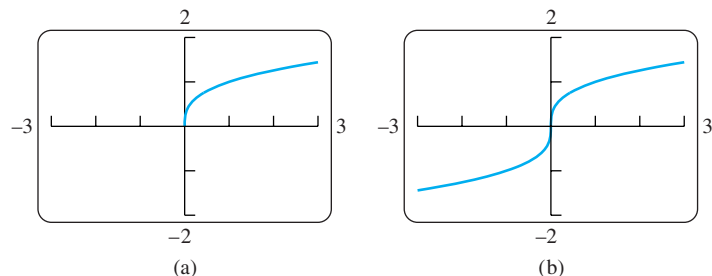


FIGURE 1.55 The graph of $y = x^{1/3}$ is missing the left branch in (a). In (b) we graph the function $f(x) = \frac{x}{|x|} \cdot |x|^{1/3}$, obtaining both branches. (See Example 6.)

$x < 0$ is missing. The reason the graphs differ is that many calculators and computer software programs calculate $x^{1/3}$ as $e^{(1/3)\ln x}$. Since the logarithmic function is not defined for negative values of x , the computing device can produce only the right branch, where $x > 0$. (Logarithmic and exponential functions are introduced in the next two sections.)

To obtain the full picture showing both branches, we can graph the function

$$f(x) = \frac{x}{|x|} \cdot |x|^{1/3}.$$

This function equals $x^{1/3}$ except at $x = 0$ (where f is undefined, although $0^{1/3} = 0$). The graph of f is shown in Figure 1.55b. ■

Exercises 1.4

Choosing a Viewing Window

T In Exercises 1–4, use a graphing calculator or computer to determine which of the given viewing windows displays the most appropriate graph of the specified function.

- $f(x) = x^4 - 7x^2 + 6x$
 - $[-1, 1]$ by $[-1, 1]$
 - $[-2, 2]$ by $[-5, 5]$
 - $[-10, 10]$ by $[-10, 10]$
 - $[-5, 5]$ by $[-25, 15]$
- $f(x) = x^3 - 4x^2 - 4x + 16$
 - $[-1, 1]$ by $[-5, 5]$
 - $[-3, 3]$ by $[-10, 10]$
 - $[-5, 5]$ by $[-10, 20]$
 - $[-20, 20]$ by $[-100, 100]$
- $f(x) = 5 + 12x - x^3$
 - $[-1, 1]$ by $[-1, 1]$
 - $[-5, 5]$ by $[-10, 10]$
 - $[-4, 4]$ by $[-20, 20]$
 - $[-4, 5]$ by $[-15, 25]$
- $f(x) = \sqrt{5 + 4x - x^2}$
 - $[-2, 2]$ by $[-2, 2]$
 - $[-2, 6]$ by $[-1, 4]$
 - $[-3, 7]$ by $[0, 10]$
 - $[-10, 10]$ by $[-10, 10]$

Finding a Viewing Window

T In Exercises 5–30, find an appropriate viewing window for the given function and use it to display its graph.

- $f(x) = x^4 - 4x^3 + 15$
- $f(x) = \frac{x^3}{3} - \frac{x^2}{2} - 2x + 1$
- $f(x) = x^5 - 5x^4 + 10$
- $f(x) = 4x^3 - x^4$
- $f(x) = x\sqrt{9 - x^2}$
- $f(x) = x^2(6 - x^3)$
- $y = 2x - 3x^{2/3}$
- $y = x^{1/3}(x^2 - 8)$
- $y = 5x^{2/5} - 2x$
- $y = x^{2/3}(5 - x)$
- $y = |x^2 - 1|$
- $y = |x^2 - x|$
- $y = \frac{x + 3}{x + 2}$
- $y = 1 - \frac{1}{x + 3}$

- $f(x) = \frac{x^2 + 2}{x^2 + 1}$
- $f(x) = \frac{x^2 - 1}{x^2 + 1}$
- $f(x) = \frac{x - 1}{x^2 - x - 6}$
- $f(x) = \frac{8}{x^2 - 9}$
- $f(x) = \frac{6x^2 - 15x + 6}{4x^2 - 10x}$
- $f(x) = \frac{x^2 - 3}{x - 2}$
- $y = \sin 250x$
- $y = 3 \cos 60x$
- $y = \cos\left(\frac{x}{50}\right)$
- $y = \frac{1}{10} \sin\left(\frac{x}{10}\right)$
- $y = x + \frac{1}{10} \sin 30x$
- $y = x^2 + \frac{1}{50} \cos 100x$
- Graph the lower half of the circle defined by the equation $x^2 + 2x = 4 + 4y - y^2$.
- Graph the upper branch of the hyperbola $y^2 - 16x^2 = 1$.
- Graph four periods of the function $f(x) = -\tan 2x$.
- Graph two periods of the function $f(x) = 3 \cot \frac{x}{2} + 1$.
- Graph the function $f(x) = \sin 2x + \cos 3x$.
- Graph the function $f(x) = \sin^3 x$.

Graphing in Dot Mode

T Another way to avoid incorrect connections when using a graphing device is through the use of a “dot mode,” which plots only the points. If your graphing utility allows that mode, use it to plot the functions in Exercises 37–40.

- $y = \frac{1}{x - 3}$
- $y = \sin \frac{1}{x}$
- $y = x[x]$
- $y = \frac{x^3 - 1}{x^2 - 1}$

1.5 Exponential Functions

Exponential functions are among the most important in mathematics and occur in a wide variety of applications, including interest rates, radioactive decay, population growth, the spread of a disease, consumption of natural resources, the earth’s atmospheric pressure, temperature change of a heated object placed in a cooler environment, and the dating of

fossils. In this section we introduce these functions informally, using an intuitive approach. We give a rigorous development of them in Chapter 7, based on important calculus ideas and results.

Exponential Behavior

When a positive quantity P doubles, it increases by a factor of 2 and the quantity becomes $2P$. If it doubles again, it becomes $2(2P) = 2^2P$, and a third doubling gives $2(2^2P) = 2^3P$. Continuing to double in this fashion leads us to the consideration of the function $f(x) = 2^x$. We call this an *exponential* function because the variable x appears in the exponent of 2^x . Functions such as $g(x) = 10^x$ and $h(x) = (1/2)^x$ are other examples of exponential functions. In general, if $a \neq 1$ is a positive constant, the function

$$f(x) = a^x$$

is the **exponential function with base a** .

Don't confuse 2^x with the power x^2 , where the variable x is the base, not the exponent.

EXAMPLE 1 In 2000, \$100 is invested in a savings account, where it grows by accruing interest that is compounded annually (once a year) at an interest rate of 5.5%. Assuming no additional funds are deposited to the account and no money is withdrawn, give a formula for a function describing the amount A in the account after x years have elapsed.

Solution If $P = 100$, at the end of the first year the amount in the account is the original amount plus the interest accrued, or

$$P + \left(\frac{5.5}{100}\right)P = (1 + 0.055)P = (1.055)P.$$

At the end of the second year the account earns interest again and grows to

$$(1 + 0.055) \cdot (1.055P) = (1.055)^2P = 100 \cdot (1.055)^2. \quad P = 100$$

Continuing this process, after x years the value of the account is

$$A = 100 \cdot (1.055)^x.$$

This is a multiple of the exponential function with base 1.055. Table 1.4 shows the amounts accrued over the first four years. Notice that the amount in the account each year is always 1.055 times its value in the previous year.

TABLE 1.4 Savings account growth

Year	Amount (dollars)	Increase (dollars)
2000	100	
2001	$100(1.055) = 105.50$	5.50
2002	$100(1.055)^2 = 111.30$	5.80
2003	$100(1.055)^3 = 117.42$	6.12
2004	$100(1.055)^4 = 123.88$	6.46

In general, the amount after x years is given by $P(1 + r)^x$, where r is the interest rate (expressed as a decimal). ■

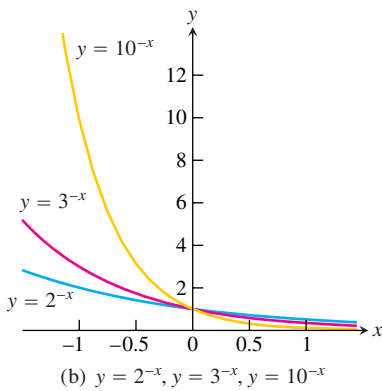
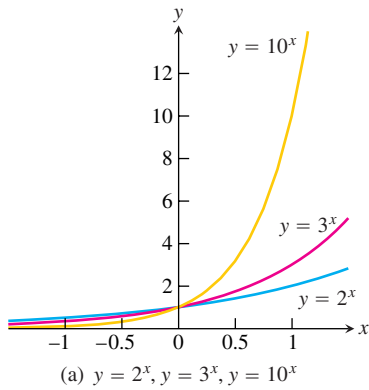


FIGURE 1.56 Graphs of exponential functions.

For integer and rational exponents, the value of an exponential function $f(x) = a^x$ is obtained arithmetically as follows. If $x = n$ is a positive integer, the number a^n is given by multiplying a by itself n times:

$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}$$

If $x = 0$, then $a^0 = 1$, and if $x = -n$ for some positive integer n , then

$$a^{-n} = \frac{1}{a^n} = \left(\frac{1}{a}\right)^n$$

If $x = 1/n$ for some positive integer n , then

$$a^{1/n} = \sqrt[n]{a},$$

which is the positive number that when multiplied by itself n times gives a . If $x = p/q$ is any rational number, then

$$a^{p/q} = \sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p.$$

If x is *irrational*, the meaning of a^x is not so clear, but its value can be defined by considering values for rational numbers that get closer and closer to x . This informal approach is based on the graph of the exponential function. In Chapter 7 we define the meaning in a rigorous way.

We displayed the graphs of several exponential functions in Section 1.1, and show them again here in Figure 1.56. These graphs describe the values of the exponential functions for all real inputs x . The value at an irrational number x is chosen so that the graph of a^x has no “holes” or “jumps.” Of course, these words are not mathematical terms, but they do convey the informal idea. We mean that the value of a^x , when x is irrational, is chosen so that the function $f(x) = a^x$ is *continuous*, a notion that will be carefully explored in the next chapter. This choice ensures the graph retains its increasing behavior when $a > 1$, or decreasing behavior when $0 < a < 1$ (see Figure 1.56).

Arithmetically, the graphical idea can be described in the following way, using the exponential $f(x) = 2^x$ as an illustration. Any particular irrational number, say $x = \sqrt{3}$, has a decimal expansion

$$\sqrt{3} = 1.732050808 \dots$$

We then consider the list of numbers, given as follows in the order of taking more and more digits in the decimal expansion,

$$2^1, 2^{1.7}, 2^{1.73}, 2^{1.732}, 2^{1.7320}, 2^{1.73205}, \dots \tag{1}$$

We know the meaning of each number in list (1) because the successive decimal approximations to $\sqrt{3}$ given by 1, 1.7, 1.73, 1.732, and so on, are all *rational* numbers. As these decimal approximations get closer and closer to $\sqrt{3}$, it seems reasonable that the list of numbers in (1) gets closer and closer to some fixed number, which we specify to be $2^{\sqrt{3}}$.

Table 1.5 illustrates how taking better approximations to $\sqrt{3}$ gives better approximations to the number $2^{\sqrt{3}} \approx 3.321997086$. It is the *completeness property* of the real numbers (discussed briefly in Appendix 6) which guarantees that this procedure gives a single number we define to be $2^{\sqrt{3}}$ (although it is beyond the scope of this text to give a proof). In a similar way, we can identify the number 2^x (or $a^x, a > 0$) for any irrational x . By identifying the number a^x for both rational and irrational x , we eliminate any “holes” or “gaps” in the graph of a^x . In practice you can use a calculator to find the number a^x for irrational x , taking successive decimal approximations to x and creating a table similar to Table 1.5.

Exponential functions obey the familiar rules of exponents listed on the next page. It is easy to check these rules using algebra when the exponents are integers or rational numbers. We prove them for all real exponents in Chapters 4 and 7.

TABLE 1.5 Values of $2^{\sqrt{3}}$ for rational r closer and closer to $\sqrt{3}$

r	2^r
1.0	2.000000000
1.7	3.249009585
1.73	3.317278183
1.732	3.321880096
1.7320	3.321880096
1.73205	3.321995226
1.732050	3.321995226
1.7320508	3.321997068
1.73205080	3.321997068
1.732050808	3.321997086

Rules for Exponents

If $a > 0$ and $b > 0$, the following rules hold true for all real numbers x and y .

1. $a^x \cdot a^y = a^{x+y}$
2. $\frac{a^x}{a^y} = a^{x-y}$
3. $(a^x)^y = (a^y)^x = a^{xy}$
4. $a^x \cdot b^x = (ab)^x$
5. $\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x$

EXAMPLE 2 We illustrate using the rules for exponents.

1. $3^{1.1} \cdot 3^{0.7} = 3^{1.1+0.7} = 3^{1.8}$
2. $\frac{(\sqrt{10})^3}{\sqrt{10}} = (\sqrt{10})^{3-1} = (\sqrt{10})^2 = 10$
3. $(5^{\sqrt{2}})^{\sqrt{2}} = 5^{\sqrt{2} \cdot \sqrt{2}} = 5^2 = 25$
4. $7^\pi \cdot 8^\pi = (56)^\pi$
5. $\left(\frac{4}{9}\right)^{1/2} = \frac{4^{1/2}}{9^{1/2}} = \frac{2}{3}$

The Natural Exponential Function e^x

The most important exponential function used for modeling natural, physical, and economic phenomena is the **natural exponential function**, whose base is the special number e . The number e is irrational, and its value is 2.718281828 to nine decimal places. It might seem strange that we would use this number for a base rather than a simple number like 2 or 10. The advantage in using e as a base is that it simplifies many of the calculations in calculus.

If you look at Figure 1.56a you can see that the graphs of the exponential functions $y = a^x$ get steeper as the base a gets larger. This idea of steepness is conveyed by the slope of the tangent line to the graph at a point. Tangent lines to graphs of functions are defined precisely in the next chapter, but intuitively the tangent line to the graph at a point is a line that just touches the graph at the point, like a tangent to a circle. Figure 1.57 shows the slope of the graph of $y = a^x$ as it crosses the y -axis for several values of a . Notice that the slope is exactly equal to 1 when a equals the number e . The slope is smaller than 1 if $a < e$, and larger than 1 if $a > e$. This is the property that makes the number e so useful in calculus: **The graph of $y = e^x$ has slope 1 when it crosses the y -axis.**

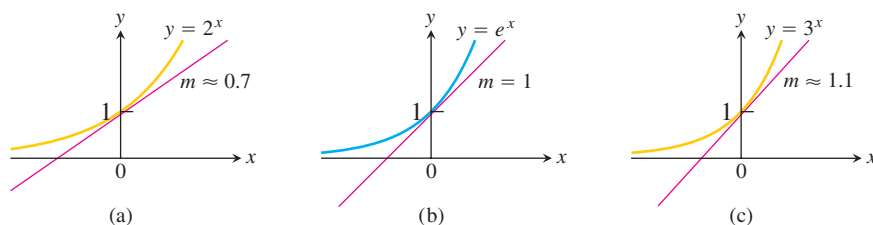


FIGURE 1.57 Among the exponential functions, the graph of $y = e^x$ has the property that the slope m of the tangent line to the graph is exactly 1 when it crosses the y -axis. The slope is smaller for a base less than e , such as 2^x , and larger for a base greater than e , such as 3^x .

In Chapter 3 we use that slope property to prove e is the number the quantity $(1 + 1/x)^x$ approaches as x becomes large without bound. That result provides one way to compute the value of e , at least approximately. The graph and table in Figure 1.58 show the behavior of this expression and how it gets closer and closer to the line $y = e \approx 2.718281828$ as x gets larger and larger. (This *limit* idea is made precise in the next chapter.) A more complete discussion of e is given in Chapter 7.

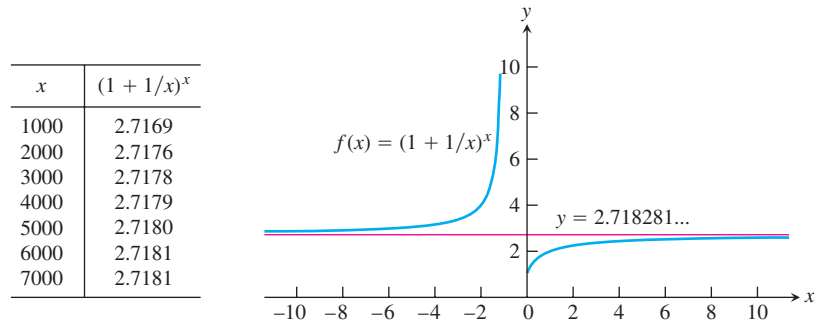


FIGURE 1.58 A graph and table of values for $f(x) = (1 + 1/x)^x$ both suggest that as x gets larger and larger, $f(x)$ gets closer and closer to $e \approx 2.7182818 \dots$

Exponential Growth and Decay

The exponential functions $y = e^{kx}$, where k is a nonzero constant, are frequently used for modeling exponential growth or decay. The function $y = y_0 e^{kx}$ is a model for **exponential growth** if $k > 0$ and a model for **exponential decay** if $k < 0$. Here y_0 represents a constant. An example of exponential growth occurs when computing interest **compounded continuously** modeled by $y = P \cdot e^{rt}$, where P is the initial investment, r is the interest rate as a decimal, and t is time in units consistent with r . An example of exponential decay is the model $y = A \cdot e^{-1.2 \times 10^{-4}t}$, which represents how the radioactive element carbon-14 decays over time. Here A is the original amount of carbon-14 and t is the time in years. Carbon-14 decay is used to date the remains of dead organisms such as shells, seeds, and wooden artifacts. Figure 1.59 shows graphs of exponential growth and exponential decay.

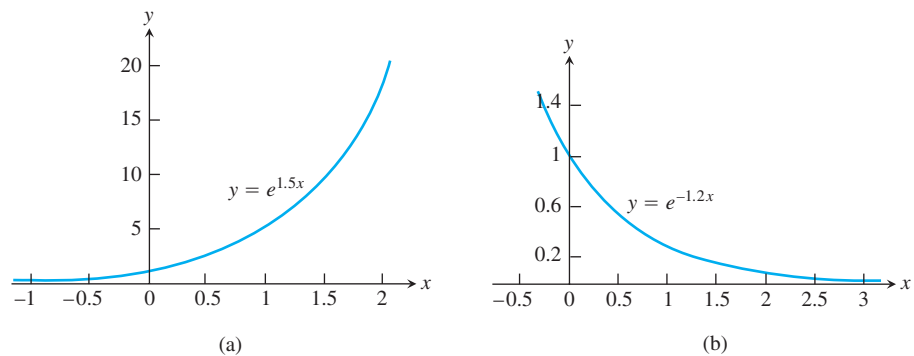


FIGURE 1.59 Graphs of (a) exponential growth, $k = 1.5 > 0$, and (b) exponential decay, $k = -1.2 < 0$.

EXAMPLE 3 Investment companies often use the model $y = Pe^{rt}$ in calculating the growth of an investment. Use this model to track the growth of \$100 invested in 2000 at an annual interest rate of 5.5%.

Solution Let $t = 0$ represent 2000, $t = 1$ represent 2001, and so on. Then the exponential growth model is $y(t) = Pe^{rt}$, where $P = 100$ (the initial investment), $r = 0.055$ (the

annual interest rate expressed as a decimal), and t is time in years. To predict the amount in the account in 2004, after four years have elapsed, we take $t = 4$ and calculate

$$\begin{aligned} y(4) &= 100e^{0.055(4)} \\ &= 100e^{0.22} \\ &= 124.61. \end{aligned} \quad \text{Nearest cent using calculator}$$

This compares with \$123.88 in the account when the interest is compounded annually from Example 1. ■

EXAMPLE 4 Laboratory experiments indicate that some atoms emit a part of their mass as radiation, with the remainder of the atom re-forming to make an atom of some new element. For example, radioactive carbon-14 decays into nitrogen; radium eventually decays into lead. If y_0 is the number of radioactive nuclei present at time zero, the number still present at any later time t will be

$$y = y_0 e^{-rt}, \quad r > 0.$$

The number r is called the **decay rate** of the radioactive substance. (We will see how this formula is obtained in Section 7.2.) For carbon-14, the decay rate has been determined experimentally to be about $r = 1.2 \times 10^{-4}$ when t is measured in years. Predict the percent of carbon-14 present after 866 years have elapsed.

Solution If we start with an amount y_0 of carbon-14 nuclei, after 866 years we are left with the amount

$$\begin{aligned} y(866) &= y_0 e^{(-1.2 \times 10^{-4})(866)} \\ &\approx (0.901)y_0. \end{aligned} \quad \text{Calculator evaluation}$$

That is, after 866 years, we are left with about 90% of the original amount of carbon-14, so about 10% of the original nuclei have decayed. In Example 7 in the next section, you will see how to find the number of years required for half of the radioactive nuclei present in a sample to decay (called the *half-life* of the substance). ■

You may wonder why we use the family of functions $y = e^{kx}$ for different values of the constant k instead of the general exponential functions $y = a^x$. In the next section, we show that the exponential function a^x is equal to e^{kx} for an appropriate value of k . So the formula $y = e^{kx}$ covers the entire range of possibilities, and we will see that it is easier to use.

Exercises 1.5

Sketching Exponential Curves

In Exercises 1–6, sketch the given curves together in the appropriate coordinate plane and label each curve with its equation.

- $y = 2^x, y = 4^x, y = 3^{-x}, y = (1/5)^x$
- $y = 3^x, y = 8^x, y = 2^{-x}, y = (1/4)^x$
- $y = 2^{-t}$ and $y = -2^t$
- $y = 3^{-t}$ and $y = -3^t$
- $y = e^x$ and $y = 1/e^x$
- $y = -e^x$ and $y = -e^{-x}$

In each of Exercises 7–10, sketch the shifted exponential curves.

- $y = 2^x - 1$ and $y = 2^{-x} - 1$
- $y = 3^x + 2$ and $y = 3^{-x} + 2$
- $y = 1 - e^x$ and $y = 1 - e^{-x}$
- $y = -1 - e^x$ and $y = -1 - e^{-x}$

Applying the Laws of Exponents

Use the laws of exponents to simplify the expressions in Exercises 11–20.

- $16^2 \cdot 16^{-1.75}$
- $9^{1/3} \cdot 9^{1/6}$
- $\frac{4^{4.2}}{4^{3.7}}$
- $\frac{3^{5/3}}{3^{2/3}}$
- $(25^{1/8})^4$
- $(13^{\sqrt{2}})^{\sqrt{2}/2}$
- $2^{\sqrt{3}} \cdot 7^{\sqrt{3}}$
- $(\sqrt{3})^{1/2} \cdot (\sqrt{12})^{1/2}$
- $\left(\frac{2}{\sqrt{2}}\right)^4$
- $\left(\frac{\sqrt{6}}{3}\right)^2$

Composites Involving Exponential Functions

Find the domain and range for each of the functions in Exercises 21–24.

21. $f(x) = \frac{1}{2 + e^x}$

22. $g(t) = \cos(e^{-t})$

23. $g(t) = \sqrt{1 + 3^{-t}}$

24. $f(x) = \frac{3}{1 - e^{2x}}$

Applications

T In Exercises 25–28, use graphs to find approximate solutions.

25. $2^x = 5$

26. $e^x = 4$

27. $3^x - 0.5 = 0$

28. $3 - 2^{-x} = 0$

T In Exercises 29–36, use an exponential model and a graphing calculator to estimate the answer in each problem.

29. Population growth The population of Knoxville is 500,000 and is increasing at the rate of 3.75% each year. Approximately when will the population reach 1 million?

30. Population growth The population of Silver Run in the year 1890 was 6250. Assume the population increased at a rate of 2.75% per year.

a. Estimate the population in 1915 and 1940.

b. Approximately when did the population reach 50,000?

31. Radioactive decay The half-life of phosphorus-32 is about 14 days. There are 6.6 grams present initially.

a. Express the amount of phosphorus-32 remaining as a function of time t .

b. When will there be 1 gram remaining?

32. If John invests \$2300 in a savings account with a 6% interest rate compounded annually, how long will it take until John's account has a balance of \$4150?

33. Doubling your money Determine how much time is required for an investment to double in value if interest is earned at the rate of 6.25% compounded annually.

34. Tripling your money Determine how much time is required for an investment to triple in value if interest is earned at the rate of 5.75% compounded continuously.

35. Cholera bacteria Suppose that a colony of bacteria starts with 1 bacterium and doubles in number every half hour. How many bacteria will the colony contain at the end of 24 hr?

36. Eliminating a disease Suppose that in any given year the number of cases of a disease is reduced by 20%. If there are 10,000 cases today, how many years will it take

a. to reduce the number of cases to 1000?

b. to eliminate the disease; that is, to reduce the number of cases to less than 1?

1.6

Inverse Functions and Logarithms

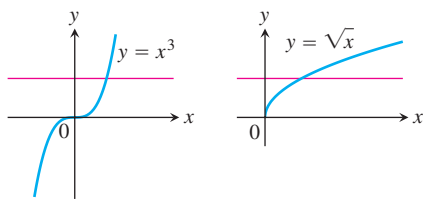
A function that undoes, or inverts, the effect of a function f is called the *inverse* of f . Many common functions, though not all, are paired with an inverse. In this section we present the natural logarithmic function $y = \ln x$ as the inverse of the exponential function $y = e^x$, and we also give examples of several inverse trigonometric functions.

One-to-One Functions

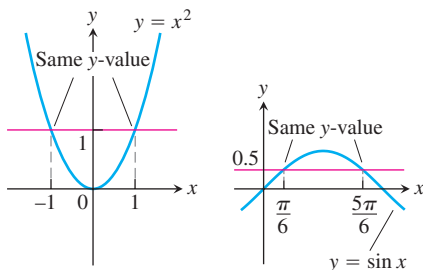
A function is a rule that assigns a value from its range to each element in its domain. Some functions assign the same range value to more than one element in the domain. The function $f(x) = x^2$ assigns the same value, 1, to both of the numbers -1 and $+1$; the sines of $\pi/3$ and $2\pi/3$ are both $\sqrt{3}/2$. Other functions assume each value in their range no more than once. The square roots and cubes of different numbers are always different. A function that has distinct values at distinct elements in its domain is called *one-to-one*. These functions take on any one value in their range exactly once.

DEFINITION A function $f(x)$ is **one-to-one** on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D .

EXAMPLE 1 Some functions are one-to-one on their entire natural domain. Other functions are not one-to-one on their entire domain, but by restricting the function to a smaller domain we can create a function that is one-to-one. The original and restricted functions are not the same functions, because they have different domains. However, the two functions have the same values on the smaller domain, so the original function is an extension of the restricted function from its smaller domain to the larger domain.



(a) One-to-one: Graph meets each horizontal line at most once.



(b) Not one-to-one: Graph meets one or more horizontal lines more than once.

FIGURE 1.60 (a) $y = x^3$ and $y = \sqrt{x}$ are one-to-one on their domains $(-\infty, \infty)$ and $[0, \infty)$. (b) $y = x^2$ and $y = \sin x$ are not one-to-one on their domains $(-\infty, \infty)$.

- (a) $f(x) = \sqrt{x}$ is one-to-one on any domain of nonnegative numbers because $\sqrt{x_1} \neq \sqrt{x_2}$ whenever $x_1 \neq x_2$.
- (b) $g(x) = \sin x$ is *not* one-to-one on the interval $[0, \pi]$ because $\sin(\pi/6) = \sin(5\pi/6)$. In fact, for each element x_1 in the subinterval $[0, \pi/2)$ there is a corresponding element x_2 in the subinterval $(\pi/2, \pi]$ satisfying $\sin x_1 = \sin x_2$, so distinct elements in the domain are assigned to the same value in the range. The sine function *is* one-to-one on $[0, \pi/2]$, however, because it is an increasing function on $[0, \pi/2]$ giving distinct outputs for distinct inputs. ■

The graph of a one-to-one function $y = f(x)$ can intersect a given horizontal line at most once. If the function intersects the line more than once, it assumes the same y -value for at least two different x -values and is therefore not one-to-one (Figure 1.60).

The Horizontal Line Test for One-to-One Functions
 A function $y = f(x)$ is one-to-one if and only if its graph intersects each horizontal line at most once.

Inverse Functions

Since each output of a one-to-one function comes from just one input, the effect of the function can be inverted to send an output back to the input from which it came.

DEFINITION Suppose that f is a one-to-one function on a domain D with range R . The **inverse function** f^{-1} is defined by

$$f^{-1}(b) = a \text{ if } f(a) = b.$$

The domain of f^{-1} is R and the range of f^{-1} is D .

The symbol f^{-1} for the inverse of f is read “ f inverse.” The “ -1 ” in f^{-1} is *not* an exponent; $f^{-1}(x)$ does not mean $1/f(x)$. Notice that the domains and ranges of f and f^{-1} are interchanged.

EXAMPLE 2 Suppose a one-to-one function $y = f(x)$ is given by a table of values

x	1	2	3	4	5	6	7	8
$f(x)$	3	4.5	7	10.5	15	20.5	27	34.5

A table for the values of $x = f^{-1}(y)$ can then be obtained by simply interchanging the values in the columns (or rows) of the table for f :

y	3	4.5	7	10.5	15	20.5	27	34.5
$f^{-1}(y)$	1	2	3	4	5	6	7	8

If we apply f to send an input x to the output $f(x)$ and follow by applying f^{-1} to $f(x)$ we get right back to x , just where we started. Similarly, if we take some number y in the range of f , apply f^{-1} to it, and then apply f to the resulting value $f^{-1}(y)$, we get back the value y with which we began. Composing a function and its inverse has the same effect as doing nothing.

$$(f^{-1} \circ f)(x) = x, \quad \text{for all } x \text{ in the domain of } f$$

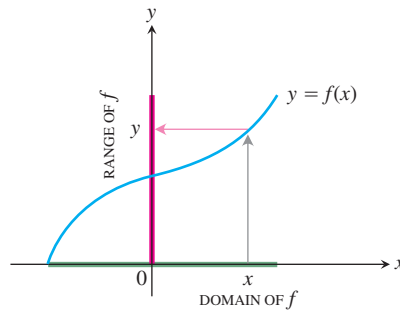
$$(f \circ f^{-1})(y) = y, \quad \text{for all } y \text{ in the domain of } f^{-1} \text{ (or range of } f)$$

Only a one-to-one function can have an inverse. The reason is that if $f(x_1) = y$ and $f(x_2) = y$ for two distinct inputs x_1 and x_2 , then there is no way to assign a value to $f^{-1}(y)$ that satisfies both $f^{-1}(f(x_1)) = x_1$ and $f^{-1}(f(x_2)) = x_2$.

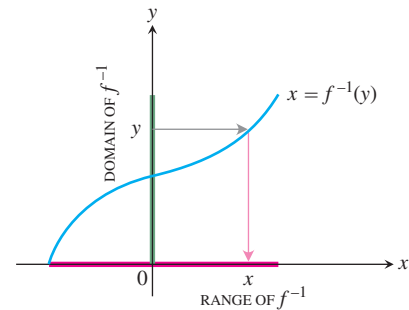
A function that is increasing on an interval so it satisfies the inequality $f(x_2) > f(x_1)$ when $x_2 > x_1$ is one-to-one and has an inverse. Decreasing functions also have an inverse. Functions that are neither increasing nor decreasing may still be one-to-one and have an inverse, as with the function $f(x) = 1/x$ for $x \neq 0$ and $f(0) = 0$, defined on $(-\infty, \infty)$ and passing the horizontal line test.

Finding Inverses

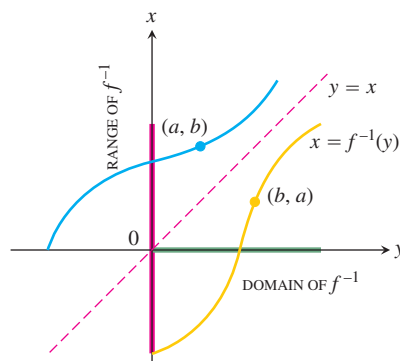
The graphs of a function and its inverse are closely related. To read the value of a function from its graph, we start at a point x on the x -axis, go vertically to the graph, and then move horizontally to the y -axis to read the value of y . The inverse function can be read from the graph by reversing this process. Start with a point y on the y -axis, go horizontally to the graph of $y = f(x)$, and then move vertically to the x -axis to read the value of $x = f^{-1}(y)$ (Figure 1.61).



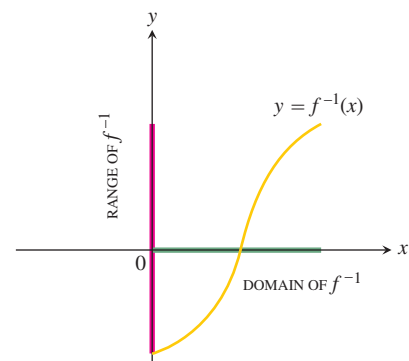
(a) To find the value of f at x , we start at x , go up to the curve, and then over to the y -axis.



(b) The graph of f^{-1} is the graph of f , but with x and y interchanged. To find the x that gave y , we start at y and go over to the curve and down to the x -axis. The domain of f^{-1} is the range of f . The range of f^{-1} is the domain of f .



(c) To draw the graph of f^{-1} in the more usual way, we reflect the system across the line $y = x$.



(d) Then we interchange the letters x and y . We now have a normal-looking graph of f^{-1} as a function of x .

FIGURE 1.61 Determining the graph of $y = f^{-1}(x)$ from the graph of $y = f(x)$. The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

We want to set up the graph of f^{-1} so that its input values lie along the x -axis, as is usually done for functions, rather than on the y -axis. To achieve this we interchange the x

and y axes by reflecting across the 45° line $y = x$. After this reflection we have a new graph that represents f^{-1} . The value of $f^{-1}(x)$ can now be read from the graph in the usual way, by starting with a point x on the x -axis, going vertically to the graph, and then horizontally to the y -axis to get the value of $f^{-1}(x)$. Figure 1.61 indicates the relationship between the graphs of f and f^{-1} . The graphs are interchanged by reflection through the line $y = x$.

The process of passing from f to f^{-1} can be summarized as a two-step procedure.

1. Solve the equation $y = f(x)$ for x . This gives a formula $x = f^{-1}(y)$ where x is expressed as a function of y .
2. Interchange x and y , obtaining a formula $y = f^{-1}(x)$ where f^{-1} is expressed in the conventional format with x as the independent variable and y as the dependent variable.

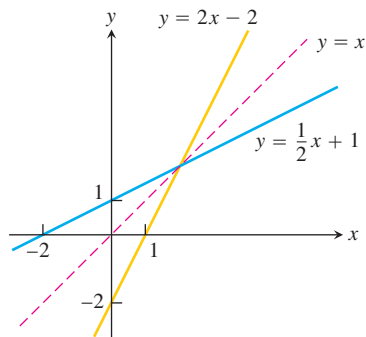


FIGURE 1.62 Graphing $f(x) = (1/2)x + 1$ and $f^{-1}(x) = 2x - 2$ together shows the graphs' symmetry with respect to the line $y = x$ (Example 3).

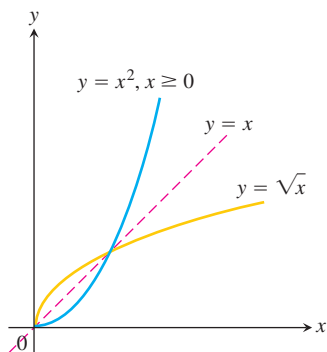


FIGURE 1.63 The functions $y = \sqrt{x}$ and $y = x^2, x \geq 0$, are inverses of one another (Example 4).

EXAMPLE 3 Find the inverse of $y = \frac{1}{2}x + 1$, expressed as a function of x .

Solution

1. Solve for x in terms of y : $y = \frac{1}{2}x + 1$
 $2y = x + 2$
 $x = 2y - 2.$
2. Interchange x and y : $y = 2x - 2.$

The inverse of the function $f(x) = (1/2)x + 1$ is the function $f^{-1}(x) = 2x - 2$. (See Figure 1.62.) To check, we verify that both composites give the identity function:

$$f^{-1}(f(x)) = 2\left(\frac{1}{2}x + 1\right) - 2 = x + 2 - 2 = x$$

$$f(f^{-1}(x)) = \frac{1}{2}(2x - 2) + 1 = x - 1 + 1 = x. \quad \blacksquare$$

EXAMPLE 4 Find the inverse of the function $y = x^2, x \geq 0$, expressed as a function of x .

Solution We first solve for x in terms of y :

$$y = x^2$$

$$\sqrt{y} = \sqrt{x^2} = |x| = x \quad |x| = x \text{ because } x \geq 0$$

We then interchange x and y , obtaining

$$y = \sqrt{x}.$$

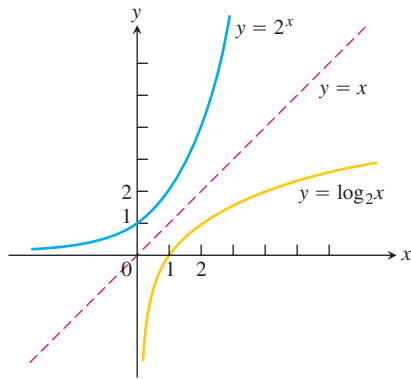
The inverse of the function $y = x^2, x \geq 0$, is the function $y = \sqrt{x}$ (Figure 1.63).

Notice that the function $y = x^2, x \geq 0$, with domain *restricted* to the nonnegative real numbers, is one-to-one (Figure 1.63) and has an inverse. On the other hand, the function $y = x^2$, with no domain restrictions, is *not* one-to-one (Figure 1.60b) and therefore has no inverse. \blacksquare

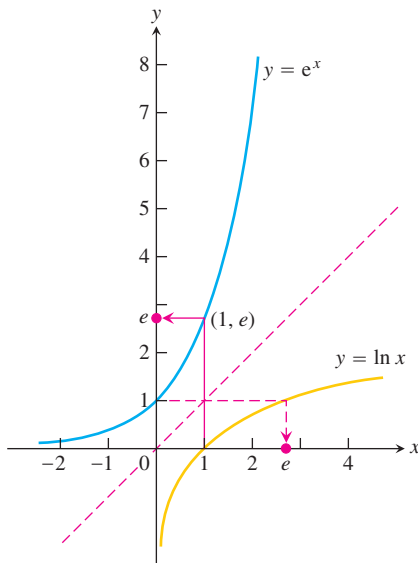
Logarithmic Functions

If a is any positive real number other than 1, the base a exponential function $f(x) = a^x$ is one-to-one. It therefore has an inverse. Its inverse is called the *logarithm function with base a* .

DEFINITION The **logarithm function with base a** , $y = \log_a x$, is the inverse of the base a exponential function $y = a^x$ ($a > 0, a \neq 1$).



(a)



(b)

FIGURE 1.64 (a) The graph of 2^x and its inverse, $\log_2 x$. (b) The graph of e^x and its inverse, $\ln x$.

HISTORICAL BIOGRAPHY*

John Napier
(1550–1617)

The domain of $\log_a x$ is $(0, \infty)$, the range of a^x . The range of $\log_a x$ is $(-\infty, \infty)$, the domain of a^x .

Figure 1.23 in Section 1.1 shows the graphs of four logarithmic functions with $a > 1$. Figure 1.64a shows the graph of $y = \log_2 x$. The graph of $y = a^x$, $a > 1$, increases rapidly for $x > 0$, so its inverse, $y = \log_a x$, increases slowly for $x > 1$.

Because we have no technique yet for solving the equation $y = a^x$ for x in terms of y , we do not have an explicit formula for computing the logarithm at a given value of x . Nevertheless, we can obtain the graph of $y = \log_a x$ by reflecting the graph of the exponential $y = a^x$ across the line $y = x$. Figure 1.64 shows the graphs for $a = 2$ and $a = e$.

Logarithms with base 2 are commonly used in computer science. Logarithms with base e and base 10 are so important in applications that calculators have special keys for them. They also have their own special notation and names:

$\log_e x$ is written as $\ln x$.

$\log_{10} x$ is written as $\log x$.

The function $y = \ln x$ is called the **natural logarithm function**, and $y = \log x$ is often called the **common logarithm function**. For the natural logarithm,

$$\ln x = y \Leftrightarrow e^y = x.$$

In particular, if we set $x = e$, we obtain

$$\ln e = 1$$

because $e^1 = e$.

Properties of Logarithms

Logarithms, invented by John Napier, were the single most important improvement in arithmetic calculation before the modern electronic computer. What made them so useful is that the properties of logarithms reduce multiplication of positive numbers to addition of their logarithms, division of positive numbers to subtraction of their logarithms, and exponentiation of a number to multiplying its logarithm by the exponent.

We summarize these properties for the natural logarithm as a series of rules that we prove in Chapter 3. Although here we state the Power Rule for all real powers r , the case when r is an irrational number cannot be dealt with properly until Chapter 4. We also establish the validity of the rules for logarithmic functions with any base a in Chapter 7.

THEOREM 1—Algebraic Properties of the Natural Logarithm For any numbers $b > 0$ and $x > 0$, the natural logarithm satisfies the following rules:

1. *Product Rule:* $\ln bx = \ln b + \ln x$
2. *Quotient Rule:* $\ln \frac{b}{x} = \ln b - \ln x$
3. *Reciprocal Rule:* $\ln \frac{1}{x} = -\ln x$ Rule 2 with $b = 1$
4. *Power Rule:* $\ln x^r = r \ln x$

*To learn more about the historical figures mentioned in the text and the development of many major elements and topics of calculus, visit www.aw.com/thomas.

EXAMPLE 5 Here are examples of the properties in Theorem 1.

- (a) $\ln 4 + \ln \sin x = \ln(4 \sin x)$ Product Rule
- (b) $\ln \frac{x+1}{2x-3} = \ln(x+1) - \ln(2x-3)$ Quotient Rule
- (c) $\ln \frac{1}{8} = -\ln 8$ Reciprocal Rule
 $= -\ln 2^3 = -3 \ln 2$ Power Rule ■

Because a^x and $\log_a x$ are inverses, composing them in either order gives the identity function.

Inverse Properties for a^x and $\log_a x$

1. Base a : $a^{\log_a x} = x$, $\log_a a^x = x$, $a > 0, a \neq 1, x > 0$
2. Base e : $e^{\ln x} = x$, $\ln e^x = x$, $x > 0$

Substituting a^x for x in the equation $x = e^{\ln x}$ enables us to rewrite a^x as a power of e :

$$\begin{aligned} a^x &= e^{\ln(a^x)} && \text{Substitute } a^x \text{ for } x \text{ in } x = e^{\ln x}. \\ &= e^{x \ln a} && \text{Power Rule for logs} \\ &= e^{(\ln a)x}. && \text{Exponent rearranged} \end{aligned}$$

Thus, the exponential function a^x is the same as e^{kx} for $k = \ln a$.

Every exponential function is a power of the natural exponential function.

$$a^x = e^{x \ln a}$$

That is, a^x is the same as e^x raised to the power $\ln a$: $a^x = e^{kx}$ for $k = \ln a$.

For example,

$$2^x = e^{(\ln 2)x} = e^{x \ln 2}, \quad \text{and} \quad 5^{-3x} = e^{(\ln 5)(-3x)} = e^{-3x \ln 5}.$$

Returning once more to the properties of a^x and $\log_a x$, we have

$$\begin{aligned} \ln x &= \ln(a^{\log_a x}) && \text{Inverse Property for } a^x \text{ and } \log_a x \\ &= (\log_a x)(\ln a). && \text{Power Rule for logarithms, with } r = \log_a x \end{aligned}$$

Rewriting this equation as $\log_a x = (\ln x)/(\ln a)$ shows that every logarithmic function is a constant multiple of the natural logarithm $\ln x$. This allows us to extend the algebraic properties for $\ln x$ to $\log_a x$. For instance, $\log_a bx = \log_a b + \log_a x$.

Change of Base Formula

Every logarithmic function is a constant multiple of the natural logarithm.

$$\log_a x = \frac{\ln x}{\ln a} \quad (a > 0, a \neq 1)$$

Applications

In Section 1.5 we looked at examples of exponential growth and decay problems. Here we use properties of logarithms to answer more questions concerning such problems.

EXAMPLE 6 If \$1000 is invested in an account that earns 5.25% interest compounded annually, how long will it take the account to reach \$2500?

Solution From Example 1, Section 1.5 with $P = 1000$ and $r = 0.0525$, the amount in the account at any time t in years is $1000(1.0525)^t$, so we need to solve the equation

$$1000(1.0525)^t = 2500.$$

Thus we have

$$(1.0525)^t = 2.5 \quad \text{Divide by 1000.}$$

$$\ln(1.0525)^t = \ln 2.5 \quad \text{Take logarithms of both sides.}$$

$$t \ln 1.0525 = \ln 2.5 \quad \text{Power Rule}$$

$$t = \frac{\ln 2.5}{\ln 1.0525} \approx 17.9 \quad \text{Values obtained by calculator}$$

The amount in the account will reach \$2500 in 18 years, when the annual interest payment is deposited for that year. ■

EXAMPLE 7 The **half-life** of a radioactive element is the time required for half of the radioactive nuclei present in a sample to decay. It is a remarkable fact that the half-life is a constant that does not depend on the number of radioactive nuclei initially present in the sample, but only on the radioactive substance.

To see why, let y_0 be the number of radioactive nuclei initially present in the sample. Then the number y present at any later time t will be $y = y_0 e^{-kt}$. We seek the value of t at which the number of radioactive nuclei present equals half the original number:

$$y_0 e^{-kt} = \frac{1}{2} y_0$$

$$e^{-kt} = \frac{1}{2}$$

$$-kt = \ln \frac{1}{2} = -\ln 2 \quad \text{Reciprocal Rule for logarithms}$$

$$t = \frac{\ln 2}{k}. \quad (1)$$

This value of t is the half-life of the element. It depends only on the value of k ; the number y_0 does not have any effect.

The effective radioactive lifetime of polonium-210 is so short that we measure it in days rather than years. The number of radioactive atoms remaining after t days in a sample that starts with y_0 radioactive atoms is

$$y = y_0 e^{-5 \times 10^{-3} t}.$$

The element's half-life is

$$\text{Half-life} = \frac{\ln 2}{k} \quad \text{Eq. (1)}$$

$$= \frac{\ln 2}{5 \times 10^{-3}} \quad \text{The } k \text{ from polonium's decay equation}$$

$$\approx 139 \text{ days.} \quad \blacksquare$$

Inverse Trigonometric Functions

The six basic trigonometric functions of a general radian angle x were reviewed in Section 1.3. These functions are not one-to-one (their values repeat periodically). However, we can restrict their domains to intervals on which they are one-to-one. The sine function

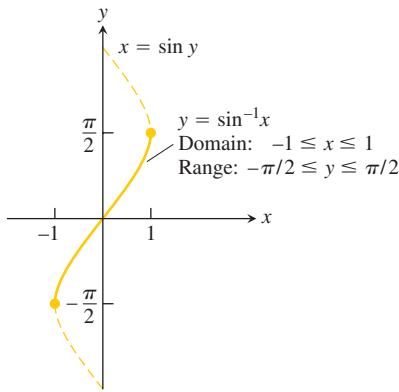
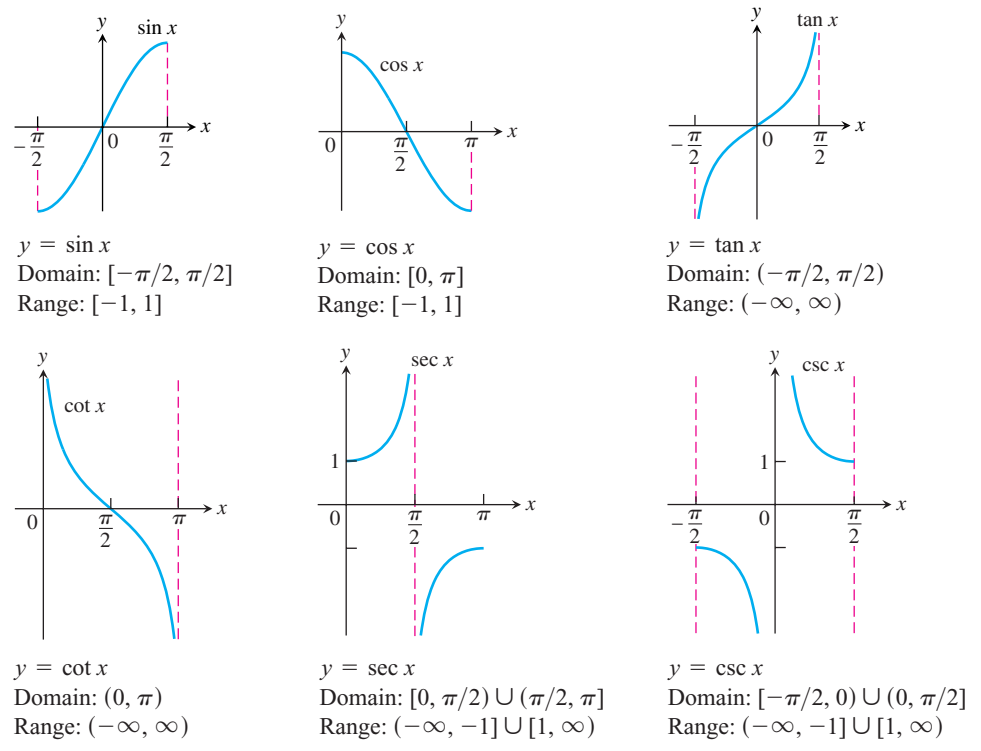


FIGURE 1.65 The graph of $y = \sin^{-1} x$.

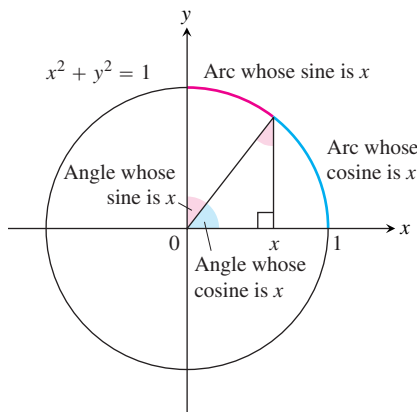
increases from -1 at $x = -\pi/2$ to $+1$ at $x = \pi/2$. By restricting its domain to the interval $[-\pi/2, \pi/2]$ we make it one-to-one, so that it has an inverse $\sin^{-1} x$ (Figure 1.65). Similar domain restrictions can be applied to all six trigonometric functions.

Domain restrictions that make the trigonometric functions one-to-one



The “Arc” in Arcsine and Arccosine

The accompanying figure gives a geometric interpretation of $y = \sin^{-1} x$ and $y = \cos^{-1} x$ for radian angles in the first quadrant. For a unit circle, the equation $s = r\theta$ becomes $s = \theta$, so central angles and the arcs they subtend have the same measure. If $x = \sin y$, then, in addition to being the angle whose sine is x , y is also the length of arc on the unit circle that subtends an angle whose sine is x . So we call y “the arc whose sine is x .”



Since these restricted functions are now one-to-one, they have inverses, which we denote by

$y = \sin^{-1} x$	or	$y = \arcsin x$
$y = \cos^{-1} x$	or	$y = \arccos x$
$y = \tan^{-1} x$	or	$y = \arctan x$
$y = \cot^{-1} x$	or	$y = \text{arccot } x$
$y = \sec^{-1} x$	or	$y = \text{arcsec } x$
$y = \csc^{-1} x$	or	$y = \text{arccsc } x$

These equations are read “ y equals the arcsine of x ” or “ y equals arcsin x ” and so on.

Caution The -1 in the expressions for the inverse means “inverse.” It does *not* mean reciprocal. For example, the *reciprocal* of $\sin x$ is $(\sin x)^{-1} = 1/\sin x = \csc x$.

The graphs of the six inverse trigonometric functions are shown in Figure 1.66. We can obtain these graphs by reflecting the graphs of the restricted trigonometric functions through the line $y = x$. We now take a closer look at two of these functions.

The Arcsine and Arccosine Functions

We define the arcsine and arccosine as functions whose values are angles (measured in radians) that belong to restricted domains of the sine and cosine functions.

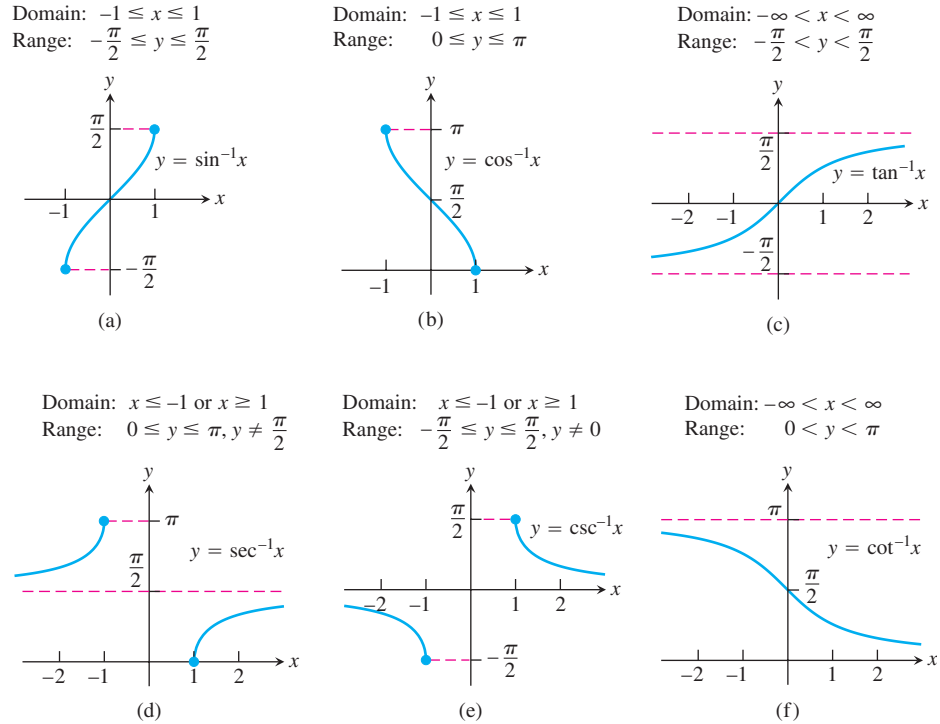


FIGURE 1.66 Graphs of the six basic inverse trigonometric functions.

DEFINITION

$y = \sin^{-1} x$ is the number in $[-\pi/2, \pi/2]$ for which $\sin y = x$.
 $y = \cos^{-1} x$ is the number in $[0, \pi]$ for which $\cos y = x$.

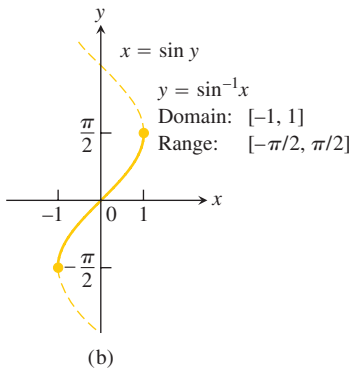
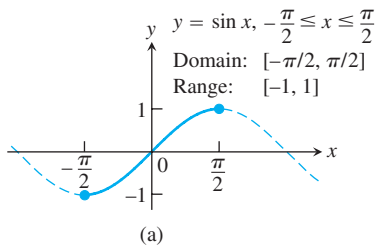


FIGURE 1.67 The graphs of (a) $y = \sin x$, $-\pi/2 \leq x \leq \pi/2$, and (b) its inverse, $y = \sin^{-1} x$. The graph of $\sin^{-1} x$, obtained by reflection across the line $y = x$, is a portion of the curve $x = \sin y$.

The graph of $y = \sin^{-1} x$ (Figure 1.67b) is symmetric about the origin (it lies along the graph of $x = \sin y$). The arcsine is therefore an odd function:

$$\sin^{-1}(-x) = -\sin^{-1} x. \tag{2}$$

The graph of $y = \cos^{-1} x$ (Figure 1.68b) has no such symmetry.

EXAMPLE 8 Evaluate (a) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ and (b) $\cos^{-1}\left(-\frac{1}{2}\right)$.

Solution

(a) We see that

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$$

because $\sin(\pi/3) = \sqrt{3}/2$ and $\pi/3$ belongs to the range $[-\pi/2, \pi/2]$ of the arcsine function. See Figure 1.69a.

(b) We have

$$\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$$

because $\cos(2\pi/3) = -1/2$ and $2\pi/3$ belongs to the range $[0, \pi]$ of the arccosine function. See Figure 1.69b. ■

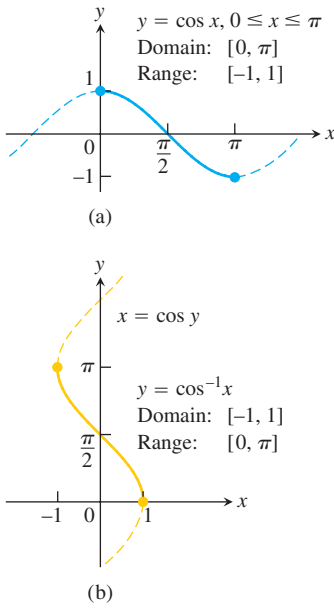


FIGURE 1.68 The graphs of (a) $y = \cos x, 0 \leq x \leq \pi$, and (b) its inverse, $y = \cos^{-1} x$. The graph of $\cos^{-1} x$, obtained by reflection across the line $y = x$, is a portion of the curve $x = \cos y$.

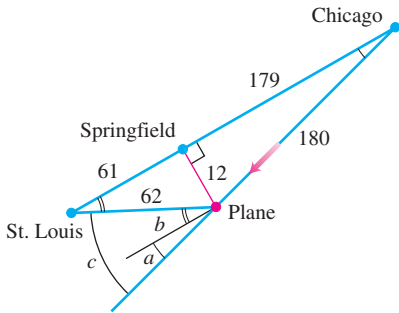


FIGURE 1.70 Diagram for drift correction (Example 9), with distances rounded to the nearest mile (drawing not to scale).

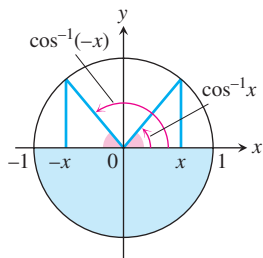


FIGURE 1.71 $\cos^{-1} x$ and $\cos^{-1}(-x)$ are supplementary angles (so their sum is π).

Using the same procedure illustrated in Example 8, we can create the following table of common values for the arcsine and arccosine functions.

x	$\sin^{-1} x$	$\cos^{-1} x$
$\sqrt{3}/2$	$\pi/3$	$\pi/6$
$\sqrt{2}/2$	$\pi/4$	$\pi/4$
$1/2$	$\pi/6$	$\pi/3$
$-1/2$	$-\pi/6$	$2\pi/3$
$-\sqrt{2}/2$	$-\pi/4$	$3\pi/4$
$-\sqrt{3}/2$	$-\pi/3$	$5\pi/6$

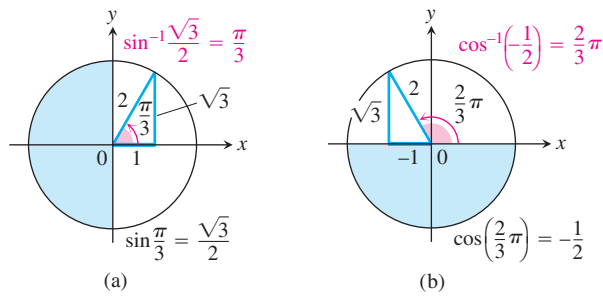


FIGURE 1.69 Values of the arcsine and arccosine functions (Example 8).

EXAMPLE 9 During an airplane flight from Chicago to St. Louis, the navigator determines that the plane is 12 mi off course, as shown in Figure 1.70. Find the angle a for a course parallel to the original correct course, the angle b , and the drift correction angle $c = a + b$.

Solution From Figure 1.70 and elementary geometry, we see that $180 \sin a = 12$ and $62 \sin b = 12$, so

$$a = \sin^{-1} \frac{12}{180} \approx 0.067 \text{ radian} \approx 3.8^\circ$$

$$b = \sin^{-1} \frac{12}{62} \approx 0.195 \text{ radian} \approx 11.2^\circ$$

$$c = a + b \approx 15^\circ. \quad \blacksquare$$

Identities Involving Arcsine and Arccosine

As we can see from Figure 1.71, the arccosine of x satisfies the identity

$$\cos^{-1} x + \cos^{-1}(-x) = \pi, \tag{3}$$

or

$$\cos^{-1}(-x) = \pi - \cos^{-1} x. \tag{4}$$

Also, we can see from the triangle in Figure 1.72 that for $x > 0$,

$$\sin^{-1} x + \cos^{-1} x = \pi/2. \tag{5}$$

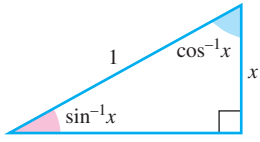


FIGURE 1.72 $\sin^{-1} x$ and $\cos^{-1} x$ are complementary angles (so their sum is $\pi/2$).

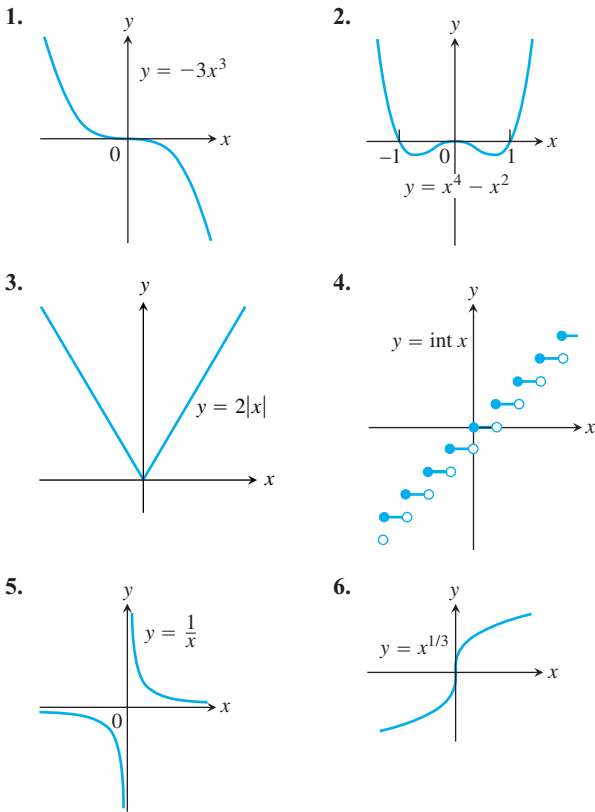
Equation (5) holds for the other values of x in $[-1, 1]$ as well, but we cannot conclude this from the triangle in Figure 1.72. It is, however, a consequence of Equations (2) and (4) (Exercise 74).

The arctangent, arccotangent, arcsecant, and arccosecant functions are defined in Section 3.9. There we develop additional properties of the inverse trigonometric functions in a calculus setting using the identities discussed here.

Exercises 1.6

Identifying One-to-One Functions Graphically

Which of the functions graphed in Exercises 1–6 are one-to-one, and which are not?

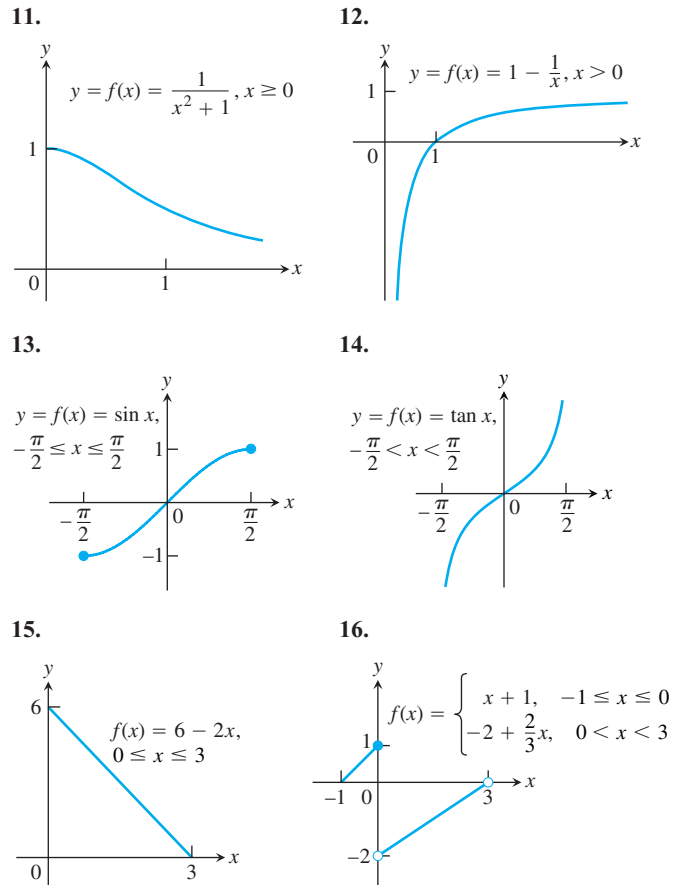


In Exercises 7–10, determine from its graph if the function is one-to-one.

7. $f(x) = \begin{cases} 3 - x, & x < 0 \\ 3, & x \geq 0 \end{cases}$
8. $f(x) = \begin{cases} 2x + 6, & x \leq -3 \\ x + 4, & x > -3 \end{cases}$
9. $f(x) = \begin{cases} 1 - \frac{x}{2}, & x \leq 0 \\ \frac{x}{x + 2}, & x > 0 \end{cases}$
10. $f(x) = \begin{cases} 2 - x^2, & x \leq 1 \\ x^2, & x > 1 \end{cases}$

Graphing Inverse Functions

Each of Exercises 11–16 shows the graph of a function $y = f(x)$. Copy the graph and draw in the line $y = x$. Then use symmetry with respect to the line $y = x$ to add the graph of f^{-1} to your sketch. (It is not necessary to find a formula for f^{-1} .) Identify the domain and range of f^{-1} .

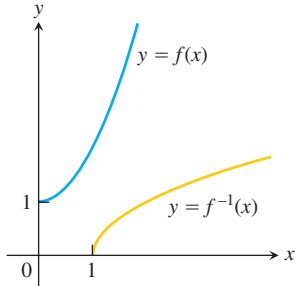


17. a. Graph the function $f(x) = \sqrt{1 - x^2}$, $0 \leq x \leq 1$. What symmetry does the graph have?
b. Show that f is its own inverse. (Remember that $\sqrt{x^2} = x$ if $x \geq 0$.)
18. a. Graph the function $f(x) = 1/x$. What symmetry does the graph have?
b. Show that f is its own inverse.

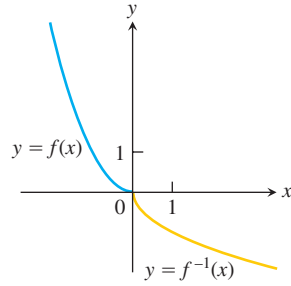
Formulas for Inverse Functions

Each of Exercises 19–24 gives a formula for a function $y = f(x)$ and shows the graphs of f and f^{-1} . Find a formula for f^{-1} in each case.

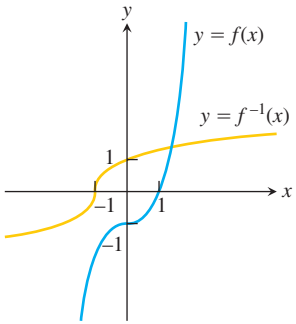
19. $f(x) = x^2 + 1, \quad x \geq 0$



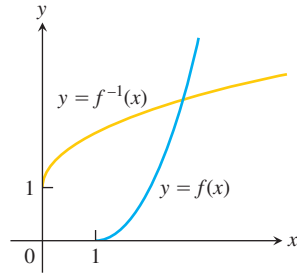
20. $f(x) = x^2, \quad x \leq 0$



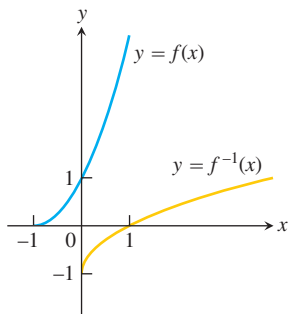
21. $f(x) = x^3 - 1$



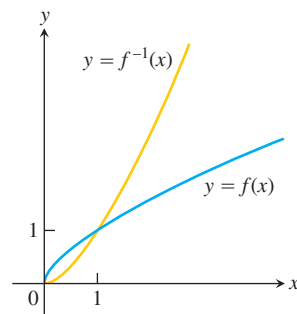
22. $f(x) = x^2 - 2x + 1, \quad x \geq 1$



23. $f(x) = (x + 1)^2, \quad x \geq -1$



24. $f(x) = x^{2/3}, \quad x \geq 0$



Each of Exercises 25–34 gives a formula for a function $y = f(x)$. In each case, find $f^{-1}(x)$ and identify the domain and range of f^{-1} . As a check, show that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

25. $f(x) = x^5$

26. $f(x) = x^4, \quad x \geq 0$

27. $f(x) = x^3 + 1$

28. $f(x) = (1/2)x - 7/2$

29. $f(x) = 1/x^2, \quad x > 0$

30. $f(x) = 1/x^3, \quad x \neq 0$

31. $f(x) = \frac{x+3}{x-2}$

32. $f(x) = \frac{\sqrt{x}}{\sqrt{x}-3}$

33. $f(x) = x^2 - 2x, \quad x \leq 1$

(Hint: Complete the square.)

34. $f(x) = (2x^3 + 1)^{1/5}$

Inverses of Lines

35. a. Find the inverse of the function $f(x) = mx$, where m is a constant different from zero.
 b. What can you conclude about the inverse of a function $y = f(x)$ whose graph is a line through the origin with a nonzero slope m ?
36. Show that the graph of the inverse of $f(x) = mx + b$, where m and b are constants and $m \neq 0$, is a line with slope $1/m$ and y -intercept $-b/m$.
37. a. Find the inverse of $f(x) = x + 1$. Graph f and its inverse together. Add the line $y = x$ to your sketch, drawing it with dashes or dots for contrast.
 b. Find the inverse of $f(x) = x + b$ (b constant). How is the graph of f^{-1} related to the graph of f ?
 c. What can you conclude about the inverses of functions whose graphs are lines parallel to the line $y = x$?
38. a. Find the inverse of $f(x) = -x + 1$. Graph the line $y = -x + 1$ together with the line $y = x$. At what angle do the lines intersect?
 b. Find the inverse of $f(x) = -x + b$ (b constant). What angle does the line $y = -x + b$ make with the line $y = x$?
 c. What can you conclude about the inverses of functions whose graphs are lines perpendicular to the line $y = x$?

Logarithms and Exponentials

39. Express the following logarithms in terms of $\ln 2$ and $\ln 3$.

a. $\ln 0.75$

b. $\ln(4/9)$

c. $\ln(1/2)$

d. $\ln \sqrt[3]{9}$

e. $\ln 3\sqrt{2}$

f. $\ln \sqrt{13.5}$

40. Express the following logarithms in terms of $\ln 5$ and $\ln 7$.

a. $\ln(1/125)$

b. $\ln 9.8$

c. $\ln 7\sqrt{7}$

d. $\ln 1225$

e. $\ln 0.056$

f. $(\ln 35 + \ln(1/7))/(\ln 25)$

Use the properties of logarithms to simplify the expressions in Exercises 41 and 42.

41. a. $\ln \sin \theta - \ln \left(\frac{\sin \theta}{5}\right)$

b. $\ln(3x^2 - 9x) + \ln \left(\frac{1}{3x}\right)$

c. $\frac{1}{2} \ln(4t^4) - \ln 2$

42. a. $\ln \sec \theta + \ln \cos \theta$

b. $\ln(8x + 4) - 2 \ln 2$

c. $3 \ln \sqrt[3]{t^2 - 1} - \ln(t + 1)$

Find simpler expressions for the quantities in Exercises 43–46.

43. a. $e^{\ln 7.2}$

b. $e^{-\ln x^2}$

c. $e^{\ln x - \ln y}$

44. a. $e^{\ln(x^2 + y^2)}$

b. $e^{-\ln 0.3}$

c. $e^{\ln \pi x - \ln 2}$

45. a. $2 \ln \sqrt{e}$

b. $\ln(\ln e^e)$

c. $\ln(e^{-x^2 - y^2})$

46. a. $\ln(e^{\sec \theta})$

b. $\ln(e^{e^y})$

c. $\ln(e^{2 \ln x})$

In Exercises 47–52, solve for y in terms of t or x , as appropriate.

47. $\ln y = 2t + 4$

48. $\ln y = -t + 5$

49. $\ln(y - 40) = 5t$

50. $\ln(1 - 2y) = t$

51. $\ln(y - 1) - \ln 2 = x + \ln x$

52. $\ln(y^2 - 1) - \ln(y + 1) = \ln(\sin x)$

In Exercises 53 and 54, solve for k .

53. a. $e^{2k} = 4$ b. $100e^{10k} = 200$ c. $e^{k/1000} = a$
 54. a. $e^{5k} = \frac{1}{4}$ b. $80e^k = 1$ c. $e^{(\ln 0.8)k} = 0.8$

In Exercises 55–58, solve for t .

55. a. $e^{-0.3t} = 27$ b. $e^{kt} = \frac{1}{2}$ c. $e^{(\ln 0.2)t} = 0.4$
 56. a. $e^{-0.01t} = 1000$ b. $e^{kt} = \frac{1}{10}$ c. $e^{(\ln 2)t} = \frac{1}{2}$
 57. $e^{\sqrt{t}} = x^2$ 58. $e^{(x^2)}e^{(2x+1)} = e^t$

Simplify the expressions in Exercises 59–62.

59. a. $5^{\log_5 7}$ b. $8^{\log_8 \sqrt{2}}$ c. $1.3^{\log_{1.3} 75}$
 d. $\log_4 16$ e. $\log_3 \sqrt{3}$ f. $\log_4 \left(\frac{1}{4}\right)$
 60. a. $2^{\log_2 3}$ b. $10^{\log_{10} (1/2)}$ c. $\pi^{\log_\pi 7}$
 d. $\log_{11} 121$ e. $\log_{121} 11$ f. $\log_3 \left(\frac{1}{9}\right)$
 61. a. $2^{\log_4 x}$ b. $9^{\log_3 x}$ c. $\log_2 (e^{(\ln 2)(\sin x)})$
 62. a. $25^{\log_5 (3x^2)}$ b. $\log_e (e^x)$ c. $\log_4 (2^{e^x \sin x})$

Express the ratios in Exercises 63 and 64 as ratios of natural logarithms and simplify.

63. a. $\frac{\log_2 x}{\log_3 x}$ b. $\frac{\log_2 x}{\log_8 x}$ c. $\frac{\log_x a}{\log_{x^2} a}$
 64. a. $\frac{\log_9 x}{\log_3 x}$ b. $\frac{\log_{\sqrt{10}} x}{\log_{\sqrt{2}} x}$ c. $\frac{\log_a b}{\log_b a}$

Arcsine and Arccosine

In Exercises 65–68, find the exact value of each expression.

65. a. $\sin^{-1} \left(\frac{-1}{2}\right)$ b. $\sin^{-1} \left(\frac{1}{\sqrt{2}}\right)$ c. $\sin^{-1} \left(\frac{-\sqrt{3}}{2}\right)$
 66. a. $\cos^{-1} \left(\frac{1}{2}\right)$ b. $\cos^{-1} \left(\frac{-1}{\sqrt{2}}\right)$ c. $\cos^{-1} \left(\frac{\sqrt{3}}{2}\right)$
 67. a. $\arccos(-1)$ b. $\arccos(0)$
 68. a. $\arcsin(-1)$ b. $\arcsin \left(-\frac{1}{\sqrt{2}}\right)$

Theory and Examples

69. If $f(x)$ is one-to-one, can anything be said about $g(x) = -f(x)$? Is it also one-to-one? Give reasons for your answer.
 70. If $f(x)$ is one-to-one and $f(x)$ is never zero, can anything be said about $h(x) = 1/f(x)$? Is it also one-to-one? Give reasons for your answer.
 71. Suppose that the range of g lies in the domain of f so that the composite $f \circ g$ is defined. If f and g are one-to-one, can anything be said about $f \circ g$? Give reasons for your answer.

72. If a composite $f \circ g$ is one-to-one, must g be one-to-one? Give reasons for your answer.

73. Find a formula for the inverse function f^{-1} and verify that $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$.

a. $f(x) = \frac{100}{1 + 2^{-x}}$ b. $f(x) = \frac{50}{1 + 1.1^{-x}}$

74. **The identity $\sin^{-1} x + \cos^{-1} x = \pi/2$** Figure 1.72 establishes the identity for $0 < x < 1$. To establish it for the rest of $[-1, 1]$, verify by direct calculation that it holds for $x = 1, 0$, and -1 . Then, for values of x in $(-1, 0)$, let $x = -a$, $a > 0$, and apply Eqs. (3) and (5) to the sum $\sin^{-1}(-a) + \cos^{-1}(-a)$.

75. Start with the graph of $y = \ln x$. Find an equation of the graph that results from

- a. shifting down 3 units.
 b. shifting right 1 unit.
 c. shifting left 1, up 3 units.
 d. shifting down 4, right 2 units.
 e. reflecting about the y -axis.
 f. reflecting about the line $y = x$.

76. Start with the graph of $y = \ln x$. Find an equation of the graph that results from

- a. vertical stretching by a factor of 2.
 b. horizontal stretching by a factor of 3.
 c. vertical compression by a factor of 4.
 d. horizontal compression by a factor of 2.

- T** 77. The equation $x^2 = 2^x$ has three solutions: $x = 2$, $x = 4$, and one other. Estimate the third solution as accurately as you can by graphing.

- T** 78. Could $x^{\ln 2}$ possibly be the same as $2^{\ln x}$ for $x > 0$? Graph the two functions and explain what you see.

79. **Radioactive decay** The half-life of a certain radioactive substance is 12 hours. There are 8 grams present initially.

- a. Express the amount of substance remaining as a function of time t .
 b. When will there be 1 gram remaining?

80. **Doubling your money** Determine how much time is required for a \$500 investment to double in value if interest is earned at the rate of 4.75% compounded annually.

81. **Population growth** The population of Glenbrook is 375,000 and is increasing at the rate of 2.25% per year. Predict when the population will be 1 million.

82. **Radon-222** The decay equation for radon-222 gas is known to be $y = y_0 e^{-0.18t}$, with t in days. About how long will it take the radon in a sealed sample of air to fall to 90% of its original value?

Chapter 1 Questions to Guide Your Review

- What is a function? What is its domain? Its range? What is an arrow diagram for a function? Give examples.
- What is the graph of a real-valued function of a real variable? What is the vertical line test?
- What is a piecewise-defined function? Give examples.
- What are the important types of functions frequently encountered in calculus? Give an example of each type.

- What is meant by an increasing function? A decreasing function? Give an example of each.
- What is an even function? An odd function? What symmetry properties do the graphs of such functions have? What advantage can we take of this? Give an example of a function that is neither even nor odd.
- If f and g are real-valued functions, how are the domains of $f + g$, $f - g$, fg , and f/g related to the domains of f and g ? Give examples.
- When is it possible to compose one function with another? Give examples of composites and their values at various points. Does the order in which functions are composed ever matter?
- How do you change the equation $y = f(x)$ to shift its graph vertically up or down by $|k|$ units? Horizontally to the left or right? Give examples.
- How do you change the equation $y = f(x)$ to compress or stretch the graph by a factor $c > 1$? Reflect the graph across a coordinate axis? Give examples.
- What is the standard equation of an ellipse with center (h, k) ? What is its major axis? Its minor axis? Give examples.
- What is radian measure? How do you convert from radians to degrees? Degrees to radians?
- Graph the six basic trigonometric functions. What symmetries do the graphs have?
- What is a periodic function? Give examples. What are the periods of the six basic trigonometric functions?
- Starting with the identity $\sin^2 \theta + \cos^2 \theta = 1$ and the formulas for $\cos(A + B)$ and $\sin(A + B)$, show how a variety of other trigonometric identities may be derived.
- How does the formula for the general sine function $f(x) = A \sin((2\pi/B)(x - C)) + D$ relate to the shifting, stretching, compressing, and reflection of its graph? Give examples. Graph the general sine curve and identify the constants A , B , C , and D .
- Name three issues that arise when functions are graphed using a calculator or computer with graphing software. Give examples.
- What is an exponential function? Give examples. What laws of exponents does it obey? How does it differ from a simple power function like $f(x) = x^n$? What kind of real-world phenomena are modeled by exponential functions?
- What is the number e , and how is it defined? What are the domain and range of $f(x) = e^x$? What does its graph look like? How do the values of e^x relate to x^2 , x^3 , and so on?
- What functions have inverses? How do you know if two functions f and g are inverses of one another? Give examples of functions that are (are not) inverses of one another.
- How are the domains, ranges, and graphs of functions and their inverses related? Give an example.
- What procedure can you sometimes use to express the inverse of a function of x as a function of x ?
- What is a logarithmic function? What properties does it satisfy? What is the natural logarithm function? What are the domain and range of $y = \ln x$? What does its graph look like?
- How is the graph of $\log_a x$ related to the graph of $\ln x$? What truth is there in the statement that there is really only one exponential function and one logarithmic function?
- How are the inverse trigonometric functions defined? How can you sometimes use right triangles to find values of these functions? Give examples.

Chapter 1 Practice Exercises

Functions and Graphs

- Express the area and circumference of a circle as functions of the circle's radius. Then express the area as a function of the circumference.
- Express the radius of a sphere as a function of the sphere's surface area. Then express the surface area as a function of the volume.
- A point P in the first quadrant lies on the parabola $y = x^2$. Express the coordinates of P as functions of the angle of inclination of the line joining P to the origin.
- A hot-air balloon rising straight up from a level field is tracked by a range finder located 500 ft from the point of liftoff. Express the balloon's height as a function of the angle the line from the range finder to the balloon makes with the ground.

In Exercises 5–8, determine whether the graph of the function is symmetric about the y -axis, the origin, or neither.

- $y = x^{1/5}$
- $y = x^{2/5}$
- $y = x^2 - 2x - 1$
- $y = e^{-x^2}$

In Exercises 9–16, determine whether the function is even, odd, or neither.

- $y = x^2 + 1$
- $y = x^5 - x^3 - x$
- $y = 1 - \cos x$
- $y = \sec x \tan x$
- $y = \frac{x^4 + 1}{x^3 - 2x}$
- $y = x - \sin x$
- $y = x + \cos x$
- $y = x \cos x$
- Suppose that f and g are both odd functions defined on the entire real line. Which of the following (where defined) are even? odd?
 - fg
 - f^3
 - $f(\sin x)$
 - $g(\sec x)$
 - $|g|$
- If $f(a - x) = f(a + x)$, show that $g(x) = f(x + a)$ is an even function.

In Exercises 19–28, find the (a) domain and (b) range.

- $y = |x| - 2$
- $y = -2 + \sqrt{1 - x}$
- $y = \sqrt{16 - x^2}$
- $y = 3^{2-x} + 1$
- $y = 2e^{-x} - 3$
- $y = \tan(2x - \pi)$
- $y = 2 \sin(3x + \pi) - 1$
- $y = x^{2/5}$

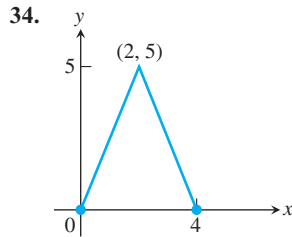
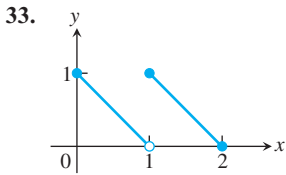
27. $y = \ln(x - 3) + 1$ 28. $y = -1 + \sqrt[3]{2 - x}$
29. State whether each function is increasing, decreasing, or neither.
- Volume of a sphere as a function of its radius
 - Greatest integer function
 - Height above Earth's sea level as a function of atmospheric pressure (assumed nonzero)
 - Kinetic energy as a function of a particle's velocity
30. Find the largest interval on which the given function is increasing.
- $f(x) = |x - 2| + 1$
 - $f(x) = (x + 1)^4$
 - $g(x) = (3x - 1)^{1/3}$
 - $R(x) = \sqrt{2x - 1}$

Piecewise-Defined Functions

In Exercises 31 and 32, find the (a) domain and (b) range.

31. $y = \begin{cases} \sqrt{-x}, & -4 \leq x \leq 0 \\ \sqrt{x}, & 0 < x \leq 4 \end{cases}$
32. $y = \begin{cases} -x - 2, & -2 \leq x \leq -1 \\ x, & -1 < x \leq 1 \\ -x + 2, & 1 < x \leq 2 \end{cases}$

In Exercises 33 and 34, write a piecewise formula for the function.



Composition of Functions

In Exercises 35 and 36, find

- $(f \circ g)(-1)$.
 - $(g \circ f)(2)$.
 - $(f \circ f)(x)$.
 - $(g \circ g)(x)$.
35. $f(x) = \frac{1}{x}$, $g(x) = \frac{1}{\sqrt{x+2}}$
36. $f(x) = 2 - x$, $g(x) = \sqrt[3]{x+1}$

In Exercises 37 and 38, (a) write formulas for $f \circ g$ and $g \circ f$ and find the (b) domain and (c) range of each.

37. $f(x) = 2 - x^2$, $g(x) = \sqrt{x+2}$
38. $f(x) = \sqrt{x}$, $g(x) = \sqrt{1-x}$

For Exercises 39 and 40, sketch the graphs of f and $f \circ f$.

39. $f(x) = \begin{cases} -x - 2, & -4 \leq x \leq -1 \\ -1, & -1 < x \leq 1 \\ x - 2, & 1 < x \leq 2 \end{cases}$
40. $f(x) = \begin{cases} x + 1, & -2 \leq x < 0 \\ x - 1, & 0 \leq x \leq 2 \end{cases}$

Composition with absolute values In Exercises 41–48, graph f_1 and f_2 together. Then describe how applying the absolute value function in f_2 affects the graph of f_1 .

$f_1(x)$	$f_2(x)$
41. x	$ x $
42. x^2	$ x ^2$
43. x^3	$ x^3 $
44. $x^2 + x$	$ x^2 + x $
45. $4 - x^2$	$ 4 - x^2 $
46. $\frac{1}{x}$	$\frac{1}{ x }$
47. \sqrt{x}	$\sqrt{ x }$
48. $\sin x$	$\sin x $

Shifting and Scaling Graphs

49. Suppose the graph of g is given. Write equations for the graphs that are obtained from the graph of g by shifting, scaling, or reflecting, as indicated.
- Up $\frac{1}{2}$ unit, right 3
 - Down 2 units, left $\frac{2}{3}$
 - Reflect about the y -axis
 - Reflect about the x -axis
 - Stretch vertically by a factor of 5
 - Compress horizontally by a factor of 5
50. Describe how each graph is obtained from the graph of $y = f(x)$.
- $y = f(x - 5)$
 - $y = f(4x)$
 - $y = f(-3x)$
 - $y = f(2x + 1)$
 - $y = f\left(\frac{x}{3}\right) - 4$
 - $y = -3f(x) + \frac{1}{4}$

In Exercises 51–54, graph each function, not by plotting points, but by starting with the graph of one of the standard functions presented in Figures 1.15–1.17, and applying an appropriate transformation.

51. $y = -\sqrt{1 + \frac{x}{2}}$ 52. $y = 1 - \frac{x}{3}$
53. $y = \frac{1}{2x^2} + 1$ 54. $y = (-5x)^{1/3}$

Trigonometry

In Exercises 55–58, sketch the graph of the given function. What is the period of the function?

55. $y = \cos 2x$ 56. $y = \sin \frac{x}{2}$
57. $y = \sin \pi x$ 58. $y = \cos \frac{\pi x}{2}$
59. Sketch the graph $y = 2 \cos\left(x - \frac{\pi}{3}\right)$.
60. Sketch the graph $y = 1 + \sin\left(x + \frac{\pi}{4}\right)$.

In Exercises 61–64, ABC is a right triangle with the right angle at C . The sides opposite angles A , B , and C are a , b , and c , respectively.

61. a. Find a and b if $c = 2$, $B = \pi/3$.
 b. Find a and c if $b = 2$, $B = \pi/3$.
62. a. Express a in terms of A and c .
 b. Express a in terms of A and b .
63. a. Express a in terms of B and b .
 b. Express c in terms of A and a .
64. a. Express $\sin A$ in terms of a and c .
 b. Express $\sin A$ in terms of b and c .
65. **Height of a pole** Two wires stretch from the top T of a vertical pole to points B and C on the ground, where C is 10 m closer to the base of the pole than is B . If wire BT makes an angle of 35° with the horizontal and wire CT makes an angle of 50° with the horizontal, how high is the pole?
66. **Height of a weather balloon** Observers at positions A and B 2 km apart simultaneously measure the angle of elevation of a weather balloon to be 40° and 70° , respectively. If the balloon is directly above a point on the line segment between A and B , find the height of the balloon.

- T** 67. a. Graph the function $f(x) = \sin x + \cos(x/2)$.
 b. What appears to be the period of this function?
 c. Confirm your finding in part (b) algebraically.
- T** 68. a. Graph $f(x) = \sin(1/x)$.
 b. What are the domain and range of f ?
 c. Is f periodic? Give reasons for your answer.

Transcendental Functions

In Exercises 69–72, find the domain of each function.

69. a. $f(x) = 1 + e^{-\sin x}$ b. $g(x) = e^x + \ln \sqrt{x}$
 70. a. $f(x) = e^{1/x^2}$ b. $g(x) = \ln|4 - x^2|$
 71. a. $h(x) = \sin^{-1}\left(\frac{x}{3}\right)$ b. $f(x) = \cos^{-1}(\sqrt{x} - 1)$
 72. a. $h(x) = \ln(\cos^{-1}x)$ b. $f(x) = \sqrt{\pi - \sin^{-1}x}$
 73. If $f(x) = \ln x$ and $g(x) = 4 - x^2$, find the functions $f \circ g$, $g \circ f$, $f \circ f$, $g \circ g$, and their domains.

74. Determine whether f is even, odd, or neither.
 a. $f(x) = e^{-x^2}$ b. $f(x) = 1 + \sin^{-1}(-x)$
 c. $f(x) = |e^x|$ d. $f(x) = e^{\ln|x|+1}$

- T** 75. Graph $\ln x$, $\ln 2x$, $\ln 4x$, $\ln 8x$, and $\ln 16x$ (as many as you can) together for $0 < x \leq 10$. What is going on? Explain.
- T** 76. Graph $y = \ln(x^2 + c)$ for $c = -4, -2, 0, 3$, and 5 . How does the graph change when c changes?
- T** 77. Graph $y = \ln|\sin x|$ in the window $0 \leq x \leq 22$, $-2 \leq y \leq 0$. Explain what you see. How could you change the formula to turn the arches upside down?
- T** 78. Graph the three functions $y = x^a$, $y = a^x$, and $y = \log_a x$ together on the same screen for $a = 2, 10$, and 20 . For large values of x , which of these functions has the largest values and which has the smallest values?

Theory and Examples

In Exercises 79 and 80, find the domain and range of each composite function. Then graph the composites on separate screens. Do the graphs make sense in each case? Give reasons for your answers and comment on any differences you see.

79. a. $y = \sin^{-1}(\sin x)$ b. $y = \sin(\sin^{-1}x)$
 80. a. $y = \cos^{-1}(\cos x)$ b. $y = \cos(\cos^{-1}x)$
 81. Use a graph to decide whether f is one-to-one.
 a. $f(x) = x^3 - \frac{x}{2}$ b. $f(x) = x^3 + \frac{x}{2}$
- T** 82. Use a graph to find to 3 decimal places the values of x for which $e^x > 10,000,000$.
83. a. Show that $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$ are inverses of one another.
T b. Graph f and g over an x -interval large enough to show the graphs intersecting at $(1, 1)$ and $(-1, -1)$. Be sure the picture shows the required symmetry in the line $y = x$.
84. a. Show that $h(x) = x^3/4$ and $k(x) = (4x)^{1/3}$ are inverses of one another.
T b. Graph h and k over an x -interval large enough to show the graphs intersecting at $(2, 2)$ and $(-2, -2)$. Be sure the picture shows the required symmetry in the line $y = x$.

Chapter 1 Additional and Advanced Exercises

Functions and Graphs

- Are there two functions f and g such that $f \circ g = g \circ f$? Give reasons for your answer.
- Are there two functions f and g with the following property? The graphs of f and g are not straight lines but the graph of $f \circ g$ is a straight line. Give reasons for your answer.
- If $f(x)$ is odd, can anything be said of $g(x) = f(x) - 2$? What if f is even instead? Give reasons for your answer.
- If $g(x)$ is an odd function defined for all values of x , can anything be said about $g(0)$? Give reasons for your answer.

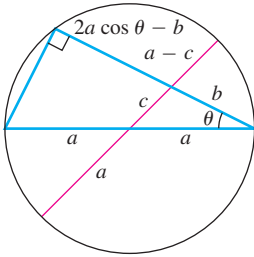
- Graph the equation $|x| + |y| = 1 + x$.
- Graph the equation $y + |y| = x + |x|$.

Derivations and Proofs

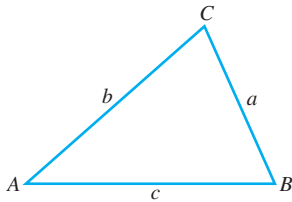
7. Prove the following identities.

$$\text{a. } \frac{1 - \cos x}{\sin x} = \frac{\sin x}{1 + \cos x} \quad \text{b. } \frac{1 - \cos x}{1 + \cos x} = \tan^2 \frac{x}{2}$$

8. Explain the following “proof without words” of the law of cosines. (Source: “Proof without Words: The Law of Cosines,” Sidney H. Kung, *Mathematics Magazine*, Vol. 63, No. 5, Dec. 1990, p. 342.)



9. Show that the area of triangle ABC is given by $(1/2)ab \sin C = (1/2)bc \sin A = (1/2)ca \sin B$.



10. Show that the area of triangle ABC is given by $\sqrt{s(s-a)(s-b)(s-c)}$ where $s = (a+b+c)/2$ is the semiperimeter of the triangle.
11. Show that if f is both even and odd, then $f(x) = 0$ for every x in the domain of f .
12. a. **Even-odd decompositions** Let f be a function whose domain is symmetric about the origin, that is, $-x$ belongs to the domain whenever x does. Show that f is the sum of an even function and an odd function:

$$f(x) = E(x) + O(x),$$

where E is an even function and O is an odd function. (Hint: Let $E(x) = (f(x) + f(-x))/2$. Show that $E(-x) = E(x)$, so that E is even. Then show that $O(x) = f(x) - E(x)$ is odd.)

- b. **Uniqueness** Show that there is only one way to write f as the sum of an even and an odd function. (Hint: One way is given in part (a). If also $f(x) = E_1(x) + O_1(x)$ where E_1 is even and O_1 is odd, show that $E - E_1 = O_1 - O$. Then use Exercise 11 to show that $E = E_1$ and $O = O_1$.)

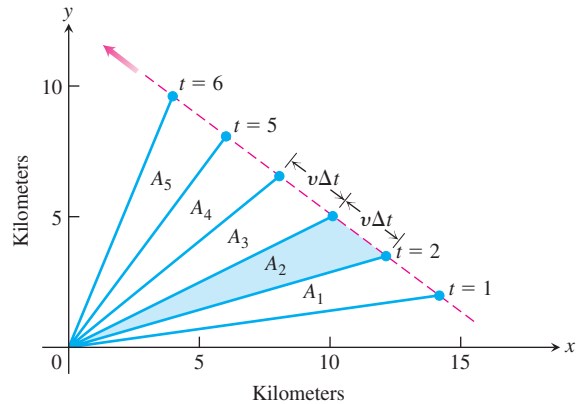
Grapher Explorations—Effects of Parameters

13. What happens to the graph of $y = ax^2 + bx + c$ as
- a changes while b and c remain fixed?
 - b changes (a and c fixed, $a \neq 0$)?
 - c changes (a and b fixed, $a \neq 0$)?
14. What happens to the graph of $y = a(x+b)^3 + c$ as
- a changes while b and c remain fixed?
 - b changes (a and c fixed, $a \neq 0$)?
 - c changes (a and b fixed, $a \neq 0$)?

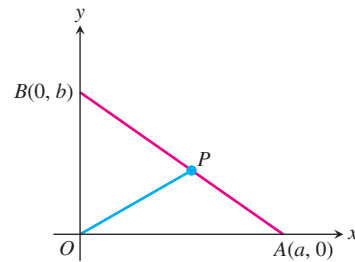
Geometry

15. An object’s center of mass moves at a constant velocity v along a straight line past the origin. The accompanying figure shows the coordinate system and the line of motion. The dots show positions that are 1 sec apart. Why are the areas A_1, A_2, \dots, A_5 in the figure all equal? As in Kepler’s equal area law (see Section 13.6), the

line that joins the object’s center of mass to the origin sweeps out equal areas in equal times.

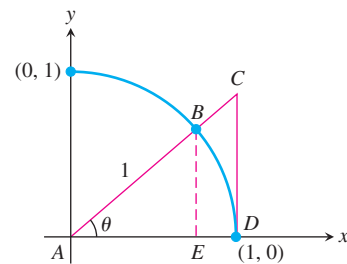


16. a. Find the slope of the line from the origin to the midpoint P of side AB in the triangle in the accompanying figure ($a, b > 0$).



- b. When is OP perpendicular to AB ?
17. Consider the quarter-circle of radius 1 and right triangles ABE and ACD given in the accompanying figure. Use standard area formulas to conclude that

$$\frac{1}{2} \sin \theta \cos \theta < \frac{\theta}{2} < \frac{1}{2} \frac{\sin \theta}{\cos \theta}.$$



18. Let $f(x) = ax + b$ and $g(x) = cx + d$. What condition must be satisfied by the constants a, b, c, d in order that $(f \circ g)(x) = (g \circ f)(x)$ for every value of x ?

Theory and Examples

19. **Domain and range** Suppose that $a \neq 0, b \neq 1$, and $b > 0$. Determine the domain and range of the function.
- $y = a(b^{c-x}) + d$
 - $y = a \log_b(x - c) + d$
20. **Inverse functions** Let

$$f(x) = \frac{ax + b}{cx + d}, \quad c \neq 0, \quad ad - bc \neq 0.$$

- Give a convincing argument that f is one-to-one.
- Find a formula for the inverse of f .

21. Depreciation Smith Hauling purchased an 18-wheel truck for \$100,000. The truck depreciates at the constant rate of \$10,000 per year for 10 years.

- Write an expression that gives the value y after x years.
- When is the value of the truck \$55,000?

22. Drug absorption A drug is administered intravenously for pain. The function

$$f(t) = 90 - 52 \ln(1 + t), \quad 0 \leq t \leq 4$$

gives the number of units of the drug remaining in the body after t hours.

- What was the initial number of units of the drug administered?
- How much is present after 2 hours?
- Draw the graph of f .

23. Finding investment time If Juanita invests \$1500 in a retirement account that earns 8% compounded annually, how long will it take this single payment to grow to \$5000?

24. The rule of 70 If you use the approximation $\ln 2 \approx 0.70$ (in place of $0.69314 \dots$), you can derive a rule of thumb that says,

“To estimate how many years it will take an amount of money to double when invested at r percent compounded continuously, divide r into 70.” For instance, an amount of money invested at 5% will double in about $70/5 = 14$ years. If you want it to double in 10 years instead, you have to invest it at $70/10 = 7\%$. Show how the rule of 70 is derived. (A similar “rule of 72” uses 72 instead of 70, because 72 has more integer factors.)

25. For what $x > 0$ does $x^{(x^x)} = (x^x)^x$? Give reasons for your answer.

T 26. a. If $(\ln x)/x = (\ln 2)/2$, must $x = 2$?

b. If $(\ln x)/x = -2 \ln 2$, must $x = 1/2$?

Give reasons for your answers.

27. The quotient $(\log_4 x)/(\log_2 x)$ has a constant value. What value? Give reasons for your answer.

T 28. $\log_x(2)$ vs. $\log_2(x)$ How does $f(x) = \log_x(2)$ compare with $g(x) = \log_2(x)$? Here is one way to find out.

a. Use the equation $\log_a b = (\ln b)/(\ln a)$ to express $f(x)$ and $g(x)$ in terms of natural logarithms.

b. Graph f and g together. Comment on the behavior of f in relation to the signs and values of g .

Chapter 1 Technology Application Projects

An Overview of Mathematica

An overview of *Mathematica* sufficient to complete the *Mathematica* modules appearing on the Web site.

Mathematica/Maple Module:

Modeling Change: Springs, Driving Safety, Radioactivity, Trees, Fish, and Mammals

Construct and interpret mathematical models, analyze and improve them, and make predictions using them.



2

LIMITS AND CONTINUITY

OVERVIEW Mathematicians of the seventeenth century were keenly interested in the study of motion for objects on or near the earth and the motion of planets and stars. This study involved both the speed of the object and its direction of motion at any instant, and they knew the direction was tangent to the path of motion. The concept of a limit is fundamental to finding the velocity of a moving object and the tangent to a curve. In this chapter we develop the limit, first intuitively and then formally. We use limits to describe the way a function varies. Some functions vary *continuously*; small changes in x produce only small changes in $f(x)$. Other functions can have values that jump, vary erratically, or tend to increase or decrease without bound. The notion of limit gives a precise way to distinguish between these behaviors.

2.1

Rates of Change and Tangents to Curves

Calculus is a tool to help us understand how functional relationships change, such as the position or speed of a moving object as a function of time, or the changing slope of a curve being traversed by a point moving along it. In this section we introduce the ideas of average and instantaneous rates of change, and show that they are closely related to the slope of a curve at a point P on the curve. We give precise developments of these important concepts in the next chapter, but for now we use an informal approach so you will see how they lead naturally to the main idea of the chapter, the *limit*. You will see that limits play a major role in calculus and the study of change.

Average and Instantaneous Speed

In the late sixteenth century, Galileo discovered that a solid object dropped from rest (not moving) near the surface of the earth and allowed to fall freely will fall a distance proportional to the square of the time it has been falling. This type of motion is called **free fall**. It assumes negligible air resistance to slow the object down, and that gravity is the only force acting on the falling body. If y denotes the distance fallen in feet after t seconds, then Galileo's law is

$$y = 16t^2,$$

where 16 is the (approximate) constant of proportionality. (If y is measured in meters, the constant is 4.9.)

A moving body's **average speed** during an interval of time is found by dividing the distance covered by the time elapsed. The unit of measure is length per unit time: kilometers per hour, feet (or meters) per second, or whatever is appropriate to the problem at hand.

HISTORICAL BIOGRAPHY

Galileo Galilei
(1564–1642)

EXAMPLE 1 A rock breaks loose from the top of a tall cliff. What is its average speed

- (a) during the first 2 sec of fall?
 (b) during the 1-sec interval between second 1 and second 2?

Solution The average speed of the rock during a given time interval is the change in distance, Δy , divided by the length of the time interval, Δt . (Increments like Δy and Δt are reviewed in Appendix 3.) Measuring distance in feet and time in seconds, we have the following calculations:

(a) For the first 2 sec:
$$\frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(0)^2}{2 - 0} = 32 \frac{\text{ft}}{\text{sec}}$$

(b) From sec 1 to sec 2:
$$\frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(1)^2}{2 - 1} = 48 \frac{\text{ft}}{\text{sec}}$$

We want a way to determine the speed of a falling object at a single instant t_0 , instead of using its average speed over an interval of time. To do this, we examine what happens when we calculate the average speed over shorter and shorter time intervals starting at t_0 . The next example illustrates this process. Our discussion is informal here, but it will be made precise in Chapter 3.

EXAMPLE 2 Find the speed of the falling rock in Example 1 at $t = 1$ and $t = 2$ sec.

Solution We can calculate the average speed of the rock over a time interval $[t_0, t_0 + h]$, having length $\Delta t = h$, as

$$\frac{\Delta y}{\Delta t} = \frac{16(t_0 + h)^2 - 16t_0^2}{h}. \quad (1)$$

We cannot use this formula to calculate the “instantaneous” speed at the exact moment t_0 by simply substituting $h = 0$, because we cannot divide by zero. But we *can* use it to calculate average speeds over increasingly short time intervals starting at $t_0 = 1$ and $t_0 = 2$. When we do so, we see a pattern (Table 2.1).

TABLE 2.1 Average speeds over short time intervals $[t_0, t_0 + h]$

Average speed: $\frac{\Delta y}{\Delta t} = \frac{16(t_0 + h)^2 - 16t_0^2}{h}$		
Length of time interval h	Average speed over interval of length h starting at $t_0 = 1$	Average speed over interval of length h starting at $t_0 = 2$
1	48	80
0.1	33.6	65.6
0.01	32.16	64.16
0.001	32.016	64.016
0.0001	32.0016	64.0016

The average speed on intervals starting at $t_0 = 1$ seems to approach a limiting value of 32 as the length of the interval decreases. This suggests that the rock is falling at a speed of 32 ft/sec at $t_0 = 1$ sec. Let's confirm this algebraically.

If we set $t_0 = 1$ and then expand the numerator in Equation (1) and simplify, we find that

$$\begin{aligned} \frac{\Delta y}{\Delta t} &= \frac{16(1+h)^2 - 16(1)^2}{h} = \frac{16(1+2h+h^2) - 16}{h} \\ &= \frac{32h + 16h^2}{h} = 32 + 16h. \end{aligned}$$

For values of h different from 0, the expressions on the right and left are equivalent and the average speed is $32 + 16h$ ft/sec. We can now see why the average speed has the limiting value $32 + 16(0) = 32$ ft/sec as h approaches 0.

Similarly, setting $t_0 = 2$ in Equation (1), the procedure yields

$$\frac{\Delta y}{\Delta t} = 64 + 16h$$

for values of h different from 0. As h gets closer and closer to 0, the average speed has the limiting value 64 ft/sec when $t_0 = 2$ sec, as suggested by Table 2.1. ■

The average speed of a falling object is an example of a more general idea which we discuss next.

Average Rates of Change and Secant Lines

Given an arbitrary function $y = f(x)$, we calculate the average rate of change of y with respect to x over the interval $[x_1, x_2]$ by dividing the change in the value of y , $\Delta y = f(x_2) - f(x_1)$, by the length $\Delta x = x_2 - x_1 = h$ of the interval over which the change occurs. (We use the symbol h for Δx to simplify the notation here and later on.)

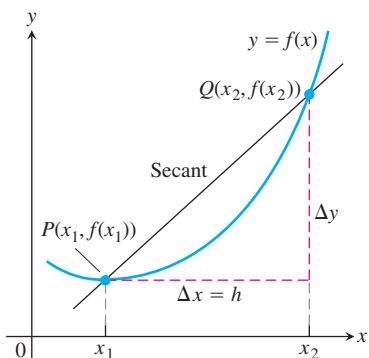


FIGURE 2.1 A secant to the graph $y = f(x)$. Its slope is $\Delta y/\Delta x$, the average rate of change of f over the interval $[x_1, x_2]$.

DEFINITION The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \quad h \neq 0.$$

Geometrically, the rate of change of f over $[x_1, x_2]$ is the slope of the line through the points $P(x_1, f(x_1))$ and $Q(x_2, f(x_2))$ (Figure 2.1). In geometry, a line joining two points of a curve is a **secant** to the curve. Thus, the average rate of change of f from x_1 to x_2 is identical with the slope of secant PQ . Let's consider what happens as the point Q approaches the point P along the curve, so the length h of the interval over which the change occurs approaches zero.

Defining the Slope of a Curve

We know what is meant by the slope of a straight line, which tells us the rate at which it rises or falls—its rate of change as the graph of a linear function. But what is meant by the *slope of a curve* at a point P on the curve? If there is a *tangent* line to the curve at P —a line that just touches the curve like the tangent to a circle—it would be reasonable to identify *the slope of the tangent* as the slope of the curve at P . So we need a precise meaning for the tangent at a point on a curve.

For circles, tangency is straightforward. A line L is tangent to a circle at a point P if L passes through P perpendicular to the radius at P (Figure 2.2). Such a line just *touches* the circle. But what does it mean to say that a line L is tangent to some other curve C at a point P ?

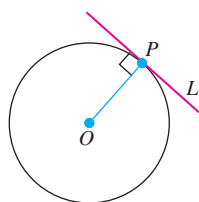


FIGURE 2.2 L is tangent to the circle at P if it passes through P perpendicular to radius OP .

To define tangency for general curves, we need an approach that takes into account the behavior of the secants through P and nearby points Q as Q moves toward P along the curve (Figure 2.3). Here is the idea:

1. Start with what we *can* calculate, namely the slope of the secant PQ .
2. Investigate the limiting value of the secant slope as Q approaches P along the curve. (We clarify the *limit* idea in the next section.)
3. If the *limit* exists, take it to be the slope of the curve at P and *define* the tangent to the curve at P to be the line through P with this slope.

This procedure is what we were doing in the falling-rock problem discussed in Example 2. The next example illustrates the geometric idea for the tangent to a curve.

HISTORICAL BIOGRAPHY

Pierre de Fermat
(1601–1665)

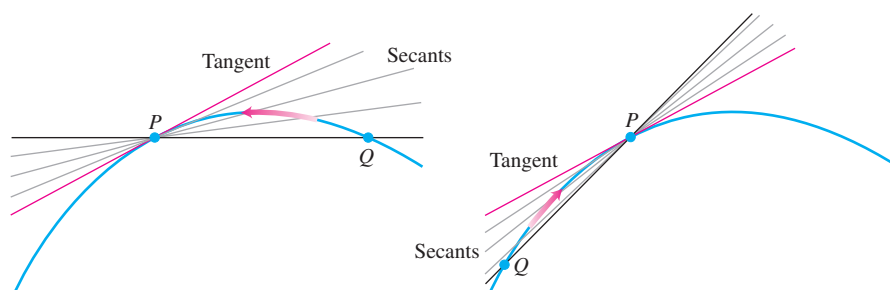


FIGURE 2.3 The tangent to the curve at P is the line through P whose slope is the limit of the secant slopes as $Q \rightarrow P$ from either side.

EXAMPLE 3 Find the slope of the parabola $y = x^2$ at the point $P(2, 4)$. Write an equation for the tangent to the parabola at this point.

Solution We begin with a secant line through $P(2, 4)$ and $Q(2 + h, (2 + h)^2)$ nearby. We then write an expression for the slope of the secant PQ and investigate what happens to the slope as Q approaches P along the curve:

$$\begin{aligned} \text{Secant slope} &= \frac{\Delta y}{\Delta x} = \frac{(2 + h)^2 - 2^2}{h} = \frac{h^2 + 4h + 4 - 4}{h} \\ &= \frac{h^2 + 4h}{h} = h + 4. \end{aligned}$$

If $h > 0$, then Q lies above and to the right of P , as in Figure 2.4. If $h < 0$, then Q lies to the left of P (not shown). In either case, as Q approaches P along the curve, h approaches zero and the secant slope $h + 4$ approaches 4. We take 4 to be the parabola's slope at P .

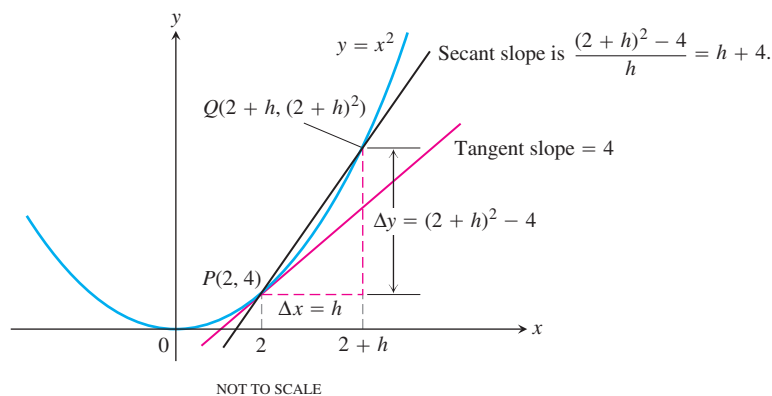


FIGURE 2.4 Finding the slope of the parabola $y = x^2$ at the point $P(2, 4)$ as the limit of secant slopes (Example 3).

The tangent to the parabola at P is the line through P with slope 4:

$$y = 4 + 4(x - 2) \quad \text{Point-slope equation}$$

$$y = 4x - 4. \quad \blacksquare$$

Instantaneous Rates of Change and Tangent Lines

The rates at which the rock in Example 2 was falling at the instants $t = 1$ and $t = 2$ are called *instantaneous rates of change*. Instantaneous rates and slopes of tangent lines are intimately connected, as we will now see in the following examples.

EXAMPLE 4 Figure 2.5 shows how a population p of fruit flies (*Drosophila*) grew in a 50-day experiment. The number of flies was counted at regular intervals, the counted values plotted with respect to time t , and the points joined by a smooth curve (colored blue in Figure 2.5). Find the average growth rate from day 23 to day 45.

Solution There were 150 flies on day 23 and 340 flies on day 45. Thus the number of flies increased by $340 - 150 = 190$ in $45 - 23 = 22$ days. The average rate of change of the population from day 23 to day 45 was

$$\text{Average rate of change: } \frac{\Delta p}{\Delta t} = \frac{340 - 150}{45 - 23} = \frac{190}{22} \approx 8.6 \text{ flies/day.}$$

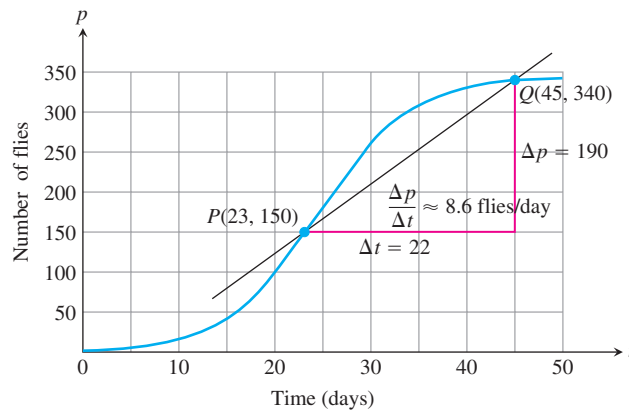


FIGURE 2.5 Growth of a fruit fly population in a controlled experiment. The average rate of change over 22 days is the slope $\Delta p/\Delta t$ of the secant line (Example 4).

This average is the slope of the secant through the points P and Q on the graph in Figure 2.5. ■

The average rate of change from day 23 to day 45 calculated in Example 4 does not tell us how fast the population was changing on day 23 itself. For that we need to examine time intervals closer to the day in question.

EXAMPLE 5 How fast was the number of flies in the population of Example 4 growing on day 23?

Solution To answer this question, we examine the average rates of change over increasingly short time intervals starting at day 23. In geometric terms, we find these rates by calculating the slopes of secants from P to Q , for a sequence of points Q approaching P along the curve (Figure 2.6).

Q	Slope of $PQ = \Delta p / \Delta t$ (flies/day)
(45, 340)	$\frac{340 - 150}{45 - 23} \approx 8.6$
(40, 330)	$\frac{330 - 150}{40 - 23} \approx 10.6$
(35, 310)	$\frac{310 - 150}{35 - 23} \approx 13.3$
(30, 265)	$\frac{265 - 150}{30 - 23} \approx 16.4$

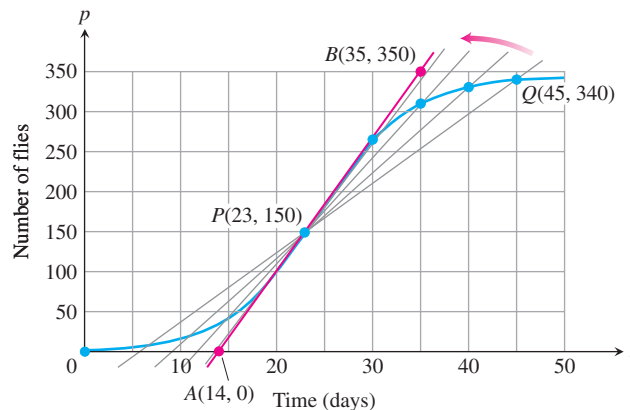


FIGURE 2.6 The positions and slopes of four secants through the point P on the fruit fly graph (Example 5).

The values in the table show that the secant slopes rise from 8.6 to 16.4 as the t -coordinate of Q decreases from 45 to 30, and we would expect the slopes to rise slightly higher as t continued on toward 23. Geometrically, the secants rotate about P and seem to approach the red tangent line in the figure. Since the line appears to pass through the points (14, 0) and (35, 350), it has slope

$$\frac{350 - 0}{35 - 14} = 16.7 \text{ flies/day (approximately).}$$

On day 23 the population was increasing at a rate of about 16.7 flies/day. ■

The instantaneous rates in Example 2 were found to be the values of the average speeds, or average rates of change, as the time interval of length h approached 0. That is, the instantaneous rate is the value the average rate approaches as the length h of the interval over which the change occurs approaches zero. The average rate of change corresponds to the slope of a secant line; the instantaneous rate corresponds to the slope of the tangent line as the independent variable approaches a fixed value. In Example 2, the independent variable t approached the values $t = 1$ and $t = 2$. In Example 3, the independent variable x approached the value $x = 2$. So we see that instantaneous rates and slopes of tangent lines are closely connected. We investigate this connection thoroughly in the next chapter, but to do so we need the concept of a *limit*.

Exercises 2.1

Average Rates of Change

In Exercises 1–6, find the average rate of change of the function over the given interval or intervals.

1. $f(x) = x^3 + 1$

a. $[2, 3]$

b. $[-1, 1]$

2. $g(x) = x^2$

a. $[-1, 1]$

b. $[-2, 0]$

3. $h(t) = \cot t$

a. $[\pi/4, 3\pi/4]$

b. $[\pi/6, \pi/2]$

4. $g(t) = 2 + \cos t$

a. $[0, \pi]$

b. $[-\pi, \pi]$

5. $R(\theta) = \sqrt{4\theta + 1}$; $[0, 2]$

6. $P(\theta) = \theta^3 - 4\theta^2 + 5\theta$; $[1, 2]$

Slope of a Curve at a Point

In Exercises 7–14, use the method in Example 3 to find (a) the slope of the curve at the given point P , and (b) an equation of the tangent line at P .

7. $y = x^2 - 3$, $P(2, 1)$

8. $y = 5 - x^2$, $P(1, 4)$

9. $y = x^2 - 2x - 3$, $P(2, -3)$

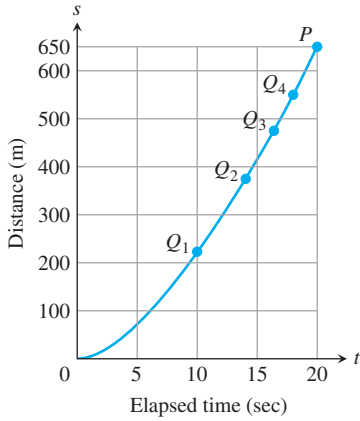
10. $y = x^2 - 4x$, $P(1, -3)$

11. $y = x^3$, $P(2, 8)$

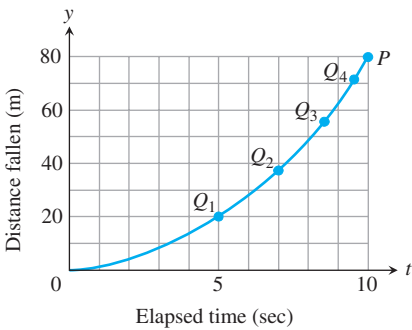
- 12. $y = 2 - x^3$, $P(1, 1)$
- 13. $y = x^3 - 12x$, $P(1, -11)$
- 14. $y = x^3 - 3x^2 + 4$, $P(2, 0)$

Instantaneous Rates of Change

15. **Speed of a car** The accompanying figure shows the time-to-distance graph for a sports car accelerating from a standstill.



- a. Estimate the slopes of secants PQ_1 , PQ_2 , PQ_3 , and PQ_4 , arranging them in order in a table like the one in Figure 2.6. What are the appropriate units for these slopes?
 - b. Then estimate the car's speed at time $t = 20$ sec.
16. The accompanying figure shows the plot of distance fallen versus time for an object that fell from the lunar landing module a distance 80 m to the surface of the moon.
- a. Estimate the slopes of the secants PQ_1 , PQ_2 , PQ_3 , and PQ_4 , arranging them in a table like the one in Figure 2.6.
 - b. About how fast was the object going when it hit the surface?



T 17. The profits of a small company for each of the first five years of its operation are given in the following table:

Year	Profit in \$1000s
2000	6
2001	27
2002	62
2003	111
2004	174

- a. Plot points representing the profit as a function of year, and join them by as smooth a curve as you can.

- b. What is the average rate of increase of the profits between 2002 and 2004?
- c. Use your graph to estimate the rate at which the profits were changing in 2002.

T 18. Make a table of values for the function $F(x) = (x + 2)/(x - 2)$ at the points $x = 1.2$, $x = 11/10$, $x = 101/100$, $x = 1001/1000$, $x = 10001/10000$, and $x = 1$.

- a. Find the average rate of change of $F(x)$ over the intervals $[1, x]$ for each $x \neq 1$ in your table.
- b. Extending the table if necessary, try to determine the rate of change of $F(x)$ at $x = 1$.

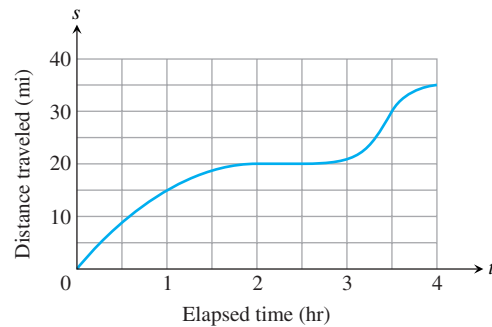
T 19. Let $g(x) = \sqrt{x}$ for $x \geq 0$.

- a. Find the average rate of change of $g(x)$ with respect to x over the intervals $[1, 2]$, $[1, 1.5]$ and $[1, 1 + h]$.
- b. Make a table of values of the average rate of change of g with respect to x over the interval $[1, 1 + h]$ for some values of h approaching zero, say $h = 0.1, 0.01, 0.001, 0.0001, 0.00001$, and 0.000001 .
- c. What does your table indicate is the rate of change of $g(x)$ with respect to x at $x = 1$?
- d. Calculate the limit as h approaches zero of the average rate of change of $g(x)$ with respect to x over the interval $[1, 1 + h]$.

T 20. Let $f(t) = 1/t$ for $t \neq 0$.

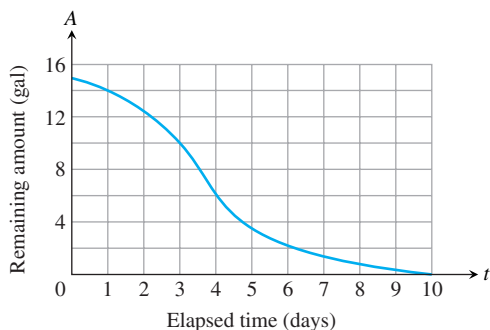
- a. Find the average rate of change of f with respect to t over the intervals (i) from $t = 2$ to $t = 3$, and (ii) from $t = 2$ to $t = T$.
- b. Make a table of values of the average rate of change of f with respect to t over the interval $[2, T]$, for some values of T approaching 2, say $T = 2.1, 2.01, 2.001, 2.0001, 2.00001$, and 2.000001 .
- c. What does your table indicate is the rate of change of f with respect to t at $t = 2$?
- d. Calculate the limit as T approaches 2 of the average rate of change of f with respect to t over the interval from 2 to T . You will have to do some algebra before you can substitute $T = 2$.

21. The accompanying graph shows the total distance s traveled by a bicyclist after t hours.



- a. Estimate the bicyclist's average speed over the time intervals $[0, 1]$, $[1, 2.5]$, and $[2.5, 3.5]$.
- b. Estimate the bicyclist's instantaneous speed at the times $t = \frac{1}{2}$, $t = 2$, and $t = 3$.
- c. Estimate the bicyclist's maximum speed and the specific time at which it occurs.

22. The accompanying graph shows the total amount of gasoline A in the gas tank of an automobile after being driven for t days.



- Estimate the average rate of gasoline consumption over the time intervals $[0, 3]$, $[0, 5]$, and $[7, 10]$.
- Estimate the instantaneous rate of gasoline consumption at the times $t = 1$, $t = 4$, and $t = 8$.
- Estimate the maximum rate of gasoline consumption and the specific time at which it occurs.

2.2 | Limit of a Function and Limit Laws

In Section 2.1 we saw that limits arise when finding the instantaneous rate of change of a function or the tangent to a curve. Here we begin with an informal definition of *limit* and show how we can calculate the values of limits. A precise definition is presented in the next section.

HISTORICAL ESSAY

Limits

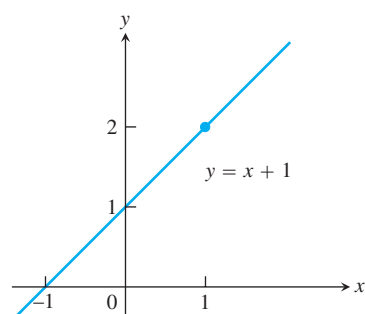
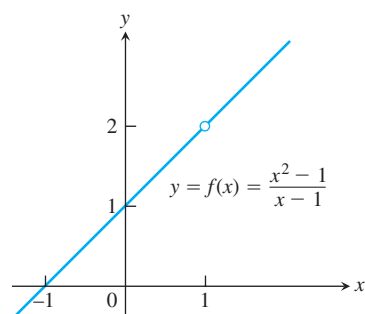


FIGURE 2.7 The graph of f is identical with the line $y = x + 1$ except at $x = 1$, where f is not defined (Example 1).

Limits of Function Values

Frequently when studying a function $y = f(x)$, we find ourselves interested in the function's behavior *near* a particular point x_0 , but *not at* x_0 . This might be the case, for instance, if x_0 is an irrational number, like π or $\sqrt{2}$, whose values can only be approximated by “close” rational numbers at which we actually evaluate the function instead. Another situation occurs when trying to evaluate a function at x_0 leads to division by zero, which is undefined. We encountered this last circumstance when seeking the instantaneous rate of change in y by considering the quotient function $\Delta y/h$ for h closer and closer to zero. Here's a specific example where we explore numerically how a function behaves near a particular point at which we cannot directly evaluate the function.

EXAMPLE 1 How does the function

$$f(x) = \frac{x^2 - 1}{x - 1}$$

behave near $x = 1$?

Solution The given formula defines f for all real numbers x except $x = 1$ (we cannot divide by zero). For any $x \neq 1$, we can simplify the formula by factoring the numerator and canceling common factors:

$$f(x) = \frac{(x - 1)(x + 1)}{x - 1} = x + 1 \quad \text{for } x \neq 1.$$

The graph of f is the line $y = x + 1$ with the point $(1, 2)$ *removed*. This removed point is shown as a “hole” in Figure 2.7. Even though $f(1)$ is not defined, it is clear that we can make the value of $f(x)$ *as close as we want* to 2 by choosing x *close enough* to 1 (Table 2.2).

TABLE 2.2 The closer x gets to 1, the closer $f(x) = (x^2 - 1)/(x - 1)$ seems to get to 2

Values of x below and above 1	$f(x) = \frac{x^2 - 1}{x - 1} = x + 1, \quad x \neq 1$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001
0.999999	1.999999
1.000001	2.000001

Let's generalize the idea illustrated in Example 1.

Suppose $f(x)$ is defined on an open interval about x_0 , *except possibly at x_0 itself*. If $f(x)$ is arbitrarily close to L (as close to L as we like) for all x sufficiently close to x_0 , we say that f approaches the **limit** L as x approaches x_0 , and write

$$\lim_{x \rightarrow x_0} f(x) = L,$$

which is read “the limit of $f(x)$ as x approaches x_0 is L .” For instance, in Example 1 we would say that $f(x)$ approaches the *limit* 2 as x approaches 1, and write

$$\lim_{x \rightarrow 1} f(x) = 2, \quad \text{or} \quad \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = 2.$$

Essentially, the definition says that the values of $f(x)$ are close to the number L whenever x is close to x_0 (on either side of x_0). This definition is “informal” because phrases like *arbitrarily close* and *sufficiently close* are imprecise; their meaning depends on the context. (To a machinist manufacturing a piston, *close* may mean *within a few thousandths of an inch*. To an astronomer studying distant galaxies, *close* may mean *within a few thousand light-years*.) Nevertheless, the definition is clear enough to enable us to recognize and evaluate limits of specific functions. We will need the precise definition of Section 2.3, however, when we set out to prove theorems about limits. Here are several more examples exploring the idea of limits.

EXAMPLE 2 This example illustrates that the limit value of a function does not depend on how the function is defined at the point being approached. Consider the three functions in Figure 2.8. The function f has limit 2 as $x \rightarrow 1$ even though f is not defined at $x = 1$.

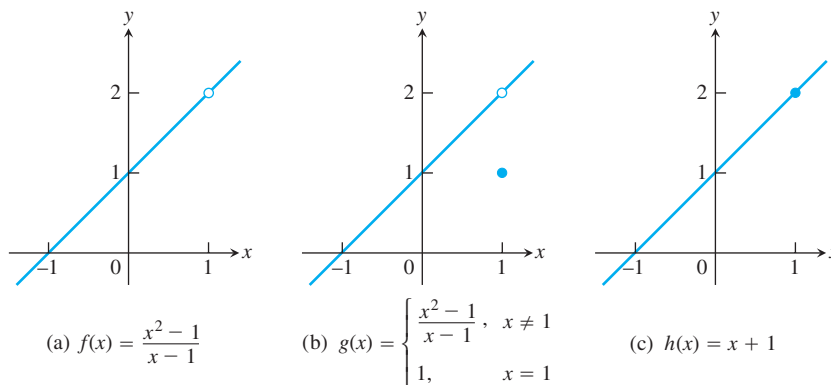
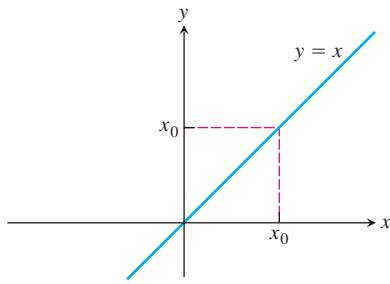
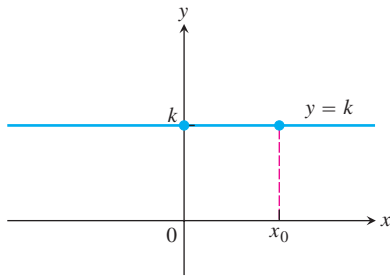


FIGURE 2.8 The limits of $f(x)$, $g(x)$, and $h(x)$ all equal 2 as x approaches 1. However, only $h(x)$ has the same function value as its limit at $x = 1$ (Example 2).



(a) Identity function



(b) Constant function

The function g has limit 2 as $x \rightarrow 1$ even though $2 \neq g(1)$. The function h is the only one of the three functions in Figure 2.8 whose limit as $x \rightarrow 1$ equals its value at $x = 1$. For h , we have $\lim_{x \rightarrow 1} h(x) = h(1)$. This equality of limit and function value is significant, and we return to it in Section 2.5. ■

EXAMPLE 3

(a) If f is the **identity function** $f(x) = x$, then for any value of x_0 (Figure 2.9a),

$$\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} x = x_0.$$

(b) If f is the **constant function** $f(x) = k$ (function with the constant value k), then for any value of x_0 (Figure 2.9b),

$$\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} k = k.$$

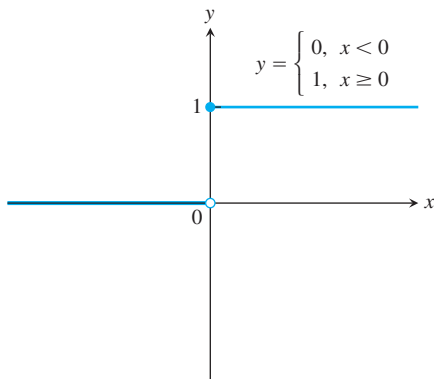
For instances of each of these rules we have

$$\lim_{x \rightarrow 3} x = 3 \quad \text{and} \quad \lim_{x \rightarrow -7} (4) = \lim_{x \rightarrow 2} (4) = 4.$$

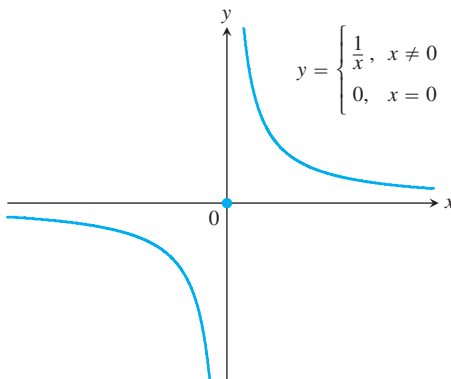
We prove these rules in Example 3 in Section 2.3. ■

FIGURE 2.9 The functions in Example 3 have limits at all points x_0 .

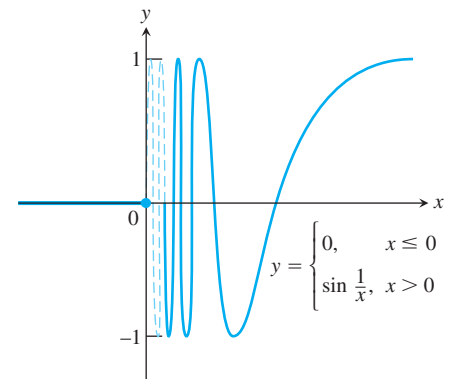
Some ways that limits can fail to exist are illustrated in Figure 2.10 and described in the next example.



(a) Unit step function $U(x)$



(b) $g(x)$



(c) $f(x)$

FIGURE 2.10 None of these functions has a limit as x approaches 0 (Example 4).

EXAMPLE 4 Discuss the behavior of the following functions as $x \rightarrow 0$.

(a) $U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$

(b) $g(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

(c) $f(x) = \begin{cases} 0, & x \leq 0 \\ \sin \frac{1}{x}, & x > 0 \end{cases}$

Solution

- (a) It *jumps*: The **unit step function** $U(x)$ has no limit as $x \rightarrow 0$ because its values jump at $x = 0$. For negative values of x arbitrarily close to zero, $U(x) = 0$. For positive values of x arbitrarily close to zero, $U(x) = 1$. There is no *single* value L approached by $U(x)$ as $x \rightarrow 0$ (Figure 2.10a).
- (b) It *grows too “large” to have a limit*: $g(x)$ has no limit as $x \rightarrow 0$ because the values of g grow arbitrarily large in absolute value as $x \rightarrow 0$ and do not stay close to *any* fixed real number (Figure 2.10b).
- (c) It *oscillates too much to have a limit*: $f(x)$ has no limit as $x \rightarrow 0$ because the function’s values oscillate between $+1$ and -1 in every open interval containing 0 . The values do not stay close to any one number as $x \rightarrow 0$ (Figure 2.10c). ■

The Limit Laws

When discussing limits, sometimes we use the notation $x \rightarrow x_0$ if we want to emphasize the point x_0 that is being approached in the limit process (usually to enhance the clarity of a particular discussion or example). Other times, such as in the statements of the following theorem, we use the simpler notation $x \rightarrow c$ or $x \rightarrow a$ which avoids the subscript in x_0 . In every case, the symbols x_0 , c , and a refer to a single point on the x -axis that may or may not belong to the domain of the function involved. To calculate limits of functions that are arithmetic combinations of functions having known limits, we can use several easy rules.

THEOREM 1—Limit Laws If L , M , c , and k are real numbers and

$$\lim_{x \rightarrow c} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow c} g(x) = M, \quad \text{then}$$

- | | |
|------------------------------------|---|
| 1. <i>Sum Rule</i> : | $\lim_{x \rightarrow c} (f(x) + g(x)) = L + M$ |
| 2. <i>Difference Rule</i> : | $\lim_{x \rightarrow c} (f(x) - g(x)) = L - M$ |
| 3. <i>Constant Multiple Rule</i> : | $\lim_{x \rightarrow c} (k \cdot f(x)) = k \cdot L$ |
| 4. <i>Product Rule</i> : | $\lim_{x \rightarrow c} (f(x) \cdot g(x)) = L \cdot M$ |
| 5. <i>Quotient Rule</i> : | $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$ |
| 6. <i>Power Rule</i> : | $\lim_{x \rightarrow c} [f(x)]^n = L^n, \quad n \text{ a positive integer}$ |
| 7. <i>Root Rule</i> : | $\lim_{x \rightarrow c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}, \quad n \text{ a positive integer}$ |

(If n is even, we assume that $\lim_{x \rightarrow c} f(x) = L > 0$.)

In words, the Sum Rule says that the limit of a sum is the sum of the limits. Similarly, the next rules say that the limit of a difference is the difference of the limits; the limit of a constant times a function is the constant times the limit of the function; the limit of a product is the product of the limits; the limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0); the limit of a positive integer power (or root) of a function is the integer power (or root) of the limit (provided that the root of the limit is a real number).

It is reasonable that the properties in Theorem 1 are true (although these intuitive arguments do not constitute proofs). If x is sufficiently close to c , then $f(x)$ is close to L and $g(x)$ is close to M , from our informal definition of a limit. It is then reasonable that $f(x) + g(x)$ is close to $L + M$; $f(x) - g(x)$ is close to $L - M$; $kf(x)$ is close to kL ; $f(x)g(x)$ is close to LM ; and $f(x)/g(x)$ is close to L/M if M is not zero. We prove the Sum Rule in Section 2.3, based on a precise definition of limit. Rules 2–5 are proved in

Appendix 4. Rule 6 is obtained by applying Rule 4 repeatedly. Rule 7 is proved in more advanced texts. The sum, difference, and product rules can be extended to any number of functions, not just two.

EXAMPLE 5 Use the observations $\lim_{x \rightarrow c} k = k$ and $\lim_{x \rightarrow c} x = c$ (Example 3) and the properties of limits to find the following limits.

$$(a) \lim_{x \rightarrow c} (x^3 + 4x^2 - 3) \quad (b) \lim_{x \rightarrow c} \frac{x^4 + x^2 - 1}{x^2 + 5} \quad (c) \lim_{x \rightarrow -2} \sqrt{4x^2 - 3}$$

Solution

$$(a) \lim_{x \rightarrow c} (x^3 + 4x^2 - 3) = \lim_{x \rightarrow c} x^3 + \lim_{x \rightarrow c} 4x^2 - \lim_{x \rightarrow c} 3 \quad \text{Sum and Difference Rules}$$

$$= c^3 + 4c^2 - 3 \quad \text{Power and Multiple Rules}$$

$$(b) \lim_{x \rightarrow c} \frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{\lim_{x \rightarrow c} (x^4 + x^2 - 1)}{\lim_{x \rightarrow c} (x^2 + 5)} \quad \text{Quotient Rule}$$

$$= \frac{\lim_{x \rightarrow c} x^4 + \lim_{x \rightarrow c} x^2 - \lim_{x \rightarrow c} 1}{\lim_{x \rightarrow c} x^2 + \lim_{x \rightarrow c} 5} \quad \text{Sum and Difference Rules}$$

$$= \frac{c^4 + c^2 - 1}{c^2 + 5} \quad \text{Power or Product Rule}$$

$$(c) \lim_{x \rightarrow -2} \sqrt{4x^2 - 3} = \sqrt{\lim_{x \rightarrow -2} (4x^2 - 3)} \quad \text{Root Rule with } n = 2$$

$$= \sqrt{\lim_{x \rightarrow -2} 4x^2 - \lim_{x \rightarrow -2} 3} \quad \text{Difference Rule}$$

$$= \sqrt{4(-2)^2 - 3} \quad \text{Product and Multiple Rules}$$

$$= \sqrt{16 - 3}$$

$$= \sqrt{13}$$

Two consequences of Theorem 1 further simplify the task of calculating limits of polynomials and rational functions. To evaluate the limit of a polynomial function as x approaches c , merely substitute c for x in the formula for the function. To evaluate the limit of a rational function as x approaches a point c at which the denominator is not zero, substitute c for x in the formula for the function. (See Examples 5a and 5b.) We state these results formally as theorems.

THEOREM 2—Limits of Polynomials

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, then

$$\lim_{x \rightarrow c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0.$$

THEOREM 3—Limits of Rational Functions

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \rightarrow c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

Identifying Common Factors

It can be shown that if $Q(x)$ is a polynomial and $Q(c) = 0$, then $(x - c)$ is a factor of $Q(x)$. Thus, if the numerator and denominator of a rational function of x are both zero at $x = c$, they have $(x - c)$ as a common factor.

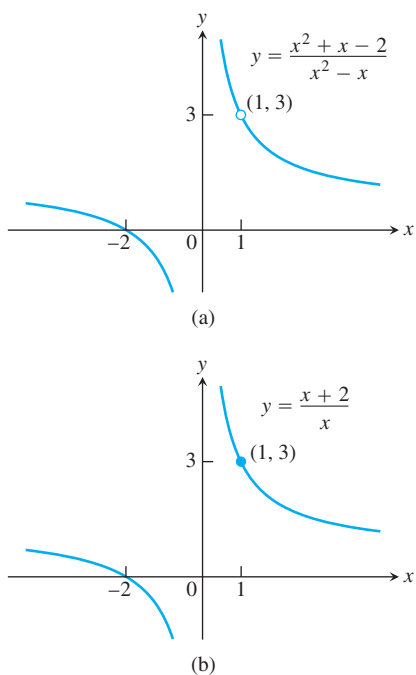


FIGURE 2.11 The graph of $f(x) = (x^2 + x - 2)/(x^2 - x)$ in part (a) is the same as the graph of $g(x) = (x + 2)/x$ in part (b) except at $x = 1$, where f is undefined. The functions have the same limit as $x \rightarrow 1$ (Example 7).

EXAMPLE 6 The following calculation illustrates Theorems 2 and 3:

$$\lim_{x \rightarrow -1} \frac{x^3 + 4x^2 - 3}{x^2 + 5} = \frac{(-1)^3 + 4(-1)^2 - 3}{(-1)^2 + 5} = \frac{0}{6} = 0$$

Eliminating Zero Denominators Algebraically

Theorem 3 applies only if the denominator of the rational function is not zero at the limit point c . If the denominator is zero, canceling common factors in the numerator and denominator may reduce the fraction to one whose denominator is no longer zero at c . If this happens, we can find the limit by substitution in the simplified fraction.

EXAMPLE 7 Evaluate

$$\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - x}.$$

Solution We cannot substitute $x = 1$ because it makes the denominator zero. We test the numerator to see if it, too, is zero at $x = 1$. It is, so it has a factor of $(x - 1)$ in common with the denominator. Canceling the $(x - 1)$'s gives a simpler fraction with the same values as the original for $x \neq 1$:

$$\frac{x^2 + x - 2}{x^2 - x} = \frac{(x - 1)(x + 2)}{x(x - 1)} = \frac{x + 2}{x}, \quad \text{if } x \neq 1.$$

Using the simpler fraction, we find the limit of these values as $x \rightarrow 1$ by substitution:

$$\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x \rightarrow 1} \frac{x + 2}{x} = \frac{1 + 2}{1} = 3.$$

See Figure 2.11. ■

Using Calculators and Computers to Estimate Limits

When we cannot use the Quotient Rule in Theorem 1 because the limit of the denominator is zero, we can try using a calculator or computer to guess the limit numerically as x gets closer and closer to c . We used this approach in Example 1, but calculators and computers can sometimes give false values and misleading impressions for functions that are undefined at a point or fail to have a limit there, as we now illustrate.

EXAMPLE 8 Estimate the value of $\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 100} - 10}{x^2}$.

Solution Table 2.3 lists values of the function for several values near $x = 0$. As x approaches 0 through the values ± 1 , ± 0.5 , ± 0.10 , and ± 0.01 , the function seems to approach the number 0.05.

As we take even smaller values of x , ± 0.0005 , ± 0.0001 , ± 0.00001 , and ± 0.000001 , the function appears to approach the value 0.

Is the answer 0.05 or 0, or some other value? We resolve this question in the next example. ■

TABLE 2.3 Computer values of $f(x) = \frac{\sqrt{x^2 + 100} - 10}{x^2}$ near $x = 0$

x	$f(x)$
± 1	0.049876
± 0.5	0.049969
± 0.1	0.049999
± 0.01	0.050000
± 0.0005	0.050000
± 0.0001	0.000000
± 0.00001	0.000000
± 0.000001	0.000000

Using a computer or calculator may give ambiguous results, as in the last example. We cannot substitute $x = 0$ in the problem, and the numerator and denominator have no obvious common factors (as they did in Example 7). Sometimes, however, we can create a common factor algebraically.

EXAMPLE 9 Evaluate

$$\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 100} - 10}{x^2}.$$

Solution This is the limit we considered in Example 8. We can create a common factor by multiplying both numerator and denominator by the conjugate radical expression $\sqrt{x^2 + 100} + 10$ (obtained by changing the sign after the square root). The preliminary algebra rationalizes the numerator:

$$\begin{aligned} \frac{\sqrt{x^2 + 100} - 10}{x^2} &= \frac{\sqrt{x^2 + 100} - 10}{x^2} \cdot \frac{\sqrt{x^2 + 100} + 10}{\sqrt{x^2 + 100} + 10} \\ &= \frac{x^2 + 100 - 100}{x^2(\sqrt{x^2 + 100} + 10)} \\ &= \frac{x^2}{x^2(\sqrt{x^2 + 100} + 10)} && \text{Common factor } x^2 \\ &= \frac{1}{\sqrt{x^2 + 100} + 10} && \text{Cancel } x^2 \text{ for } x \neq 0 \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 100} - 10}{x^2} &= \lim_{x \rightarrow 0} \frac{1}{\sqrt{x^2 + 100} + 10} \\ &= \frac{1}{\sqrt{0^2 + 100} + 10} && \text{Denominator not 0 at } x = 0; \text{ substitute} \\ &= \frac{1}{20} = 0.05. \end{aligned}$$

This calculation provides the correct answer, in contrast to the ambiguous computer results in Example 8. ■

We cannot always algebraically resolve the problem of finding the limit of a quotient where the denominator becomes zero. In some cases the limit might then be found with the

9. If $\lim_{x \rightarrow 1} f(x) = 5$, must f be defined at $x = 1$? If it is, must $f(1) = 5$? Can we conclude *anything* about the values of f at $x = 1$? Explain.
10. If $f(1) = 5$, must $\lim_{x \rightarrow 1} f(x)$ exist? If it does, then must $\lim_{x \rightarrow 1} f(x) = 5$? Can we conclude *anything* about $\lim_{x \rightarrow 1} f(x)$? Explain.

Calculating Limits

Find the limits in Exercises 11–22.

11. $\lim_{x \rightarrow -7} (2x + 5)$ 12. $\lim_{x \rightarrow 2} (-x^2 + 5x - 2)$
13. $\lim_{t \rightarrow 6} 8(t - 5)(t - 7)$ 14. $\lim_{x \rightarrow -2} (x^3 - 2x^2 + 4x + 8)$
15. $\lim_{x \rightarrow 2} \frac{x + 3}{x + 6}$ 16. $\lim_{s \rightarrow 2/3} 3s(2s - 1)$
17. $\lim_{x \rightarrow -1} 3(2x - 1)^2$ 18. $\lim_{y \rightarrow 2} \frac{y + 2}{y^2 + 5y + 6}$
19. $\lim_{y \rightarrow -3} (5 - y)^{4/3}$ 20. $\lim_{z \rightarrow 0} (2z - 8)^{1/3}$
21. $\lim_{h \rightarrow 0} \frac{3}{\sqrt{3h + 1} + 1}$ 22. $\lim_{h \rightarrow 0} \frac{\sqrt{5h + 4} - 2}{h}$

Limits of quotients Find the limits in Exercises 23–42.

23. $\lim_{x \rightarrow 5} \frac{x - 5}{x^2 - 25}$ 24. $\lim_{x \rightarrow -3} \frac{x + 3}{x^2 + 4x + 3}$
25. $\lim_{x \rightarrow -5} \frac{x^2 + 3x - 10}{x + 5}$ 26. $\lim_{x \rightarrow 2} \frac{x^2 - 7x + 10}{x - 2}$
27. $\lim_{t \rightarrow 1} \frac{t^2 + t - 2}{t^2 - 1}$ 28. $\lim_{t \rightarrow -1} \frac{t^2 + 3t + 2}{t^2 - t - 2}$
29. $\lim_{x \rightarrow -2} \frac{-2x - 4}{x^3 + 2x^2}$ 30. $\lim_{y \rightarrow 0} \frac{5y^3 + 8y^2}{3y^4 - 16y^2}$
31. $\lim_{x \rightarrow 1} \frac{\frac{1}{x} - 1}{x - 1}$ 32. $\lim_{x \rightarrow 0} \frac{\frac{1}{x-1} + \frac{1}{x+1}}{x}$
33. $\lim_{u \rightarrow 1} \frac{u^4 - 1}{u^3 - 1}$ 34. $\lim_{v \rightarrow 2} \frac{v^3 - 8}{v^4 - 16}$
35. $\lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x - 9}$ 36. $\lim_{x \rightarrow 4} \frac{4x - x^2}{2 - \sqrt{x}}$
37. $\lim_{x \rightarrow 1} \frac{x - 1}{\sqrt{x + 3} - 2}$ 38. $\lim_{x \rightarrow -1} \frac{\sqrt{x^2 + 8} - 3}{x + 1}$
39. $\lim_{x \rightarrow 2} \frac{\sqrt{x^2 + 12} - 4}{x - 2}$ 40. $\lim_{x \rightarrow -2} \frac{x + 2}{\sqrt{x^2 + 5} - 3}$
41. $\lim_{x \rightarrow -3} \frac{2 - \sqrt{x^2 - 5}}{x + 3}$ 42. $\lim_{x \rightarrow 4} \frac{4 - x}{5 - \sqrt{x^2 + 9}}$

Limits with trigonometric functions Find the limits in Exercises 43–50.

43. $\lim_{x \rightarrow 0} (2 \sin x - 1)$ 44. $\lim_{x \rightarrow 0} \sin^2 x$
45. $\lim_{x \rightarrow 0} \sec x$ 46. $\lim_{x \rightarrow 0} \tan x$
47. $\lim_{x \rightarrow 0} \frac{1 + x + \sin x}{3 \cos x}$ 48. $\lim_{x \rightarrow 0} (x^2 - 1)(2 - \cos x)$
49. $\lim_{x \rightarrow -\pi} \sqrt{x + 4} \cos(x + \pi)$ 50. $\lim_{x \rightarrow 0} \sqrt{7 + \sec^2 x}$

Using Limit Rules

51. Suppose $\lim_{x \rightarrow 0} f(x) = 1$ and $\lim_{x \rightarrow 0} g(x) = -5$. Name the rules in Theorem 1 that are used to accomplish steps (a), (b), and (c) of the following calculation.

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{2f(x) - g(x)}{(f(x) + 7)^{2/3}} &= \frac{\lim_{x \rightarrow 0} (2f(x) - g(x))}{\lim_{x \rightarrow 0} (f(x) + 7)^{2/3}} & \text{(a)} \\ &= \frac{\lim_{x \rightarrow 0} 2f(x) - \lim_{x \rightarrow 0} g(x)}{\left(\lim_{x \rightarrow 0} (f(x) + 7)\right)^{2/3}} & \text{(b)} \\ &= \frac{2 \lim_{x \rightarrow 0} f(x) - \lim_{x \rightarrow 0} g(x)}{\left(\lim_{x \rightarrow 0} f(x) + \lim_{x \rightarrow 0} 7\right)^{2/3}} & \text{(c)} \\ &= \frac{(2)(1) - (-5)}{(1 + 7)^{2/3}} = \frac{7}{4} \end{aligned}$$

52. Let $\lim_{x \rightarrow 1} h(x) = 5$, $\lim_{x \rightarrow 1} p(x) = 1$, and $\lim_{x \rightarrow 1} r(x) = 2$. Name the rules in Theorem 1 that are used to accomplish steps (a), (b), and (c) of the following calculation.

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{\sqrt{5h(x)}}{p(x)(4 - r(x))} &= \frac{\lim_{x \rightarrow 1} \sqrt{5h(x)}}{\lim_{x \rightarrow 1} (p(x)(4 - r(x)))} & \text{(a)} \\ &= \frac{\sqrt{\lim_{x \rightarrow 1} 5h(x)}}{\left(\lim_{x \rightarrow 1} p(x)\right)\left(\lim_{x \rightarrow 1} (4 - r(x))\right)} & \text{(b)} \\ &= \frac{\sqrt{5 \lim_{x \rightarrow 1} h(x)}}{\left(\lim_{x \rightarrow 1} p(x)\right)\left(\lim_{x \rightarrow 1} 4 - \lim_{x \rightarrow 1} r(x)\right)} & \text{(c)} \\ &= \frac{\sqrt{(5)(5)}}{(1)(4 - 2)} = \frac{5}{2} \end{aligned}$$

53. Suppose $\lim_{x \rightarrow c} f(x) = 5$ and $\lim_{x \rightarrow c} g(x) = -2$. Find

a. $\lim_{x \rightarrow c} f(x)g(x)$ b. $\lim_{x \rightarrow c} 2f(x)g(x)$

c. $\lim_{x \rightarrow c} (f(x) + 3g(x))$ d. $\lim_{x \rightarrow c} \frac{f(x)}{f(x) - g(x)}$

54. Suppose $\lim_{x \rightarrow 4} f(x) = 0$ and $\lim_{x \rightarrow 4} g(x) = -3$. Find

a. $\lim_{x \rightarrow 4} (g(x) + 3)$ b. $\lim_{x \rightarrow 4} xf(x)$

c. $\lim_{x \rightarrow 4} (g(x))^2$ d. $\lim_{x \rightarrow 4} \frac{g(x)}{f(x) - 1}$

55. Suppose $\lim_{x \rightarrow b} f(x) = 7$ and $\lim_{x \rightarrow b} g(x) = -3$. Find

a. $\lim_{x \rightarrow b} (f(x) + g(x))$ b. $\lim_{x \rightarrow b} f(x) \cdot g(x)$

c. $\lim_{x \rightarrow b} 4g(x)$ d. $\lim_{x \rightarrow b} f(x)/g(x)$

56. Suppose that $\lim_{x \rightarrow -2} p(x) = 4$, $\lim_{x \rightarrow -2} r(x) = 0$, and $\lim_{x \rightarrow -2} s(x) = -3$. Find

a. $\lim_{x \rightarrow -2} (p(x) + r(x) + s(x))$

b. $\lim_{x \rightarrow -2} p(x) \cdot r(x) \cdot s(x)$

c. $\lim_{x \rightarrow -2} (-4p(x) + 5r(x))/s(x)$

Limits of Average Rates of Change

Because of their connection with secant lines, tangents, and instantaneous rates, limits of the form

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

occur frequently in calculus. In Exercises 57–62, evaluate this limit for the given value of x and function f .

57. $f(x) = x^2$, $x = 1$ 58. $f(x) = x^2$, $x = -2$
 59. $f(x) = 3x - 4$, $x = 2$ 60. $f(x) = 1/x$, $x = -2$
 61. $f(x) = \sqrt{x}$, $x = 7$ 62. $f(x) = \sqrt{3x+1}$, $x = 0$

Using the Sandwich Theorem

63. If $\sqrt{5-2x^2} \leq f(x) \leq \sqrt{5-x^2}$ for $-1 \leq x \leq 1$, find $\lim_{x \rightarrow 0} f(x)$.

64. If $2 - x^2 \leq g(x) \leq 2 \cos x$ for all x , find $\lim_{x \rightarrow 0} g(x)$.

65. a. It can be shown that the inequalities

$$1 - \frac{x^2}{6} < \frac{x \sin x}{2 - 2 \cos x} < 1$$

hold for all values of x close to zero. What, if anything, does this tell you about

$$\lim_{x \rightarrow 0} \frac{x \sin x}{2 - 2 \cos x}?$$

Give reasons for your answer.

T b. Graph $y = 1 - (x^2/6)$, $y = (x \sin x)/(2 - 2 \cos x)$, and $y = 1$ together for $-2 \leq x \leq 2$. Comment on the behavior of the graphs as $x \rightarrow 0$.

66. a. Suppose that the inequalities

$$\frac{1}{2} - \frac{x^2}{24} < \frac{1 - \cos x}{x^2} < \frac{1}{2}$$

hold for values of x close to zero. (They do, as you will see in Section 10.9.) What, if anything, does this tell you about

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}?$$

Give reasons for your answer.

T b. Graph the equations $y = (1/2) - (x^2/24)$, $y = (1 - \cos x)/x^2$, and $y = 1/2$ together for $-2 \leq x \leq 2$. Comment on the behavior of the graphs as $x \rightarrow 0$.

Estimating Limits

T You will find a graphing calculator useful for Exercises 67–76.

67. Let $f(x) = (x^2 - 9)/(x + 3)$.

a. Make a table of the values of f at the points $x = -3.1, -3.01, -3.001$, and so on as far as your calculator can go. Then estimate $\lim_{x \rightarrow -3} f(x)$. What estimate do you arrive at if you evaluate f at $x = -2.9, -2.99, -2.999, \dots$ instead?

b. Support your conclusions in part (a) by graphing f near $x_0 = -3$ and using Zoom and Trace to estimate y -values on the graph as $x \rightarrow -3$.

c. Find $\lim_{x \rightarrow -3} f(x)$ algebraically, as in Example 7.

68. Let $g(x) = (x^2 - 2)/(x - \sqrt{2})$.

a. Make a table of the values of g at the points $x = 1.4, 1.41, 1.414$, and so on through successive decimal approximations of $\sqrt{2}$. Estimate $\lim_{x \rightarrow \sqrt{2}} g(x)$.

b. Support your conclusion in part (a) by graphing g near $x_0 = \sqrt{2}$ and using Zoom and Trace to estimate y -values on the graph as $x \rightarrow \sqrt{2}$.

c. Find $\lim_{x \rightarrow \sqrt{2}} g(x)$ algebraically.

69. Let $G(x) = (x + 6)/(x^2 + 4x - 12)$.

a. Make a table of the values of G at $x = -5.9, -5.99, -5.999$, and so on. Then estimate $\lim_{x \rightarrow -6} G(x)$. What estimate do you arrive at if you evaluate G at $x = -6.1, -6.01, -6.001, \dots$ instead?

b. Support your conclusions in part (a) by graphing G and using Zoom and Trace to estimate y -values on the graph as $x \rightarrow -6$.

c. Find $\lim_{x \rightarrow -6} G(x)$ algebraically.

70. Let $h(x) = (x^2 - 2x - 3)/(x^2 - 4x + 3)$.

a. Make a table of the values of h at $x = 2.9, 2.99, 2.999$, and so on. Then estimate $\lim_{x \rightarrow 3} h(x)$. What estimate do you arrive at if you evaluate h at $x = 3.1, 3.01, 3.001, \dots$ instead?

b. Support your conclusions in part (a) by graphing h near $x_0 = 3$ and using Zoom and Trace to estimate y -values on the graph as $x \rightarrow 3$.

c. Find $\lim_{x \rightarrow 3} h(x)$ algebraically.

71. Let $f(x) = (x^2 - 1)/(|x| - 1)$.

a. Make tables of the values of f at values of x that approach $x_0 = -1$ from above and below. Then estimate $\lim_{x \rightarrow -1} f(x)$.

b. Support your conclusion in part (a) by graphing f near $x_0 = -1$ and using Zoom and Trace to estimate y -values on the graph as $x \rightarrow -1$.

c. Find $\lim_{x \rightarrow -1} f(x)$ algebraically.

72. Let $F(x) = (x^2 + 3x + 2)/(2 - |x|)$.

a. Make tables of values of F at values of x that approach $x_0 = -2$ from above and below. Then estimate $\lim_{x \rightarrow -2} F(x)$.

b. Support your conclusion in part (a) by graphing F near $x_0 = -2$ and using Zoom and Trace to estimate y -values on the graph as $x \rightarrow -2$.

c. Find $\lim_{x \rightarrow -2} F(x)$ algebraically.

73. Let $g(\theta) = (\sin \theta)/\theta$.

a. Make a table of the values of g at values of θ that approach $\theta_0 = 0$ from above and below. Then estimate $\lim_{\theta \rightarrow 0} g(\theta)$.

b. Support your conclusion in part (a) by graphing g near $\theta_0 = 0$.

74. Let $G(t) = (1 - \cos t)/t^2$.

a. Make tables of values of G at values of t that approach $t_0 = 0$ from above and below. Then estimate $\lim_{t \rightarrow 0} G(t)$.

b. Support your conclusion in part (a) by graphing G near $t_0 = 0$.

75. Let $f(x) = x^{1/(1-x)}$.

a. Make tables of values of f at values of x that approach $x_0 = 1$ from above and below. Does f appear to have a limit as $x \rightarrow 1$? If so, what is it? If not, why not?

b. Support your conclusions in part (a) by graphing f near $x_0 = 1$.

76. Let $f(x) = (3^x - 1)/x$.
- Make tables of values of f at values of x that approach $x_0 = 0$ from above and below. Does f appear to have a limit as $x \rightarrow 0$? If so, what is it? If not, why not?
 - Support your conclusions in part (a) by graphing f near $x_0 = 0$.

Theory and Examples

77. If $x^4 \leq f(x) \leq x^2$ for x in $[-1, 1]$ and $x^2 \leq f(x) \leq x^4$ for $x < -1$ and $x > 1$, at what points c do you automatically know $\lim_{x \rightarrow c} f(x)$? What can you say about the value of the limit at these points?

78. Suppose that $g(x) \leq f(x) \leq h(x)$ for all $x \neq 2$ and suppose that $\lim_{x \rightarrow 2} g(x) = \lim_{x \rightarrow 2} h(x) = -5$.

Can we conclude anything about the values of f , g , and h at $x = 2$? Could $f(2) = 0$? Could $\lim_{x \rightarrow 2} f(x) = 0$? Give reasons for your answers.

79. If $\lim_{x \rightarrow 4} \frac{f(x) - 5}{x - 2} = 1$, find $\lim_{x \rightarrow 4} f(x)$.
80. If $\lim_{x \rightarrow -2} \frac{f(x)}{x^2} = 1$, find
- $\lim_{x \rightarrow -2} f(x)$
 - $\lim_{x \rightarrow -2} \frac{f(x)}{x}$
81. a. If $\lim_{x \rightarrow 2} \frac{f(x) - 5}{x - 2} = 3$, find $\lim_{x \rightarrow 2} f(x)$.

b. If $\lim_{x \rightarrow 2} \frac{f(x) - 5}{x - 2} = 4$, find $\lim_{x \rightarrow 2} f(x)$.

82. If $\lim_{x \rightarrow 0} \frac{f(x)}{x^2} = 1$, find

- $\lim_{x \rightarrow 0} f(x)$
- $\lim_{x \rightarrow 0} \frac{f(x)}{x}$

T 83. a. Graph $g(x) = x \sin(1/x)$ to estimate $\lim_{x \rightarrow 0} g(x)$, zooming in on the origin as necessary.

b. Confirm your estimate in part (a) with a proof.

T 84. a. Graph $h(x) = x^2 \cos(1/x^3)$ to estimate $\lim_{x \rightarrow 0} h(x)$, zooming in on the origin as necessary.

b. Confirm your estimate in part (a) with a proof.

COMPUTER EXPLORATIONS

Graphical Estimates of Limits

In Exercises 85–90, use a CAS to perform the following steps:

- Plot the function near the point x_0 being approached.
- From your plot guess the value of the limit.

85. $\lim_{x \rightarrow 2} \frac{x^4 - 16}{x - 2}$

86. $\lim_{x \rightarrow -1} \frac{x^3 - x^2 - 5x - 3}{(x + 1)^2}$

87. $\lim_{x \rightarrow 0} \frac{\sqrt[3]{1 + x} - 1}{x}$

88. $\lim_{x \rightarrow 3} \frac{x^2 - 9}{\sqrt{x^2 + 7} - 4}$

89. $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x \sin x}$

90. $\lim_{x \rightarrow 0} \frac{2x^2}{3 - 3 \cos x}$

2.3 The Precise Definition of a Limit

We now turn our attention to the precise definition of a limit. We replace vague phrases like “gets arbitrarily close to” in the informal definition with specific conditions that can be applied to any particular example. With a precise definition, we can prove the limit properties given in the preceding section and establish many important limits.

To show that the limit of $f(x)$ as $x \rightarrow x_0$ equals the number L , we need to show that the gap between $f(x)$ and L can be made “as small as we choose” if x is kept “close enough” to x_0 . Let us see what this would require if we specified the size of the gap between $f(x)$ and L .

EXAMPLE 1 Consider the function $y = 2x - 1$ near $x_0 = 4$. Intuitively it appears that y is close to 7 when x is close to 4, so $\lim_{x \rightarrow 4} (2x - 1) = 7$. However, how close to $x_0 = 4$ does x have to be so that $y = 2x - 1$ differs from 7 by, say, less than 2 units?

Solution We are asked: For what values of x is $|y - 7| < 2$? To find the answer we first express $|y - 7|$ in terms of x :

$$|y - 7| = |(2x - 1) - 7| = |2x - 8|.$$

The question then becomes: what values of x satisfy the inequality $|2x - 8| < 2$? To find out, we solve the inequality:

$$\begin{aligned} |2x - 8| &< 2 \\ -2 &< 2x - 8 < 2 \\ 6 &< 2x < 10 \\ 3 &< x < 5 \\ -1 &< x - 4 < 1. \end{aligned}$$

Keeping x within 1 unit of $x_0 = 4$ will keep y within 2 units of $y_0 = 7$ (Figure 2.15). ■

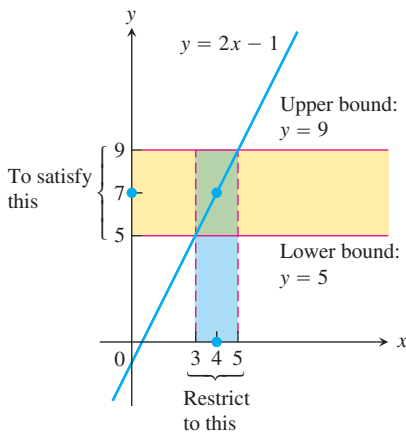


FIGURE 2.15 Keeping x within 1 unit of $x_0 = 4$ will keep y within 2 units of $y_0 = 7$ (Example 1).

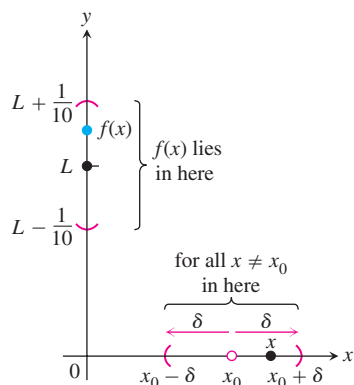


FIGURE 2.16 How should we define $\delta > 0$ so that keeping x within the interval $(x_0 - \delta, x_0 + \delta)$ will keep $f(x)$ within the interval $(L - \frac{1}{10}, L + \frac{1}{10})$?

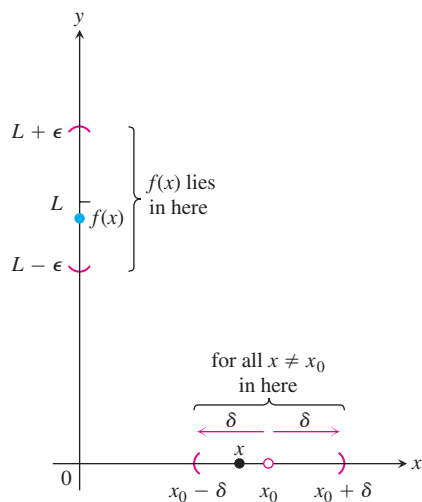


FIGURE 2.17 The relation of δ and ϵ in the definition of limit.

In the previous example we determined how close x must be to a particular value x_0 to ensure that the outputs $f(x)$ of some function lie within a prescribed interval about a limit value L . To show that the limit of $f(x)$ as $x \rightarrow x_0$ actually equals L , we must be able to show that the gap between $f(x)$ and L can be made less than *any prescribed error*, no matter how small, by holding x close enough to x_0 .

Definition of Limit

Suppose we are watching the values of a function $f(x)$ as x approaches x_0 (without taking on the value of x_0 itself). Certainly we want to be able to say that $f(x)$ stays within one-tenth of a unit from L as soon as x stays within some distance δ of x_0 (Figure 2.16). But that in itself is not enough, because as x continues on its course toward x_0 , what is to prevent $f(x)$ from jittering about within the interval from $L - (1/10)$ to $L + (1/10)$ without tending toward L ?

We can be told that the error can be no more than $1/100$ or $1/1000$ or $1/100,000$. Each time, we find a new δ -interval about x_0 so that keeping x within that interval satisfies the new error tolerance. And each time the possibility exists that $f(x)$ jitters away from L at some stage.

The figures on the next page illustrate the problem. You can think of this as a quarrel between a skeptic and a scholar. The skeptic presents ϵ -challenges to prove that the limit does not exist or, more precisely, that there is room for doubt. The scholar answers every challenge with a δ -interval around x_0 that keeps the function values within ϵ of L .

How do we stop this seemingly endless series of challenges and responses? By proving that for every error tolerance ϵ that the challenger can produce, we can find, calculate, or conjure a matching distance δ that keeps x “close enough” to x_0 to keep $f(x)$ within that tolerance of L (Figure 2.17). This leads us to the precise definition of a limit.

DEFINITION Let $f(x)$ be defined on an open interval about x_0 , except possibly at x_0 itself. We say that the **limit of $f(x)$ as x approaches x_0 is the number L** , and write

$$\lim_{x \rightarrow x_0} f(x) = L,$$

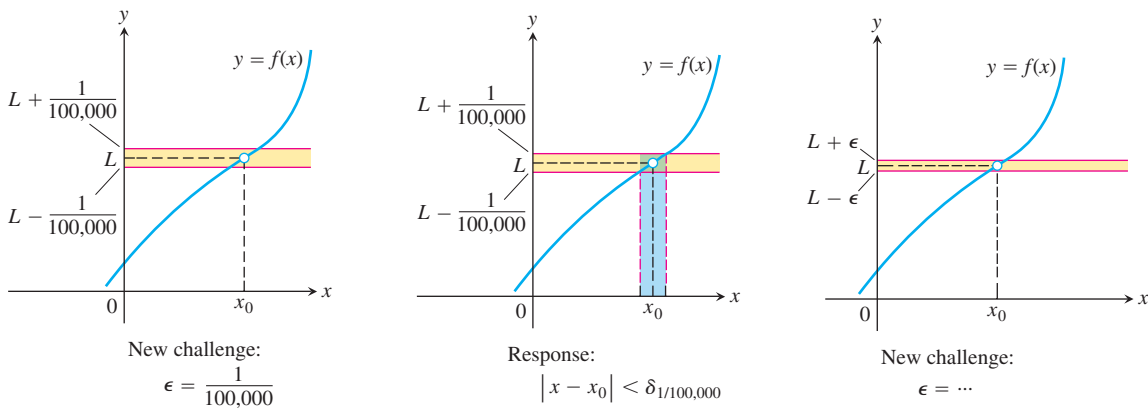
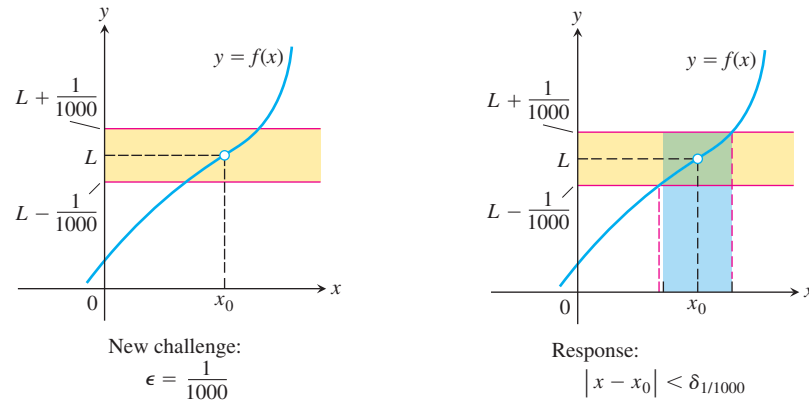
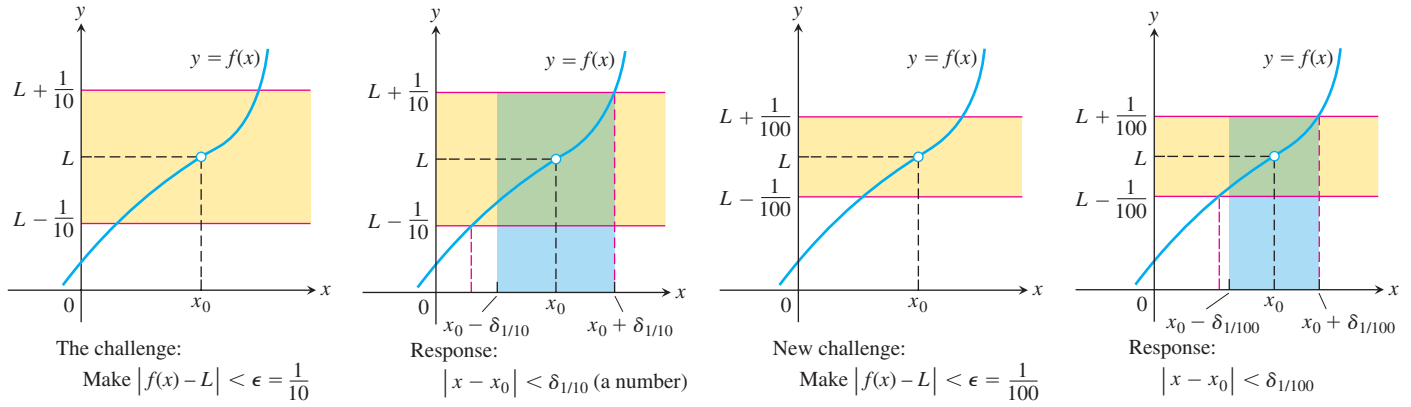
if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x ,

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

One way to think about the definition is to suppose we are machining a generator shaft to a close tolerance. We may try for diameter L , but since nothing is perfect, we must be satisfied with a diameter $f(x)$ somewhere between $L - \epsilon$ and $L + \epsilon$. The δ is the measure of how accurate our control setting for x must be to guarantee this degree of accuracy in the diameter of the shaft. Notice that as the tolerance for error becomes stricter, we may have to adjust δ . That is, the value of δ , how tight our control setting must be, depends on the value of ϵ , the error tolerance.

Examples: Testing the Definition

The formal definition of limit does not tell how to find the limit of a function, but it enables us to verify that a suspected limit is correct. The following examples show how the definition can be used to verify limit statements for specific functions. However, the real purpose of the definition is not to do calculations like this, but rather to prove general theorems so that the calculation of specific limits can be simplified.



EXAMPLE 2 Show that

$$\lim_{x \rightarrow 1} (5x - 3) = 2.$$

Solution Set $x_0 = 1$, $f(x) = 5x - 3$, and $L = 2$ in the definition of limit. For any given $\epsilon > 0$, we have to find a suitable $\delta > 0$ so that if $x \neq 1$ and x is within distance δ of $x_0 = 1$, that is, whenever

$$0 < |x - 1| < \delta,$$

it is true that $f(x)$ is within distance ϵ of $L = 2$, so

$$|f(x) - 2| < \epsilon.$$

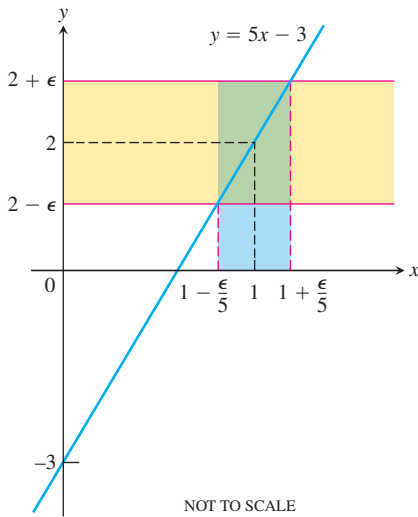


FIGURE 2.18 If $f(x) = 5x - 3$, then $0 < |x - 1| < \epsilon/5$ guarantees that $|f(x) - 2| < \epsilon$ (Example 2).

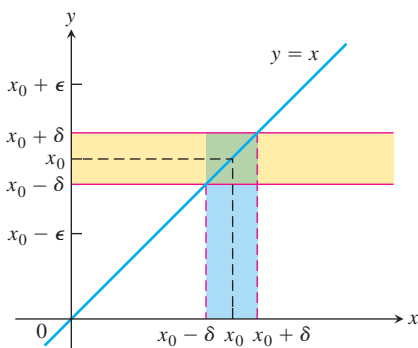


FIGURE 2.19 For the function $f(x) = x$, we find that $0 < |x - x_0| < \delta$ will guarantee $|f(x) - x_0| < \epsilon$ whenever $\delta \leq \epsilon$ (Example 3a).

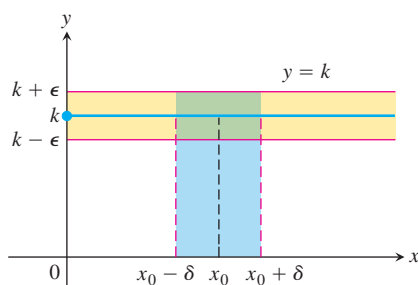


FIGURE 2.20 For the function $f(x) = k$, we find that $|f(x) - k| < \epsilon$ for any positive δ (Example 3b).

We find δ by working backward from the ϵ -inequality:

$$\begin{aligned} |(5x - 3) - 2| &= |5x - 5| < \epsilon \\ 5|x - 1| &< \epsilon \\ |x - 1| &< \epsilon/5. \end{aligned}$$

Thus, we can take $\delta = \epsilon/5$ (Figure 2.18). If $0 < |x - 1| < \delta = \epsilon/5$, then

$$|(5x - 3) - 2| = |5x - 5| = 5|x - 1| < 5(\epsilon/5) = \epsilon,$$

which proves that $\lim_{x \rightarrow 1}(5x - 3) = 2$.

The value of $\delta = \epsilon/5$ is not the only value that will make $0 < |x - 1| < \delta$ imply $|5x - 5| < \epsilon$. Any smaller positive δ will do as well. The definition does not ask for a “best” positive δ , just one that will work. ■

EXAMPLE 3 Prove the following results presented graphically in Section 2.2.

- (a) $\lim_{x \rightarrow x_0} x = x_0$ (b) $\lim_{x \rightarrow x_0} k = k$ (k constant)

Solution

- (a) Let $\epsilon > 0$ be given. We must find $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \quad \text{implies} \quad |x - x_0| < \epsilon.$$

The implication will hold if δ equals ϵ or any smaller positive number (Figure 2.19). This proves that $\lim_{x \rightarrow x_0} x = x_0$.

- (b) Let $\epsilon > 0$ be given. We must find $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \quad \text{implies} \quad |k - k| < \epsilon.$$

Since $k - k = 0$, we can use any positive number for δ and the implication will hold (Figure 2.20). This proves that $\lim_{x \rightarrow x_0} k = k$. ■

Finding Deltas Algebraically for Given Epsilons

In Examples 2 and 3, the interval of values about x_0 for which $|f(x) - L|$ was less than ϵ was symmetric about x_0 and we could take δ to be half the length of that interval. When such symmetry is absent, as it usually is, we can take δ to be the distance from x_0 to the interval's *nearer* endpoint.

EXAMPLE 4 For the limit $\lim_{x \rightarrow 5} \sqrt{x - 1} = 2$, find a $\delta > 0$ that works for $\epsilon = 1$. That is, find a $\delta > 0$ such that for all x

$$0 < |x - 5| < \delta \quad \Rightarrow \quad |\sqrt{x - 1} - 2| < 1.$$

Solution We organize the search into two steps, as discussed below.

1. Solve the inequality $|\sqrt{x - 1} - 2| < 1$ to find an interval containing $x_0 = 5$ on which the inequality holds for all $x \neq x_0$.

$$\begin{aligned} |\sqrt{x - 1} - 2| &< 1 \\ -1 &< \sqrt{x - 1} - 2 < 1 \\ 1 &< \sqrt{x - 1} < 3 \\ 1 &< x - 1 < 9 \\ 2 &< x < 10 \end{aligned}$$

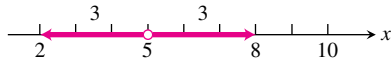


FIGURE 2.21 An open interval of radius 3 about $x_0 = 5$ will lie inside the open interval $(2, 10)$.

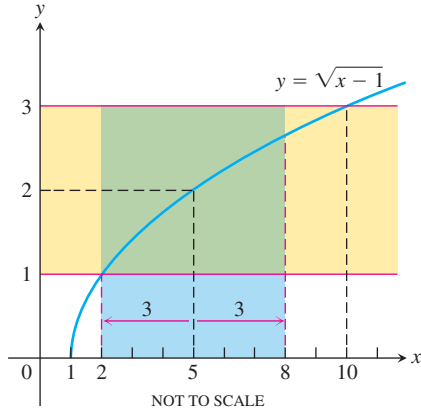


FIGURE 2.22 The function and intervals in Example 4.

The inequality holds for all x in the open interval $(2, 10)$, so it holds for all $x \neq 5$ in this interval as well.

- Find a value of $\delta > 0$ to place the centered interval $5 - \delta < x < 5 + \delta$ (centered at $x_0 = 5$) inside the interval $(2, 10)$. The distance from 5 to the nearer endpoint of $(2, 10)$ is 3 (Figure 2.21). If we take $\delta = 3$ or any smaller positive number, then the inequality $0 < |x - 5| < \delta$ will automatically place x between 2 and 10 to make $|\sqrt{x - 1} - 2| < 1$ (Figure 2.22):

$$0 < |x - 5| < 3 \quad \Rightarrow \quad |\sqrt{x - 1} - 2| < 1. \quad \blacksquare$$

How to Find Algebraically a δ for a Given f, L, x_0 , and $\epsilon > 0$

The process of finding a $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon$$

can be accomplished in two steps.

- Solve the inequality $|f(x) - L| < \epsilon$ to find an open interval (a, b) containing x_0 on which the inequality holds for all $x \neq x_0$.
- Find a value of $\delta > 0$ that places the open interval $(x_0 - \delta, x_0 + \delta)$ centered at x_0 inside the interval (a, b) . The inequality $|f(x) - L| < \epsilon$ will hold for all $x \neq x_0$ in this δ -interval.

EXAMPLE 5 Prove that $\lim_{x \rightarrow 2} f(x) = 4$ if

$$f(x) = \begin{cases} x^2, & x \neq 2 \\ 1, & x = 2. \end{cases}$$

Solution Our task is to show that given $\epsilon > 0$ there exists a $\delta > 0$ such that for all x

$$0 < |x - 2| < \delta \quad \Rightarrow \quad |f(x) - 4| < \epsilon.$$

- Solve the inequality $|f(x) - 4| < \epsilon$ to find an open interval containing $x_0 = 2$ on which the inequality holds for all $x \neq x_0$.

For $x \neq x_0 = 2$, we have $f(x) = x^2$, and the inequality to solve is $|x^2 - 4| < \epsilon$:

$$\begin{aligned} |x^2 - 4| &< \epsilon \\ -\epsilon &< x^2 - 4 < \epsilon \\ 4 - \epsilon &< x^2 < 4 + \epsilon \\ \sqrt{4 - \epsilon} &< |x| < \sqrt{4 + \epsilon} \\ \sqrt{4 - \epsilon} &< x < \sqrt{4 + \epsilon}. \end{aligned}$$

Assumes $\epsilon < 4$; see below.
An open interval about $x_0 = 2$ that solves the inequality

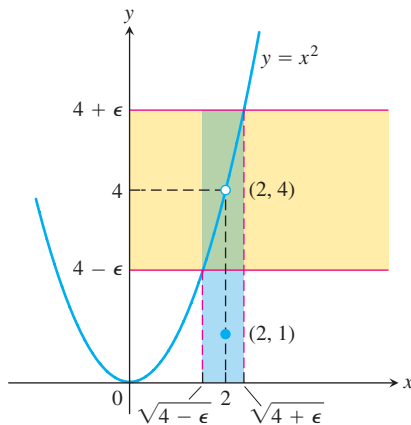


FIGURE 2.23 An interval containing $x = 2$ so that the function in Example 5 satisfies $|f(x) - 4| < \epsilon$.

The inequality $|f(x) - 4| < \epsilon$ holds for all $x \neq 2$ in the open interval $(\sqrt{4 - \epsilon}, \sqrt{4 + \epsilon})$ (Figure 2.23).

- Find a value of $\delta > 0$ that places the centered interval $(2 - \delta, 2 + \delta)$ inside the interval $(\sqrt{4 - \epsilon}, \sqrt{4 + \epsilon})$.

Take δ to be the distance from $x_0 = 2$ to the nearer endpoint of $(\sqrt{4 - \epsilon}, \sqrt{4 + \epsilon})$. In other words, take $\delta = \min \{2 - \sqrt{4 - \epsilon}, \sqrt{4 + \epsilon} - 2\}$, the *minimum* (the

smaller) of the two numbers $2 - \sqrt{4 - \epsilon}$ and $\sqrt{4 + \epsilon} - 2$. If δ has this or any smaller positive value, the inequality $0 < |x - 2| < \delta$ will automatically place x between $\sqrt{4 - \epsilon}$ and $\sqrt{4 + \epsilon}$ to make $|f(x) - 4| < \epsilon$. For all x ,

$$0 < |x - 2| < \delta \quad \Rightarrow \quad |f(x) - 4| < \epsilon.$$

This completes the proof for $\epsilon < 4$.

If $\epsilon \geq 4$, then we take δ to be the distance from $x_0 = 2$ to the nearer endpoint of the interval $(0, \sqrt{4 + \epsilon})$. In other words, take $\delta = \min\{2, \sqrt{4 + \epsilon} - 2\}$. (See Figure 2.23.) ■

Using the Definition to Prove Theorems

We do not usually rely on the formal definition of limit to verify specific limits such as those in the preceding examples. Rather we appeal to general theorems about limits, in particular the theorems of Section 2.2. The definition is used to prove these theorems (Appendix 4). As an example, we prove part 1 of Theorem 1, the Sum Rule.

EXAMPLE 6 Given that $\lim_{x \rightarrow c} f(x) = L$ and $\lim_{x \rightarrow c} g(x) = M$, prove that

$$\lim_{x \rightarrow c} (f(x) + g(x)) = L + M.$$

Solution Let $\epsilon > 0$ be given. We want to find a positive number δ such that for all x

$$0 < |x - c| < \delta \quad \Rightarrow \quad |f(x) + g(x) - (L + M)| < \epsilon.$$

Regrouping terms, we get

$$\begin{aligned} |f(x) + g(x) - (L + M)| &= |(f(x) - L) + (g(x) - M)| && \text{Triangle Inequality:} \\ &\leq |f(x) - L| + |g(x) - M|. && |a + b| \leq |a| + |b| \end{aligned}$$

Since $\lim_{x \rightarrow c} f(x) = L$, there exists a number $\delta_1 > 0$ such that for all x

$$0 < |x - c| < \delta_1 \quad \Rightarrow \quad |f(x) - L| < \epsilon/2.$$

Similarly, since $\lim_{x \rightarrow c} g(x) = M$, there exists a number $\delta_2 > 0$ such that for all x

$$0 < |x - c| < \delta_2 \quad \Rightarrow \quad |g(x) - M| < \epsilon/2.$$

Let $\delta = \min\{\delta_1, \delta_2\}$, the smaller of δ_1 and δ_2 . If $0 < |x - c| < \delta$ then $|x - c| < \delta_1$, so $|f(x) - L| < \epsilon/2$, and $|x - c| < \delta_2$, so $|g(x) - M| < \epsilon/2$. Therefore

$$|f(x) + g(x) - (L + M)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This shows that $\lim_{x \rightarrow c} (f(x) + g(x)) = L + M$. ■

Next we prove Theorem 5 of Section 2.2.

EXAMPLE 7 Given that $\lim_{x \rightarrow c} f(x) = L$ and $\lim_{x \rightarrow c} g(x) = M$, and that $f(x) \leq g(x)$ for all x in an open interval containing c (except possibly c itself), prove that $L \leq M$.

Solution We use the method of proof by contradiction. Suppose, on the contrary, that $L > M$. Then by the limit of a difference property in Theorem 1,

$$\lim_{x \rightarrow c} (g(x) - f(x)) = M - L.$$

Therefore, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|(g(x) - f(x)) - (M - L)| < \epsilon \quad \text{whenever } 0 < |x - c| < \delta.$$

Since $L - M > 0$ by hypothesis, we take $\epsilon = L - M$ in particular and we have a number $\delta > 0$ such that

$$|(g(x) - f(x)) - (M - L)| < L - M \quad \text{whenever } 0 < |x - c| < \delta.$$

Since $a \leq |a|$ for any number a , we have

$$(g(x) - f(x)) - (M - L) < L - M \quad \text{whenever } 0 < |x - c| < \delta$$

which simplifies to

$$g(x) < f(x) \quad \text{whenever } 0 < |x - c| < \delta.$$

But this contradicts $f(x) \leq g(x)$. Thus the inequality $L > M$ must be false. Therefore $L \leq M$. ■

Exercises 2.3

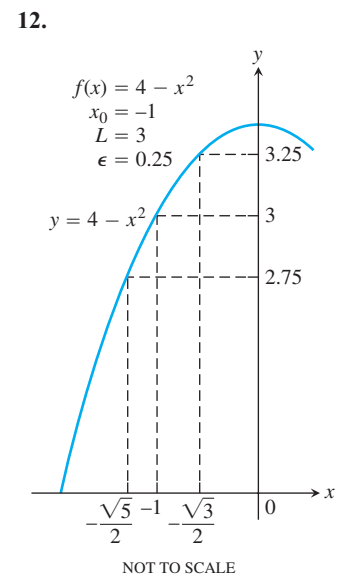
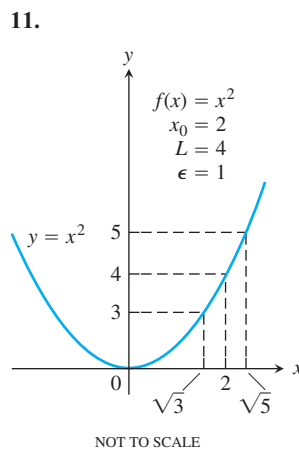
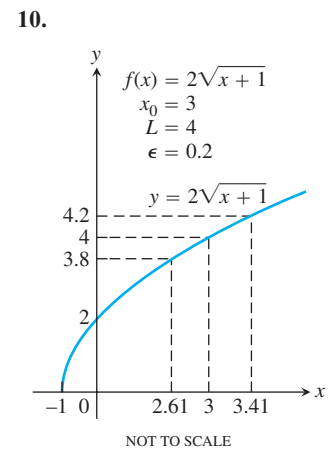
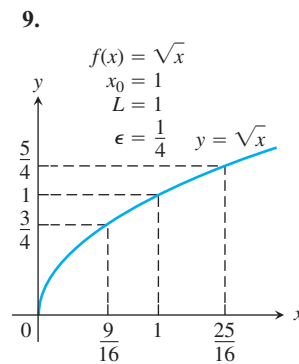
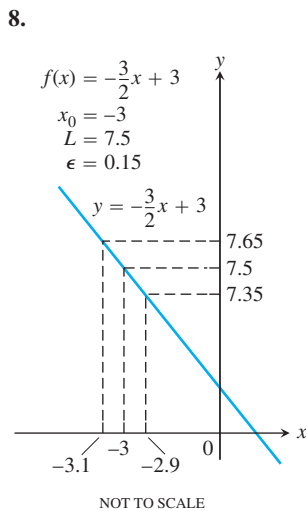
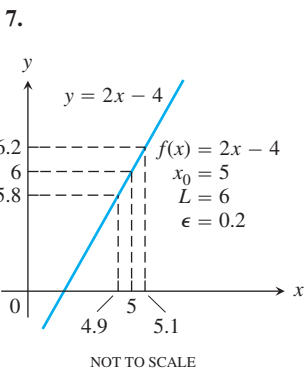
Centering Intervals About a Point

In Exercises 1–6, sketch the interval (a, b) on the x -axis with the point x_0 inside. Then find a value of $\delta > 0$ such that for all x , $0 < |x - x_0| < \delta \Rightarrow a < x < b$.

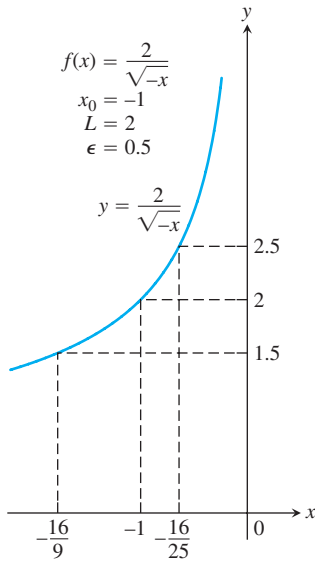
1. $a = 1, b = 7, x_0 = 5$
2. $a = 1, b = 7, x_0 = 2$
3. $a = -7/2, b = -1/2, x_0 = -3$
4. $a = -7/2, b = -1/2, x_0 = -3/2$
5. $a = 4/9, b = 4/7, x_0 = 1/2$
6. $a = 2.7591, b = 3.2391, x_0 = 3$

Finding Deltas Graphically

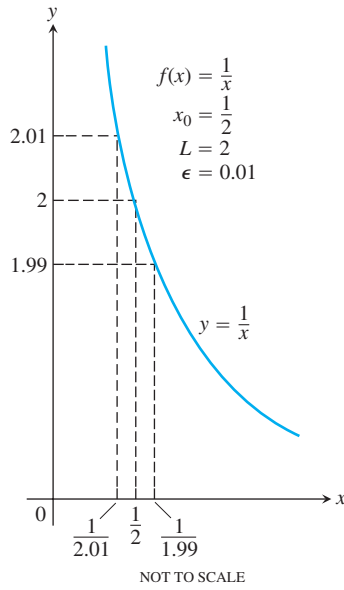
In Exercises 7–14, use the graphs to find a $\delta > 0$ such that for all x , $0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon$.



13.



14.



Finding Deltas Algebraically

Each of Exercises 15–30 gives a function $f(x)$ and numbers L, x_0 , and $\epsilon > 0$. In each case, find an open interval about x_0 on which the inequality $|f(x) - L| < \epsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - x_0| < \delta$ the inequality $|f(x) - L| < \epsilon$ holds.

- 15. $f(x) = x + 1, \quad L = 5, \quad x_0 = 4, \quad \epsilon = 0.01$
- 16. $f(x) = 2x - 2, \quad L = -6, \quad x_0 = -2, \quad \epsilon = 0.02$
- 17. $f(x) = \sqrt{x + 1}, \quad L = 1, \quad x_0 = 0, \quad \epsilon = 0.1$
- 18. $f(x) = \sqrt{x}, \quad L = 1/2, \quad x_0 = 1/4, \quad \epsilon = 0.1$
- 19. $f(x) = \sqrt{19 - x}, \quad L = 3, \quad x_0 = 10, \quad \epsilon = 1$
- 20. $f(x) = \sqrt{x - 7}, \quad L = 4, \quad x_0 = 23, \quad \epsilon = 1$
- 21. $f(x) = 1/x, \quad L = 1/4, \quad x_0 = 4, \quad \epsilon = 0.05$
- 22. $f(x) = x^2, \quad L = 3, \quad x_0 = \sqrt{3}, \quad \epsilon = 0.1$
- 23. $f(x) = x^2, \quad L = 4, \quad x_0 = -2, \quad \epsilon = 0.5$
- 24. $f(x) = 1/x, \quad L = -1, \quad x_0 = -1, \quad \epsilon = 0.1$
- 25. $f(x) = x^2 - 5, \quad L = 11, \quad x_0 = 4, \quad \epsilon = 1$
- 26. $f(x) = 120/x, \quad L = 5, \quad x_0 = 24, \quad \epsilon = 1$
- 27. $f(x) = mx, \quad m > 0, \quad L = 2m, \quad x_0 = 2, \quad \epsilon = 0.03$
- 28. $f(x) = mx, \quad m > 0, \quad L = 3m, \quad x_0 = 3, \quad \epsilon = c > 0$
- 29. $f(x) = mx + b, \quad m > 0, \quad L = (m/2) + b, \quad x_0 = 1/2, \quad \epsilon = c > 0$
- 30. $f(x) = mx + b, \quad m > 0, \quad L = m + b, \quad x_0 = 1, \quad \epsilon = 0.05$

Using the Formal Definition

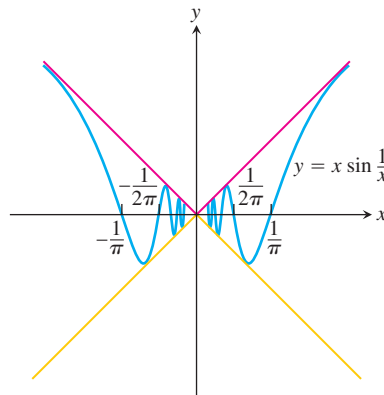
Each of Exercises 31–36 gives a function $f(x)$, a point x_0 , and a positive number ϵ . Find $L = \lim_{x \rightarrow x_0} f(x)$. Then find a number $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \implies |f(x) - L| < \epsilon.$$

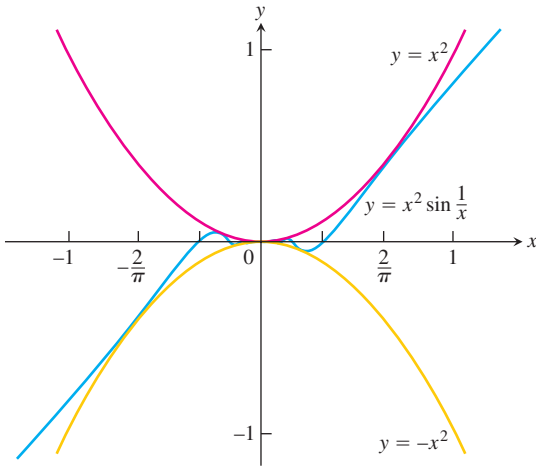
- 31. $f(x) = 3 - 2x, \quad x_0 = 3, \quad \epsilon = 0.02$
- 32. $f(x) = -3x - 2, \quad x_0 = -1, \quad \epsilon = 0.03$
- 33. $f(x) = \frac{x^2 - 4}{x - 2}, \quad x_0 = 2, \quad \epsilon = 0.05$
- 34. $f(x) = \frac{x^2 + 6x + 5}{x + 5}, \quad x_0 = -5, \quad \epsilon = 0.05$
- 35. $f(x) = \sqrt{1 - 5x}, \quad x_0 = -3, \quad \epsilon = 0.5$
- 36. $f(x) = 4/x, \quad x_0 = 2, \quad \epsilon = 0.4$

Prove the limit statements in Exercises 37–50.

- 37. $\lim_{x \rightarrow 4} (9 - x) = 5$
- 38. $\lim_{x \rightarrow 3} (3x - 7) = 2$
- 39. $\lim_{x \rightarrow 9} \sqrt{x - 5} = 2$
- 40. $\lim_{x \rightarrow 0} \sqrt{4 - x} = 2$
- 41. $\lim_{x \rightarrow 1} f(x) = 1$ if $f(x) = \begin{cases} x^2, & x \neq 1 \\ 2, & x = 1 \end{cases}$
- 42. $\lim_{x \rightarrow -2} f(x) = 4$ if $f(x) = \begin{cases} x^2, & x \neq -2 \\ 1, & x = -2 \end{cases}$
- 43. $\lim_{x \rightarrow 1} \frac{1}{x} = 1$
- 44. $\lim_{x \rightarrow \sqrt{3}} \frac{1}{x^2} = \frac{1}{3}$
- 45. $\lim_{x \rightarrow -3} \frac{x^2 - 9}{x + 3} = -6$
- 46. $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = 2$
- 47. $\lim_{x \rightarrow 1} f(x) = 2$ if $f(x) = \begin{cases} 4 - 2x, & x < 1 \\ 6x - 4, & x \geq 1 \end{cases}$
- 48. $\lim_{x \rightarrow 0} f(x) = 0$ if $f(x) = \begin{cases} 2x, & x < 0 \\ x/2, & x \geq 0 \end{cases}$
- 49. $\lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0$



50. $\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x} = 0$



Theory and Examples

- 51. Define what it means to say that $\lim_{x \rightarrow 0} g(x) = k$.
- 52. Prove that $\lim_{x \rightarrow c} f(x) = L$ if and only if $\lim_{h \rightarrow 0} f(h + c) = L$.
- 53. **A wrong statement about limits** Show by example that the following statement is wrong.

The number L is the limit of $f(x)$ as x approaches x_0 if $f(x)$ gets closer to L as x approaches x_0 .

Explain why the function in your example does not have the given value of L as a limit as $x \rightarrow x_0$.

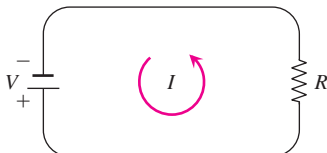
- 54. **Another wrong statement about limits** Show by example that the following statement is wrong.

The number L is the limit of $f(x)$ as x approaches x_0 if, given any $\epsilon > 0$, there exists a value of x for which $|f(x) - L| < \epsilon$.

Explain why the function in your example does not have the given value of L as a limit as $x \rightarrow x_0$.

T 55. Grinding engine cylinders Before contracting to grind engine cylinders to a cross-sectional area of 9 in^2 , you need to know how much deviation from the ideal cylinder diameter of $x_0 = 3.385 \text{ in.}$ you can allow and still have the area come within 0.01 in^2 of the required 9 in^2 . To find out, you let $A = \pi(x/2)^2$ and look for the interval in which you must hold x to make $|A - 9| \leq 0.01$. What interval do you find?

- 56. **Manufacturing electrical resistors** Ohm's law for electrical circuits like the one shown in the accompanying figure states that $V = RI$. In this equation, V is a constant voltage, I is the current in amperes, and R is the resistance in ohms. Your firm has been asked to supply the resistors for a circuit in which V will be 120 volts and I is to be 5 ± 0.1 amp. In what interval does R have to lie for I to be within 0.1 amp of the value $I_0 = 5$?



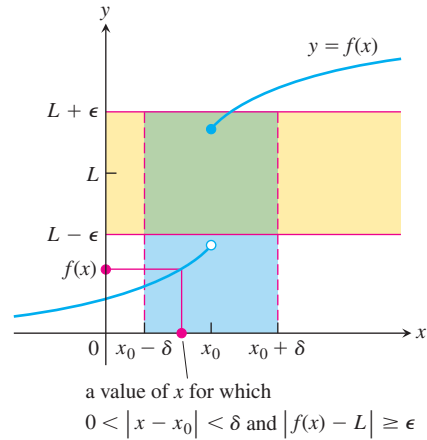
When Is a Number L Not the Limit of $f(x)$ as $x \rightarrow x_0$?

Showing L is not a limit We can prove that $\lim_{x \rightarrow x_0} f(x) \neq L$ by providing an $\epsilon > 0$ such that no possible $\delta > 0$ satisfies the condition

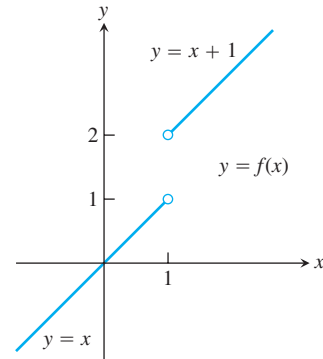
$$\text{for all } x, \quad 0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

We accomplish this for our candidate ϵ by showing that for each $\delta > 0$ there exists a value of x such that

$$0 < |x - x_0| < \delta \quad \text{and} \quad |f(x) - L| \geq \epsilon.$$



57. Let $f(x) = \begin{cases} x, & x < 1 \\ x + 1, & x > 1 \end{cases}$



- a. Let $\epsilon = 1/2$. Show that no possible $\delta > 0$ satisfies the following condition:

$$\text{For all } x, \quad 0 < |x - 1| < \delta \quad \Rightarrow \quad |f(x) - 2| < 1/2.$$

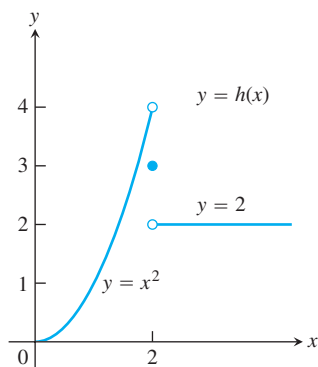
That is, for each $\delta > 0$ show that there is a value of x such that

$$0 < |x - 1| < \delta \quad \text{and} \quad |f(x) - 2| \geq 1/2.$$

This will show that $\lim_{x \rightarrow 1} f(x) \neq 2$.

- b. Show that $\lim_{x \rightarrow 1} f(x) \neq 1$.
- c. Show that $\lim_{x \rightarrow 1} f(x) \neq 1.5$.

58. Let $h(x) = \begin{cases} x^2, & x < 2 \\ 3, & x = 2 \\ 2, & x > 2. \end{cases}$

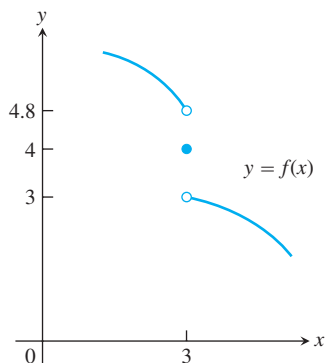


Show that

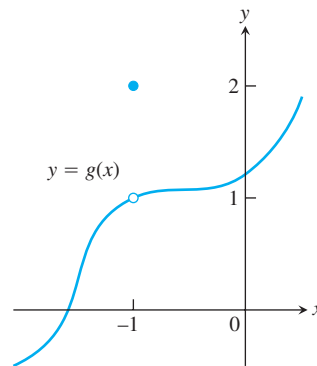
- $\lim_{x \rightarrow 2} h(x) \neq 4$
- $\lim_{x \rightarrow 2} h(x) \neq 3$
- $\lim_{x \rightarrow 2} h(x) \neq 2$

59. For the function graphed here, explain why

- $\lim_{x \rightarrow 3} f(x) \neq 4$
- $\lim_{x \rightarrow 3} f(x) \neq 4.8$
- $\lim_{x \rightarrow 3} f(x) \neq 3$



- For the function graphed here, show that $\lim_{x \rightarrow -1} g(x) \neq 2$.
- Does $\lim_{x \rightarrow -1} g(x)$ appear to exist? If so, what is the value of the limit? If not, why not?



COMPUTER EXPLORATIONS

In Exercises 61–66, you will further explore finding deltas graphically. Use a CAS to perform the following steps:

- Plot the function $y = f(x)$ near the point x_0 being approached.
- Guess the value of the limit L and then evaluate the limit symbolically to see if you guessed correctly.
- Using the value $\epsilon = 0.2$, graph the banding lines $y_1 = L - \epsilon$ and $y_2 = L + \epsilon$ together with the function f near x_0 .
- From your graph in part (c), estimate a $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

Test your estimate by plotting f , y_1 , and y_2 over the interval $0 < |x - x_0| < \delta$. For your viewing window use $x_0 - 2\delta \leq x \leq x_0 + 2\delta$ and $L - 2\epsilon \leq y \leq L + 2\epsilon$. If any function values lie outside the interval $[L - \epsilon, L + \epsilon]$, your choice of δ was too large. Try again with a smaller estimate.

- Repeat parts (c) and (d) successively for $\epsilon = 0.1, 0.05$, and 0.001 .

61. $f(x) = \frac{x^4 - 81}{x - 3}, \quad x_0 = 3$

62. $f(x) = \frac{5x^3 + 9x^2}{2x^5 + 3x^2}, \quad x_0 = 0$

63. $f(x) = \frac{\sin 2x}{3x}, \quad x_0 = 0$

64. $f(x) = \frac{x(1 - \cos x)}{x - \sin x}, \quad x_0 = 0$

65. $f(x) = \frac{\sqrt[3]{x} - 1}{x - 1}, \quad x_0 = 1$

66. $f(x) = \frac{3x^2 - (7x + 1)\sqrt{x} + 5}{x - 1}, \quad x_0 = 1$

2.4

One-Sided Limits

In this section we extend the limit concept to *one-sided limits*, which are limits as x approaches the number c from the left-hand side (where $x < c$) or the right-hand side ($x > c$) only.

One-Sided Limits

To have a limit L as x approaches c , a function f must be defined on *both sides* of c and its values $f(x)$ must approach L as x approaches c from either side. Because of this, ordinary limits are called **two-sided**.

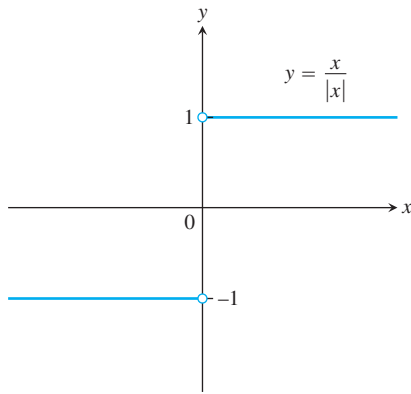


FIGURE 2.24 Different right-hand and left-hand limits at the origin.

If f fails to have a two-sided limit at c , it may still have a one-sided limit, that is, a limit if the approach is only from one side. If the approach is from the right, the limit is a **right-hand limit**. From the left, it is a **left-hand limit**.

The function $f(x) = x/|x|$ (Figure 2.24) has limit 1 as x approaches 0 from the right, and limit -1 as x approaches 0 from the left. Since these one-sided limit values are not the same, there is no single number that $f(x)$ approaches as x approaches 0. So $f(x)$ does not have a (two-sided) limit at 0.

Intuitively, if $f(x)$ is defined on an interval (c, b) , where $c < b$, and approaches arbitrarily close to L as x approaches c from within that interval, then f has **right-hand limit** L at c . We write

$$\lim_{x \rightarrow c^+} f(x) = L.$$

The symbol “ $x \rightarrow c^+$ ” means that we consider only values of x greater than c .

Similarly, if $f(x)$ is defined on an interval (a, c) , where $a < c$ and approaches arbitrarily close to M as x approaches c from within that interval, then f has **left-hand limit** M at c . We write

$$\lim_{x \rightarrow c^-} f(x) = M.$$

The symbol “ $x \rightarrow c^-$ ” means that we consider only x values less than c .

These informal definitions of one-sided limits are illustrated in Figure 2.25. For the function $f(x) = x/|x|$ in Figure 2.24 we have

$$\lim_{x \rightarrow 0^+} f(x) = 1 \quad \text{and} \quad \lim_{x \rightarrow 0^-} f(x) = -1.$$

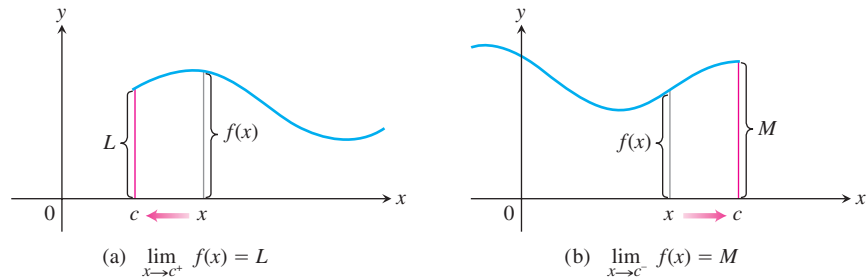


FIGURE 2.25 (a) Right-hand limit as x approaches c . (b) Left-hand limit as x approaches c .

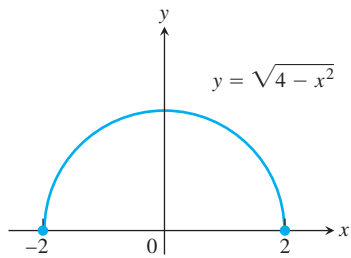


FIGURE 2.26 $\lim_{x \rightarrow 2^-} \sqrt{4 - x^2} = 0$ and $\lim_{x \rightarrow -2^+} \sqrt{4 - x^2} = 0$ (Example 1).

EXAMPLE 1 The domain of $f(x) = \sqrt{4 - x^2}$ is $[-2, 2]$; its graph is the semicircle in Figure 2.26. We have

$$\lim_{x \rightarrow -2^+} \sqrt{4 - x^2} = 0 \quad \text{and} \quad \lim_{x \rightarrow 2^-} \sqrt{4 - x^2} = 0.$$

The function does not have a left-hand limit at $x = -2$ or a right-hand limit at $x = 2$. It does not have ordinary two-sided limits at either -2 or 2 . ■

One-sided limits have all the properties listed in Theorem 1 in Section 2.2. The right-hand limit of the sum of two functions is the sum of their right-hand limits, and so on. The theorems for limits of polynomials and rational functions hold with one-sided limits, as do the Sandwich Theorem and Theorem 5. One-sided limits are related to limits in the following way.

THEOREM 6 A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \rightarrow c} f(x) = L \quad \Leftrightarrow \quad \lim_{x \rightarrow c^-} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow c^+} f(x) = L.$$

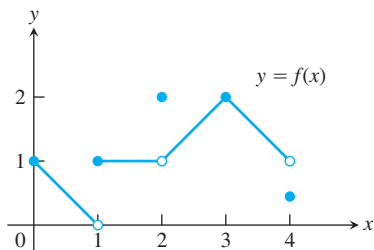


FIGURE 2.27 Graph of the function in Example 2.

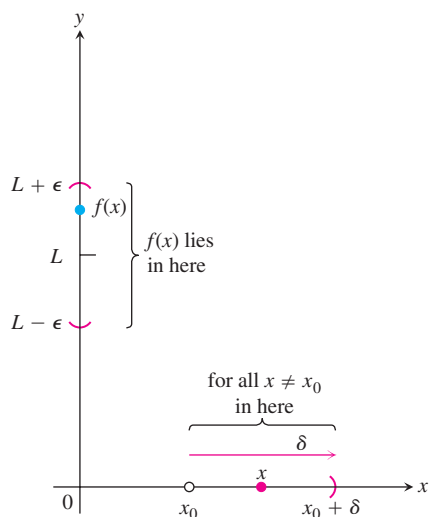


FIGURE 2.28 Intervals associated with the definition of right-hand limit.

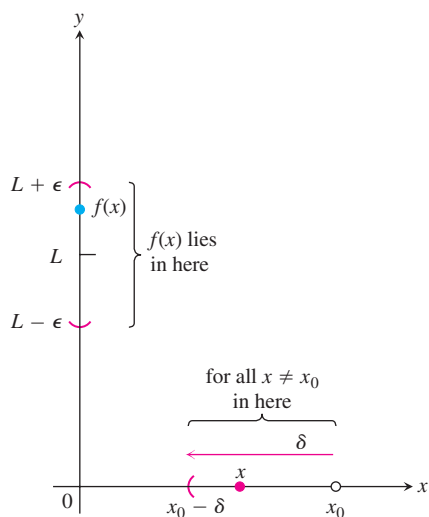


FIGURE 2.29 Intervals associated with the definition of left-hand limit.

EXAMPLE 2 For the function graphed in Figure 2.27,

At $x = 0$: $\lim_{x \rightarrow 0^+} f(x) = 1$,
 $\lim_{x \rightarrow 0^-} f(x)$ and $\lim_{x \rightarrow 0} f(x)$ do not exist. The function is not defined to the left of $x = 0$.

At $x = 1$: $\lim_{x \rightarrow 1^-} f(x) = 0$ even though $f(1) = 1$,
 $\lim_{x \rightarrow 1^+} f(x) = 1$,
 $\lim_{x \rightarrow 1} f(x)$ does not exist. The right- and left-hand limits are not equal.

At $x = 2$: $\lim_{x \rightarrow 2^-} f(x) = 1$,
 $\lim_{x \rightarrow 2^+} f(x) = 1$,
 $\lim_{x \rightarrow 2} f(x) = 1$ even though $f(2) = 2$.

At $x = 3$: $\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3} f(x) = f(3) = 2$.

At $x = 4$: $\lim_{x \rightarrow 4^-} f(x) = 1$ even though $f(4) \neq 1$,
 $\lim_{x \rightarrow 4^+} f(x)$ and $\lim_{x \rightarrow 4} f(x)$ do not exist. The function is not defined to the right of $x = 4$.

At every other point c in $[0, 4]$, $f(x)$ has limit $f(c)$. ■

Precise Definitions of One-Sided Limits

The formal definition of the limit in Section 2.3 is readily modified for one-sided limits.

DEFINITIONS We say that $f(x)$ has **right-hand limit** L at x_0 , and write

$$\lim_{x \rightarrow x_0^+} f(x) = L \quad (\text{see Figure 2.28})$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 < x < x_0 + \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

We say that f has **left-hand limit** L at x_0 , and write

$$\lim_{x \rightarrow x_0^-} f(x) = L \quad (\text{see Figure 2.29})$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 - \delta < x < x_0 \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

EXAMPLE 3 Prove that

$$\lim_{x \rightarrow 0^+} \sqrt{x} = 0.$$

Solution Let $\epsilon > 0$ be given. Here $x_0 = 0$ and $L = 0$, so we want to find a $\delta > 0$ such that for all x

$$0 < x < \delta \quad \Rightarrow \quad |\sqrt{x} - 0| < \epsilon,$$

or

$$0 < x < \delta \quad \Rightarrow \quad \sqrt{x} < \epsilon.$$

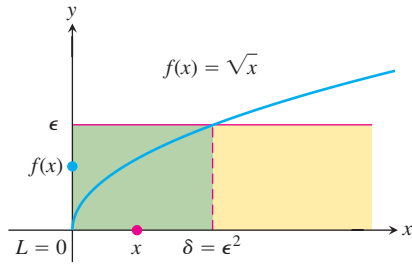


FIGURE 2.30 $\lim_{x \rightarrow 0^+} \sqrt{x} = 0$ in Example 3.

Squaring both sides of this last inequality gives

$$x < \epsilon^2 \quad \text{if} \quad 0 < x < \delta.$$

If we choose $\delta = \epsilon^2$ we have

$$0 < x < \delta = \epsilon^2 \quad \Rightarrow \quad \sqrt{x} < \epsilon,$$

or

$$0 < x < \epsilon^2 \quad \Rightarrow \quad |\sqrt{x} - 0| < \epsilon.$$

According to the definition, this shows that $\lim_{x \rightarrow 0^+} \sqrt{x} = 0$ (Figure 2.30). ■

The functions examined so far have had some kind of limit at each point of interest. In general, that need not be the case.

EXAMPLE 4 Show that $y = \sin(1/x)$ has no limit as x approaches zero from either side (Figure 2.31).

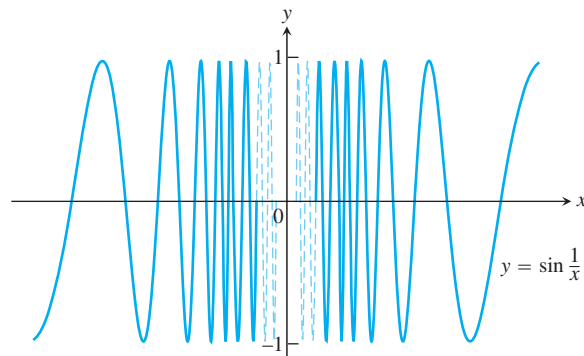


FIGURE 2.31 The function $y = \sin(1/x)$ has neither a right-hand nor a left-hand limit as x approaches zero (Example 4). The graph here omits values very near the y -axis.

Solution As x approaches zero, its reciprocal, $1/x$, grows without bound and the values of $\sin(1/x)$ cycle repeatedly from -1 to 1 . There is no single number L that the function's values stay increasingly close to as x approaches zero. This is true even if we restrict x to positive values or to negative values. The function has neither a right-hand limit nor a left-hand limit at $x = 0$. ■

Limits Involving $(\sin \theta)/\theta$

A central fact about $(\sin \theta)/\theta$ is that in radian measure its limit as $\theta \rightarrow 0$ is 1 . We can see this in Figure 2.32 and confirm it algebraically using the Sandwich Theorem. You will see the importance of this limit in Section 3.5, where instantaneous rates of change of the trigonometric functions are studied.

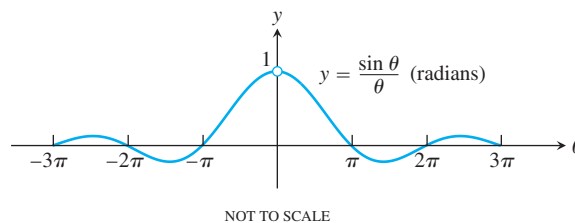


FIGURE 2.32 The graph of $f(\theta) = (\sin \theta)/\theta$ suggests that the right- and left-hand limits as θ approaches 0 are both 1 .

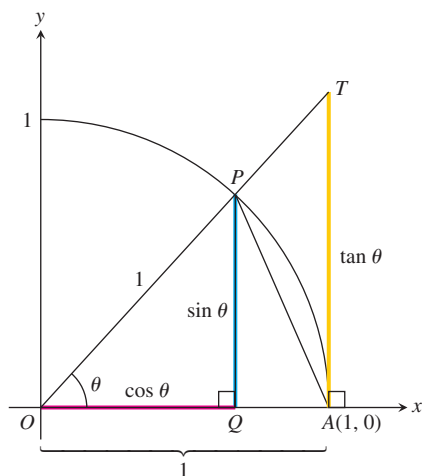


FIGURE 2.33 The figure for the proof of Theorem 7. By definition, $TA/OA = \tan \theta$, but $OA = 1$, so $TA = \tan \theta$.

Equation (2) is where radian measure comes in: The area of sector OAP is $\theta/2$ only if θ is measured in radians.

THEOREM 7

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \quad (\theta \text{ in radians}) \quad (1)$$

Proof The plan is to show that the right-hand and left-hand limits are both 1. Then we will know that the two-sided limit is 1 as well.

To show that the right-hand limit is 1, we begin with positive values of θ less than $\pi/2$ (Figure 2.33). Notice that

$$\text{Area } \triangle OAP < \text{area sector } OAP < \text{area } \triangle OAT.$$

We can express these areas in terms of θ as follows:

$$\begin{aligned} \text{Area } \triangle OAP &= \frac{1}{2} \text{base} \times \text{height} = \frac{1}{2} (1)(\sin \theta) = \frac{1}{2} \sin \theta \\ \text{Area sector } OAP &= \frac{1}{2} r^2 \theta = \frac{1}{2} (1)^2 \theta = \frac{\theta}{2} \\ \text{Area } \triangle OAT &= \frac{1}{2} \text{base} \times \text{height} = \frac{1}{2} (1)(\tan \theta) = \frac{1}{2} \tan \theta. \end{aligned} \quad (2)$$

Thus,

$$\frac{1}{2} \sin \theta < \frac{1}{2} \theta < \frac{1}{2} \tan \theta.$$

This last inequality goes the same way if we divide all three terms by the number $(1/2) \sin \theta$, which is positive since $0 < \theta < \pi/2$:

$$1 < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta}.$$

Taking reciprocals reverses the inequalities:

$$1 > \frac{\sin \theta}{\theta} > \cos \theta.$$

Since $\lim_{\theta \rightarrow 0^+} \cos \theta = 1$ (Example 11b, Section 2.2), the Sandwich Theorem gives

$$\lim_{\theta \rightarrow 0^+} \frac{\sin \theta}{\theta} = 1.$$

Recall that $\sin \theta$ and θ are both *odd functions* (Section 1.1). Therefore, $f(\theta) = (\sin \theta)/\theta$ is an *even function*, with a graph symmetric about the y -axis (see Figure 2.32). This symmetry implies that the left-hand limit at 0 exists and has the same value as the right-hand limit:

$$\lim_{\theta \rightarrow 0^-} \frac{\sin \theta}{\theta} = 1 = \lim_{\theta \rightarrow 0^+} \frac{\sin \theta}{\theta},$$

so $\lim_{\theta \rightarrow 0} (\sin \theta)/\theta = 1$ by Theorem 6. ■

EXAMPLE 5 Show that (a) $\lim_{h \rightarrow 0} \frac{\cos h - 1}{h} = 0$ and (b) $\lim_{x \rightarrow 0} \frac{\sin 2x}{5x} = \frac{2}{5}$.

Solution

(a) Using the half-angle formula $\cos h = 1 - 2 \sin^2(h/2)$, we calculate

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} &= \lim_{h \rightarrow 0} -\frac{2 \sin^2(h/2)}{h} \\ &= -\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \sin \theta \quad \text{Let } \theta = h/2. \\ &= -(1)(0) = 0. \quad \text{Eq. (1) and Example 11a in Section 2.2} \end{aligned}$$

(b) Equation (1) does not apply to the original fraction. We need a $2x$ in the denominator, not a $5x$. We produce it by multiplying numerator and denominator by $2/5$:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin 2x}{5x} &= \lim_{x \rightarrow 0} \frac{(2/5) \cdot \sin 2x}{(2/5) \cdot 5x} \\ &= \frac{2}{5} \lim_{x \rightarrow 0} \frac{\sin 2x}{2x} \quad \text{Now, Eq. (1) applies with } \theta = 2x. \\ &= \frac{2}{5}(1) = \frac{2}{5} \end{aligned}$$

EXAMPLE 6 Find $\lim_{t \rightarrow 0} \frac{\tan t \sec 2t}{3t}$.

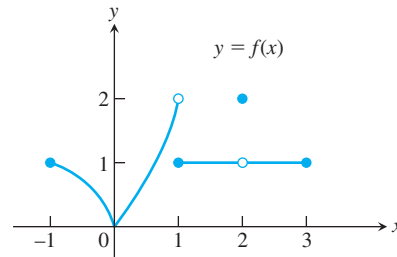
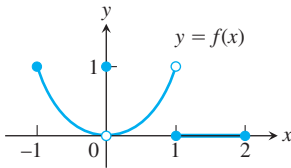
Solution From the definition of $\tan t$ and $\sec 2t$, we have

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{\tan t \sec 2t}{3t} &= \frac{1}{3} \lim_{t \rightarrow 0} \frac{\sin t}{t} \cdot \frac{1}{\cos t} \cdot \frac{1}{\cos 2t} \\ &= \frac{1}{3}(1)(1)(1) = \frac{1}{3}. \quad \text{Eq. (1) and Example 11b in Section 2.2} \end{aligned}$$

Exercises 2.4

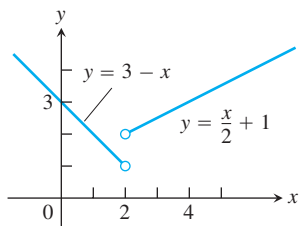
Finding Limits Graphically

1. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?



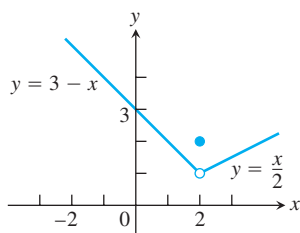
- | | | | |
|--|--|---|--|
| a. $\lim_{x \rightarrow -1^+} f(x) = 1$ | b. $\lim_{x \rightarrow 0^-} f(x) = 0$ | a. $\lim_{x \rightarrow -1^+} f(x) = 1$ | b. $\lim_{x \rightarrow 2} f(x)$ does not exist. |
| c. $\lim_{x \rightarrow 0^-} f(x) = 1$ | d. $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x)$ | c. $\lim_{x \rightarrow 2} f(x) = 2$ | d. $\lim_{x \rightarrow 1^-} f(x) = 2$ |
| e. $\lim_{x \rightarrow 0} f(x)$ exists. | f. $\lim_{x \rightarrow 0} f(x) = 0$ | e. $\lim_{x \rightarrow 1^+} f(x) = 1$ | f. $\lim_{x \rightarrow 1} f(x)$ does not exist. |
| g. $\lim_{x \rightarrow 0} f(x) = 1$ | h. $\lim_{x \rightarrow 1} f(x) = 1$ | g. $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x)$ | |
| i. $\lim_{x \rightarrow 1} f(x) = 0$ | j. $\lim_{x \rightarrow 2^-} f(x) = 2$ | h. $\lim_{x \rightarrow c} f(x)$ exists at every c in the open interval $(-1, 1)$. | |
| k. $\lim_{x \rightarrow -1} f(x)$ does not exist. | l. $\lim_{x \rightarrow 2^+} f(x) = 0$ | i. $\lim_{x \rightarrow c} f(x)$ exists at every c in the open interval $(1, 3)$. | |
| 2. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false? | | j. $\lim_{x \rightarrow -1^-} f(x) = 0$ | k. $\lim_{x \rightarrow 3^+} f(x)$ does not exist. |

3. Let $f(x) = \begin{cases} 3 - x, & x < 2 \\ \frac{x}{2} + 1, & x > 2. \end{cases}$



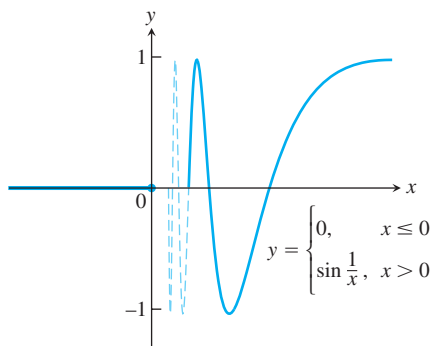
- Find $\lim_{x \rightarrow 2^+} f(x)$ and $\lim_{x \rightarrow 2^-} f(x)$.
- Does $\lim_{x \rightarrow 2} f(x)$ exist? If so, what is it? If not, why not?
- Find $\lim_{x \rightarrow 4^-} f(x)$ and $\lim_{x \rightarrow 4^+} f(x)$.
- Does $\lim_{x \rightarrow 4} f(x)$ exist? If so, what is it? If not, why not?

4. Let $f(x) = \begin{cases} 3 - x, & x < 2 \\ 2, & x = 2 \\ \frac{x}{2}, & x > 2. \end{cases}$



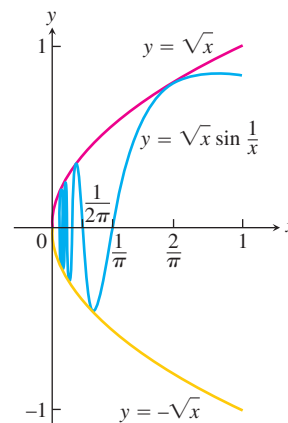
- Find $\lim_{x \rightarrow 2^+} f(x)$, $\lim_{x \rightarrow 2^-} f(x)$, and $f(2)$.
- Does $\lim_{x \rightarrow 2} f(x)$ exist? If so, what is it? If not, why not?
- Find $\lim_{x \rightarrow -1^-} f(x)$ and $\lim_{x \rightarrow -1^+} f(x)$.
- Does $\lim_{x \rightarrow -1} f(x)$ exist? If so, what is it? If not, why not?

5. Let $f(x) = \begin{cases} 0, & x \leq 0 \\ \sin \frac{1}{x}, & x > 0. \end{cases}$



- Does $\lim_{x \rightarrow 0^+} f(x)$ exist? If so, what is it? If not, why not?
- Does $\lim_{x \rightarrow 0^-} f(x)$ exist? If so, what is it? If not, why not?
- Does $\lim_{x \rightarrow 0} f(x)$ exist? If so, what is it? If not, why not?

6. Let $g(x) = \sqrt{x} \sin(1/x)$.



- Does $\lim_{x \rightarrow 0^+} g(x)$ exist? If so, what is it? If not, why not?
 - Does $\lim_{x \rightarrow 0^-} g(x)$ exist? If so, what is it? If not, why not?
 - Does $\lim_{x \rightarrow 0} g(x)$ exist? If so, what is it? If not, why not?
7. a. Graph $f(x) = \begin{cases} x^3, & x \neq 1 \\ 0, & x = 1. \end{cases}$
- Find $\lim_{x \rightarrow 1^-} f(x)$ and $\lim_{x \rightarrow 1^+} f(x)$.
 - Does $\lim_{x \rightarrow 1} f(x)$ exist? If so, what is it? If not, why not?
8. a. Graph $f(x) = \begin{cases} 1 - x^2, & x \neq 1 \\ 2, & x = 1. \end{cases}$
- Find $\lim_{x \rightarrow 1^+} f(x)$ and $\lim_{x \rightarrow 1^-} f(x)$.
 - Does $\lim_{x \rightarrow 1} f(x)$ exist? If so, what is it? If not, why not?

Graph the functions in Exercises 9 and 10. Then answer these questions.

- What are the domain and range of f ?
- At what points c , if any, does $\lim_{x \rightarrow c} f(x)$ exist?
- At what points does only the left-hand limit exist?
- At what points does only the right-hand limit exist?

9. $f(x) = \begin{cases} \sqrt{1 - x^2}, & 0 \leq x < 1 \\ 1, & 1 \leq x < 2 \\ 2, & x = 2 \end{cases}$

10. $f(x) = \begin{cases} x, & -1 \leq x < 0, \text{ or } 0 < x \leq 1 \\ 1, & x = 0 \\ 0, & x < -1 \text{ or } x > 1 \end{cases}$

Finding One-Sided Limits Algebraically

Find the limits in Exercises 11–18.

11. $\lim_{x \rightarrow -0.5^-} \sqrt{\frac{x+2}{x+1}}$ 12. $\lim_{x \rightarrow 1^+} \sqrt{\frac{x-1}{x+2}}$

13. $\lim_{x \rightarrow -2^+} \left(\frac{x}{x+1} \right) \left(\frac{2x+5}{x^2+x} \right)$

14. $\lim_{x \rightarrow 1^-} \left(\frac{1}{x+1} \right) \left(\frac{x+6}{x} \right) \left(\frac{3-x}{7} \right)$

15. $\lim_{h \rightarrow 0^+} \frac{\sqrt{h^2 + 4h + 5} - \sqrt{5}}{h}$

16. $\lim_{h \rightarrow 0^-} \frac{\sqrt{6} - \sqrt{5h^2 + 11h + 6}}{h}$
17. a. $\lim_{x \rightarrow -2^+} (x + 3) \frac{|x + 2|}{x + 2}$ b. $\lim_{x \rightarrow -2^-} (x + 3) \frac{|x + 2|}{x + 2}$
18. a. $\lim_{x \rightarrow 1^+} \frac{\sqrt{2x}(x - 1)}{|x - 1|}$ b. $\lim_{x \rightarrow 1^-} \frac{\sqrt{2x}(x - 1)}{|x - 1|}$

Use the graph of the greatest integer function $y = \lfloor x \rfloor$, Figure 1.10 in Section 1.1, to help you find the limits in Exercises 19 and 20.

19. a. $\lim_{\theta \rightarrow 3^+} \frac{\lfloor \theta \rfloor}{\theta}$ b. $\lim_{\theta \rightarrow 3^-} \frac{\lfloor \theta \rfloor}{\theta}$
20. a. $\lim_{t \rightarrow 4^+} (t - \lfloor t \rfloor)$ b. $\lim_{t \rightarrow 4^-} (t - \lfloor t \rfloor)$

Using $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$

Find the limits in Exercises 21–42.

21. $\lim_{\theta \rightarrow 0} \frac{\sin \sqrt{2\theta}}{\sqrt{2\theta}}$ 22. $\lim_{t \rightarrow 0} \frac{\sin kt}{t}$ (k constant)
23. $\lim_{y \rightarrow 0} \frac{\sin 3y}{4y}$ 24. $\lim_{h \rightarrow 0} \frac{h}{\sin 3h}$
25. $\lim_{x \rightarrow 0} \frac{\tan 2x}{x}$ 26. $\lim_{t \rightarrow 0} \frac{2t}{\tan t}$
27. $\lim_{x \rightarrow 0} \frac{x \csc 2x}{\cos 5x}$ 28. $\lim_{x \rightarrow 0} 6x^2(\cot x)(\csc 2x)$
29. $\lim_{x \rightarrow 0} \frac{x + x \cos x}{\sin x \cos x}$ 30. $\lim_{x \rightarrow 0} \frac{x^2 - x + \sin x}{2x}$
31. $\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\sin 2\theta}$ 32. $\lim_{x \rightarrow 0} \frac{x - x \cos x}{\sin^2 3x}$
33. $\lim_{t \rightarrow 0} \frac{\sin(1 - \cos t)}{1 - \cos t}$ 34. $\lim_{h \rightarrow 0} \frac{\sin(\sin h)}{\sin h}$
35. $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\sin 2\theta}$ 36. $\lim_{x \rightarrow 0} \frac{\sin 5x}{\sin 4x}$
37. $\lim_{\theta \rightarrow 0} \theta \cos \theta$ 38. $\lim_{\theta \rightarrow 0} \sin \theta \cot 2\theta$
39. $\lim_{x \rightarrow 0} \frac{\tan 3x}{\sin 8x}$ 40. $\lim_{y \rightarrow 0} \frac{\sin 3y \cot 5y}{y \cot 4y}$

41. $\lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta^2 \cot 3\theta}$ 42. $\lim_{\theta \rightarrow 0} \frac{\theta \cot 4\theta}{\sin^2 \theta \cot^2 2\theta}$

Theory and Examples

43. Once you know $\lim_{x \rightarrow a^+} f(x)$ and $\lim_{x \rightarrow a^-} f(x)$ at an interior point of the domain of f , do you then know $\lim_{x \rightarrow a} f(x)$? Give reasons for your answer.
44. If you know that $\lim_{x \rightarrow c} f(x)$ exists, can you find its value by calculating $\lim_{x \rightarrow c^+} f(x)$? Give reasons for your answer.
45. Suppose that f is an odd function of x . Does knowing that $\lim_{x \rightarrow 0^+} f(x) = 3$ tell you anything about $\lim_{x \rightarrow 0^-} f(x)$? Give reasons for your answer.
46. Suppose that f is an even function of x . Does knowing that $\lim_{x \rightarrow -2^-} f(x) = 7$ tell you anything about either $\lim_{x \rightarrow -2^-} f(x)$ or $\lim_{x \rightarrow -2^+} f(x)$? Give reasons for your answer.

Formal Definitions of One-Sided Limits

47. Given $\epsilon > 0$, find an interval $I = (5, 5 + \delta)$, $\delta > 0$, such that if x lies in I , then $\sqrt{x} - 5 < \epsilon$. What limit is being verified and what is its value?
48. Given $\epsilon > 0$, find an interval $I = (4 - \delta, 4)$, $\delta > 0$, such that if x lies in I , then $\sqrt{4 - x} < \epsilon$. What limit is being verified and what is its value?

Use the definitions of right-hand and left-hand limits to prove the limit statements in Exercises 49 and 50.

49. $\lim_{x \rightarrow 0^-} \frac{x}{|x|} = -1$ 50. $\lim_{x \rightarrow 2^+} \frac{x - 2}{|x - 2|} = 1$

51. **Greatest integer function** Find (a) $\lim_{x \rightarrow 400^+} \lfloor x \rfloor$ and (b) $\lim_{x \rightarrow 400^-} \lfloor x \rfloor$; then use limit definitions to verify your findings. (c) Based on your conclusions in parts (a) and (b), can you say anything about $\lim_{x \rightarrow 400} \lfloor x \rfloor$? Give reasons for your answer.

52. **One-sided limits** Let $f(x) = \begin{cases} x^2 \sin(1/x), & x < 0 \\ \sqrt{x}, & x > 0. \end{cases}$

Find (a) $\lim_{x \rightarrow 0^+} f(x)$ and (b) $\lim_{x \rightarrow 0^-} f(x)$; then use limit definitions to verify your findings. (c) Based on your conclusions in parts (a) and (b), can you say anything about $\lim_{x \rightarrow 0} f(x)$? Give reasons for your answer.

2.5 | Continuity

When we plot function values generated in a laboratory or collected in the field, we often connect the plotted points with an unbroken curve to show what the function's values are likely to have been at the times we did not measure (Figure 2.34). In doing so, we are assuming that we are working with a *continuous function*, so its outputs vary continuously with the inputs and do not jump from one value to another without taking on the values in between. The limit of a continuous function as x approaches c can be found simply by calculating the value of the function at c . (We found this to be true for polynomials in Theorem 2.)

Intuitively, any function $y = f(x)$ whose graph can be sketched over its domain in one continuous motion without lifting the pencil is an example of a continuous function. In this section we investigate more precisely what it means for a function to be continuous.

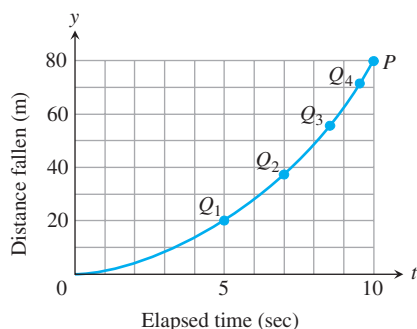


FIGURE 2.34 Connecting plotted points by an unbroken curve from experimental data Q_1, Q_2, Q_3, \dots for a falling object.

We also study the properties of continuous functions, and see that many of the function types presented in Section 1.1 are continuous.

Continuity at a Point

To understand continuity, it helps to consider a function like that in Figure 2.35, whose limits we investigated in Example 2 in the last section.

EXAMPLE 1 Find the points at which the function f in Figure 2.35 is continuous and the points at which f is not continuous.

Solution The function f is continuous at every point in its domain $[0, 4]$ except at $x = 1, x = 2,$ and $x = 4$. At these points, there are breaks in the graph. Note the relationship between the limit of f and the value of f at each point of the function's domain.

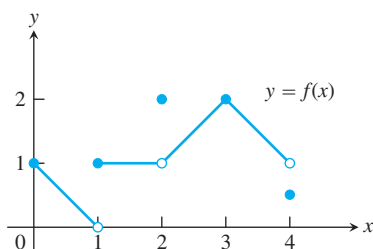


FIGURE 2.35 The function is continuous on $[0, 4]$ except at $x = 1, x = 2,$ and $x = 4$ (Example 1).

Points at which f is continuous:

$$\begin{aligned} \text{At } x = 0, & \quad \lim_{x \rightarrow 0^+} f(x) = f(0). \\ \text{At } x = 3, & \quad \lim_{x \rightarrow 3} f(x) = f(3). \\ \text{At } 0 < c < 4, c \neq 1, 2, & \quad \lim_{x \rightarrow c} f(x) = f(c). \end{aligned}$$

Points at which f is not continuous:

$$\begin{aligned} \text{At } x = 1, & \quad \lim_{x \rightarrow 1} f(x) \text{ does not exist.} \\ \text{At } x = 2, & \quad \lim_{x \rightarrow 2} f(x) = 1, \text{ but } 1 \neq f(2). \\ \text{At } x = 4, & \quad \lim_{x \rightarrow 4^-} f(x) = 1, \text{ but } 1 \neq f(4). \\ \text{At } c < 0, c > 4, & \quad \text{these points are not in the domain of } f. \end{aligned}$$

To define continuity at a point in a function's domain, we need to define continuity at an interior point (which involves a two-sided limit) and continuity at an endpoint (which involves a one-sided limit) (Figure 2.36).

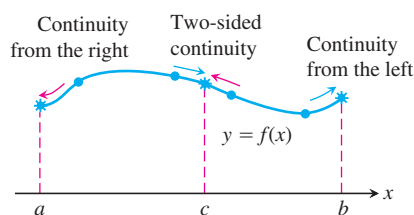


FIGURE 2.36 Continuity at points $a, b,$ and c .

DEFINITION

Interior point: A function $y = f(x)$ is **continuous at an interior point c** of its domain if

$$\lim_{x \rightarrow c} f(x) = f(c).$$

Endpoint: A function $y = f(x)$ is **continuous at a left endpoint a** or is **continuous at a right endpoint b** of its domain if

$$\lim_{x \rightarrow a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x \rightarrow b^-} f(x) = f(b), \text{ respectively.}$$

If a function f is not continuous at a point c , we say that f is **discontinuous** at c and that c is a **point of discontinuity** of f . Note that c need not be in the domain of f .

A function f is **right-continuous (continuous from the right)** at a point $x = c$ in its domain if $\lim_{x \rightarrow c^+} f(x) = f(c)$. It is **left-continuous (continuous from the left)** at c if $\lim_{x \rightarrow c^-} f(x) = f(c)$. Thus, a function is continuous at a left endpoint a of its domain if it

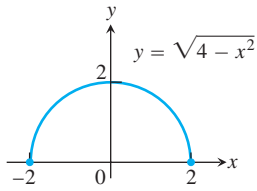


FIGURE 2.37 A function that is continuous at every domain point (Example 2).

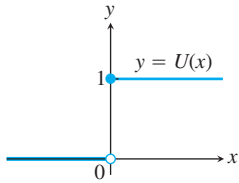


FIGURE 2.38 A function that has a jump discontinuity at the origin (Example 3).

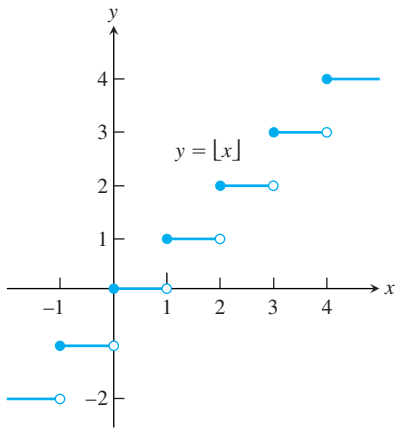


FIGURE 2.39 The greatest integer function is continuous at every noninteger point. It is right-continuous, but not left-continuous, at every integer point (Example 4).

is right-continuous at a and continuous at a right endpoint b of its domain if it is left-continuous at b . A function is continuous at an interior point c of its domain if and only if it is both right-continuous and left-continuous at c (Figure 2.36).

EXAMPLE 2 The function $f(x) = \sqrt{4 - x^2}$ is continuous at every point of its domain $[-2, 2]$ (Figure 2.37), including $x = -2$, where f is right-continuous, and $x = 2$, where f is left-continuous. ■

EXAMPLE 3 The unit step function $U(x)$, graphed in Figure 2.38, is right-continuous at $x = 0$, but is neither left-continuous nor continuous there. It has a jump discontinuity at $x = 0$. ■

We summarize continuity at a point in the form of a test.

Continuity Test

A function $f(x)$ is continuous at an interior point $x = c$ of its domain if and only if it meets the following three conditions.

1. $f(c)$ exists (c lies in the domain of f).
2. $\lim_{x \rightarrow c} f(x)$ exists (f has a limit as $x \rightarrow c$).
3. $\lim_{x \rightarrow c} f(x) = f(c)$ (the limit equals the function value).

For one-sided continuity and continuity at an endpoint, the limits in parts 2 and 3 of the test should be replaced by the appropriate one-sided limits.

EXAMPLE 4 The function $y = \lfloor x \rfloor$ introduced in Section 1.1 is graphed in Figure 2.39. It is discontinuous at every integer because the left-hand and right-hand limits are not equal as $x \rightarrow n$:

$$\lim_{x \rightarrow n^-} \lfloor x \rfloor = n - 1 \quad \text{and} \quad \lim_{x \rightarrow n^+} \lfloor x \rfloor = n.$$

Since $\lfloor n \rfloor = n$, the greatest integer function is right-continuous at every integer n (but not left-continuous).

The greatest integer function is continuous at every real number other than the integers. For example,

$$\lim_{x \rightarrow 1.5} \lfloor x \rfloor = 1 = \lfloor 1.5 \rfloor.$$

In general, if $n - 1 < c < n$, n an integer, then

$$\lim_{x \rightarrow c} \lfloor x \rfloor = n - 1 = \lfloor c \rfloor. \quad \blacksquare$$

Figure 2.40 displays several common types of discontinuities. The function in Figure 2.40a is continuous at $x = 0$. The function in Figure 2.40b would be continuous if it had $f(0) = 1$. The function in Figure 2.40c would be continuous if $f(0)$ were 1 instead of 2. The discontinuities in Figure 2.40b and c are **removable**. Each function has a limit as $x \rightarrow 0$, and we can remove the discontinuity by setting $f(0)$ equal to this limit.

The discontinuities in Figure 2.40d through f are more serious: $\lim_{x \rightarrow 0} f(x)$ does not exist, and there is no way to improve the situation by changing f at 0. The step function in Figure 2.40d has a **jump discontinuity**: The one-sided limits exist but have different values. The function $f(x) = 1/x^2$ in Figure 2.40e has an **infinite discontinuity**. The function in Figure 2.40f has an **oscillating discontinuity**: It oscillates too much to have a limit as $x \rightarrow 0$.

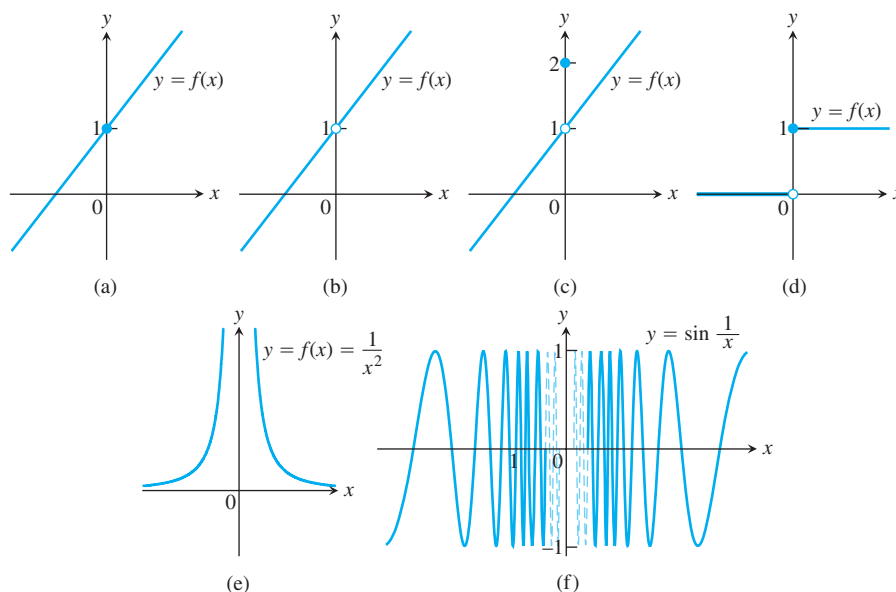


FIGURE 2.40 The function in (a) is continuous at $x = 0$; the functions in (b) through (f) are not.

Continuous Functions

A function is **continuous on an interval** if and only if it is continuous at every point of the interval. For example, the semicircle function graphed in Figure 2.37 is continuous on the interval $[-2, 2]$, which is its domain. A **continuous function** is one that is continuous at every point of its domain. A continuous function need not be continuous on every interval.

EXAMPLE 5

- (a) The function $y = 1/x$ (Figure 2.41) is a continuous function because it is continuous at every point of its domain. It has a point of discontinuity at $x = 0$, however, because it is not defined there; that is, it is discontinuous on any interval containing $x = 0$.
- (b) The identity function $f(x) = x$ and constant functions are continuous everywhere by Example 3, Section 2.3. ■

Algebraic combinations of continuous functions are continuous wherever they are defined.

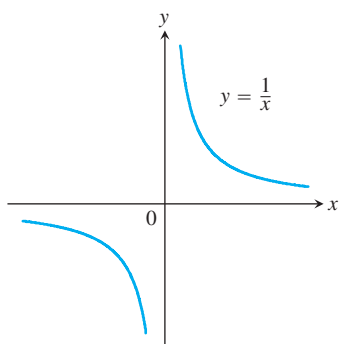


FIGURE 2.41 The function $y = 1/x$ is continuous at every value of x except $x = 0$. It has a point of discontinuity at $x = 0$ (Example 5).

THEOREM 8—Properties of Continuous Functions If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

1. *Sums:* $f + g$
2. *Differences:* $f - g$
3. *Constant multiples:* $k \cdot f$, for any number k
4. *Products:* $f \cdot g$
5. *Quotients:* f/g , provided $g(c) \neq 0$
6. *Powers:* f^n , n a positive integer
7. *Roots:* $\sqrt[n]{f}$, provided it is defined on an open interval containing c , where n is a positive integer

Most of the results in Theorem 8 follow from the limit rules in Theorem 1, Section 2.2. For instance, to prove the sum property we have

$$\begin{aligned} \lim_{x \rightarrow c} (f + g)(x) &= \lim_{x \rightarrow c} (f(x) + g(x)) \\ &= \lim_{x \rightarrow c} f(x) + \lim_{x \rightarrow c} g(x), && \text{Sum Rule, Theorem 1} \\ &= f(c) + g(c) && \text{Continuity of } f, g \text{ at } c \\ &= (f + g)(c). \end{aligned}$$

This shows that $f + g$ is continuous.

EXAMPLE 6

- (a) Every polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$ is continuous because $\lim_{x \rightarrow c} P(x) = P(c)$ by Theorem 2, Section 2.2.
- (b) If $P(x)$ and $Q(x)$ are polynomials, then the rational function $P(x)/Q(x)$ is continuous wherever it is defined ($Q(c) \neq 0$) by Theorem 3, Section 2.2. ■

EXAMPLE 7 The function $f(x) = |x|$ is continuous at every value of x . If $x > 0$, we have $f(x) = x$, a polynomial. If $x < 0$, we have $f(x) = -x$, another polynomial. Finally, at the origin, $\lim_{x \rightarrow 0} |x| = 0 = |0|$. ■

The functions $y = \sin x$ and $y = \cos x$ are continuous at $x = 0$ by Example 11 of Section 2.2. Both functions are, in fact, continuous everywhere (see Exercise 70). It follows from Theorem 8 that all six trigonometric functions are then continuous wherever they are defined. For example, $y = \tan x$ is continuous on $\dots \cup (-\pi/2, \pi/2) \cup (\pi/2, 3\pi/2) \cup \dots$.

Inverse Functions and Continuity

The inverse function of any function continuous on an interval is continuous over its domain. This result is suggested from the observation that the graph of f^{-1} , being the reflection of the graph of f across the line $y = x$, cannot have any breaks in it when the graph of f has no breaks. A rigorous proof that f^{-1} is continuous whenever f is continuous on an interval is given in more advanced texts. It follows that the inverse trigonometric functions are all continuous over their domains.

We defined the exponential function $y = a^x$ in Section 1.5 informally by its graph. Recall that the graph was obtained from the graph of $y = a^x$ for x a rational number by filling in the holes at the irrational points x , so the function $y = a^x$ was defined to be continuous over the entire real line. The inverse function $y = \log_a x$ is also continuous. In particular, the natural exponential function $y = e^x$ and the natural logarithm function $y = \ln x$ are both continuous over their domains.

Composites

All composites of continuous functions are continuous. The idea is that if $f(x)$ is continuous at $x = c$ and $g(x)$ is continuous at $x = f(c)$, then $g \circ f$ is continuous at $x = c$ (Figure 2.42). In this case, the limit as $x \rightarrow c$ is $g(f(c))$.

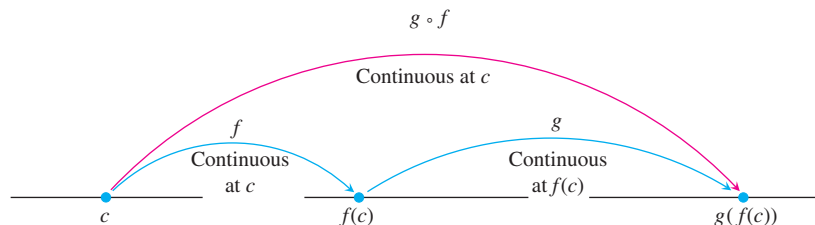


FIGURE 2.42 Composites of continuous functions are continuous.

THEOREM 9—Composite of Continuous Functions If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c .

Intuitively, Theorem 9 is reasonable because if x is close to c , then $f(x)$ is close to $f(c)$, and since g is continuous at $f(c)$, it follows that $g(f(x))$ is close to $g(f(c))$.

The continuity of composites holds for any finite number of functions. The only requirement is that each function be continuous where it is applied. For an outline of the proof of Theorem 9, see Exercise 6 in Appendix 4.

EXAMPLE 8 Show that the following functions are continuous everywhere on their respective domains.

$$\begin{array}{ll} \text{(a)} y = \sqrt{x^2 - 2x - 5} & \text{(b)} y = \frac{x^{2/3}}{1 + x^4} \\ \text{(c)} y = \left| \frac{x - 2}{x^2 - 2} \right| & \text{(d)} y = \left| \frac{x \sin x}{x^2 + 2} \right| \end{array}$$

Solution

(a) The square root function is continuous on $[0, \infty)$ because it is a root of the continuous identity function $f(x) = x$ (Part 7, Theorem 8). The given function is then the composite of the polynomial $f(x) = x^2 - 2x - 5$ with the square root function $g(t) = \sqrt{t}$, and is continuous on its domain.

(b) The numerator is the cube root of the identity function squared; the denominator is an everywhere-positive polynomial. Therefore, the quotient is continuous.

(c) The quotient $(x - 2)/(x^2 - 2)$ is continuous for all $x \neq \pm\sqrt{2}$, and the function is the composition of this quotient with the continuous absolute value function (Example 7).

(d) Because the sine function is everywhere-continuous (Exercise 70), the numerator term $x \sin x$ is the product of continuous functions, and the denominator term $x^2 + 2$ is an everywhere-positive polynomial. The given function is the composite of a quotient of continuous functions with the continuous absolute value function (Figure 2.43). ■

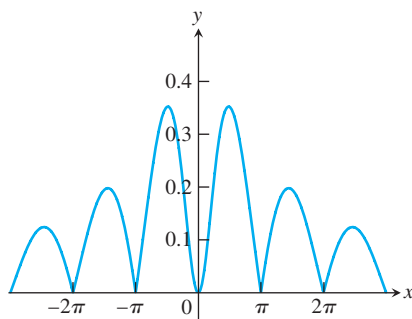


FIGURE 2.43 The graph suggests that $y = |(x \sin x)/(x^2 + 2)|$ is continuous (Example 8d).

Theorem 9 is actually a consequence of a more general result which we now state and prove.

THEOREM 10—Limits of Continuous Functions If g is continuous at the point b and $\lim_{x \rightarrow c} f(x) = b$, then

$$\lim_{x \rightarrow c} g(f(x)) = g(b) = g(\lim_{x \rightarrow c} f(x)).$$

Proof Let $\epsilon > 0$ be given. Since g is continuous at b , there exists a number $\delta_1 > 0$ such that

$$|g(y) - g(b)| < \epsilon \quad \text{whenever} \quad 0 < |y - b| < \delta_1.$$

Since $\lim_{x \rightarrow c} f(x) = b$, there exists a $\delta > 0$ such that

$$|f(x) - b| < \delta_1 \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

If we let $y = f(x)$, we then have that

$$|y - b| < \delta_1 \quad \text{whenever} \quad 0 < |x - c| < \delta,$$

which implies from the first statement that $|g(y) - g(b)| = |g(f(x)) - g(b)| < \epsilon$ whenever $0 < |x - c| < \delta$. From the definition of limit, this proves that $\lim_{x \rightarrow c} g(f(x)) = g(b)$. ■

EXAMPLE 9 As an application of Theorem 10, we have the following calculations.

$$\begin{aligned} \text{(a)} \quad \lim_{x \rightarrow \pi/2} \cos\left(2x + \sin\left(\frac{3\pi}{2} + x\right)\right) &= \cos\left(\lim_{x \rightarrow \pi/2} 2x + \lim_{x \rightarrow \pi/2} \sin\left(\frac{3\pi}{2} + x\right)\right) \\ &= \cos(\pi + \sin 2\pi) = \cos \pi = -1. \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \lim_{x \rightarrow 1} \sin^{-1}\left(\frac{1-x}{1-x^2}\right) &= \sin^{-1}\left(\lim_{x \rightarrow 1} \frac{1-x}{1-x^2}\right) && \text{Arcsine is} \\ & && \text{continuous.} \\ &= \sin^{-1}\left(\lim_{x \rightarrow 1} \frac{1}{1+x}\right) && \text{Cancel common} \\ & && \text{factor } (1-x). \\ &= \sin^{-1}\frac{1}{2} = \frac{\pi}{6} \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad \lim_{x \rightarrow 0} \sqrt{x+1} e^{\tan x} &= \lim_{x \rightarrow 0} \sqrt{x+1} \cdot \exp\left(\lim_{x \rightarrow 0} \tan x\right) && \text{Exponential is} \\ & && \text{continuous.} \\ &= 1 \cdot e^0 = 1 \end{aligned}$$

We sometimes denote e^u by $\exp u$ when u is a complicated mathematical expression.

Continuous Extension to a Point

The function $y = f(x) = (\sin x)/x$ is continuous at every point except $x = 0$. In this it is like the function $y = 1/x$. But $y = (\sin x)/x$ is different from $y = 1/x$ in that it has a finite limit as $x \rightarrow 0$ (Theorem 7). It is therefore possible to extend the function's domain to include the point $x = 0$ in such a way that the extended function is continuous at $x = 0$. We define a new function

$$F(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0. \end{cases}$$

The function $F(x)$ is continuous at $x = 0$ because

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = F(0)$$

(Figure 2.44).

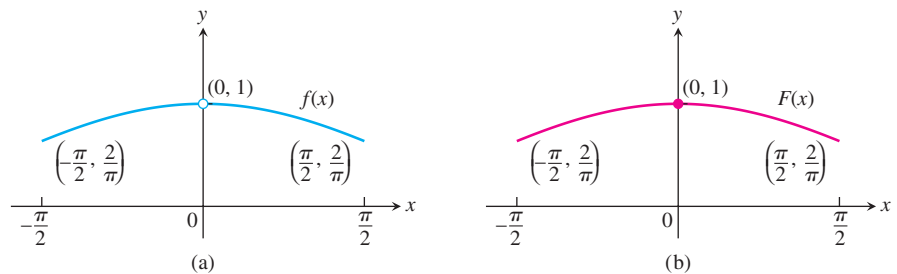


FIGURE 2.44 The graph (a) of $f(x) = (\sin x)/x$ for $-\pi/2 \leq x \leq \pi/2$ does not include the point $(0, 1)$ because the function is not defined at $x = 0$. (b) We can remove the discontinuity from the graph by defining the new function $F(x)$ with $F(0) = 1$ and $F(x) = f(x)$ everywhere else. Note that $F(0) = \lim_{x \rightarrow 0} f(x)$.

More generally, a function (such as a rational function) may have a limit even at a point where it is not defined. If $f(c)$ is not defined, but $\lim_{x \rightarrow c} f(x) = L$ exists, we can define a new function $F(x)$ by the rule

$$F(x) = \begin{cases} f(x), & \text{if } x \text{ is in the domain of } f \\ L, & \text{if } x = c. \end{cases}$$

The function F is continuous at $x = c$. It is called the **continuous extension of f** to $x = c$. For rational functions f , continuous extensions are usually found by canceling common factors.

EXAMPLE 10 Show that

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4}, \quad x \neq 2$$

has a continuous extension to $x = 2$, and find that extension.

Solution Although $f(2)$ is not defined, if $x \neq 2$ we have

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4} = \frac{(x - 2)(x + 3)}{(x - 2)(x + 2)} = \frac{x + 3}{x + 2}.$$

The new function

$$F(x) = \frac{x + 3}{x + 2}$$

is equal to $f(x)$ for $x \neq 2$, but is continuous at $x = 2$, having there the value of $5/4$. Thus F is the continuous extension of f to $x = 2$, and

$$\lim_{x \rightarrow 2} \frac{x^2 + x - 6}{x^2 - 4} = \lim_{x \rightarrow 2} f(x) = \frac{5}{4}.$$

The graph of f is shown in Figure 2.45. The continuous extension F has the same graph except with no hole at $(2, 5/4)$. Effectively, F is the function f with its point of discontinuity at $x = 2$ removed. ■

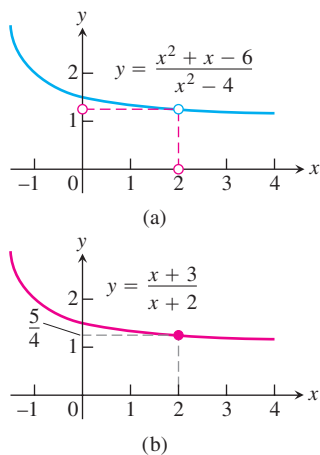
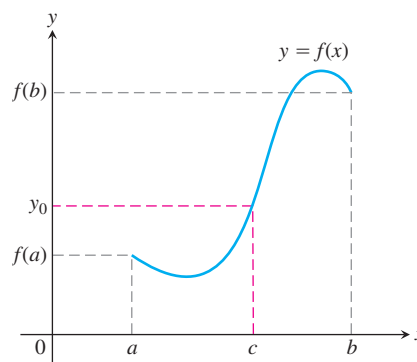


FIGURE 2.45 (a) The graph of $f(x)$ and (b) the graph of its continuous extension $F(x)$ (Example 10).

Intermediate Value Theorem for Continuous Functions

Functions that are continuous on intervals have properties that make them particularly useful in mathematics and its applications. One of these is the *Intermediate Value Property*. A function is said to have the **Intermediate Value Property** if whenever it takes on two values, it also takes on all the values in between.

THEOREM 11—The Intermediate Value Theorem for Continuous Functions If f is a continuous function on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.



Theorem 11 says that continuous functions over *finite closed* intervals have the Intermediate Value Property. Geometrically, the Intermediate Value Theorem says that any horizontal line $y = y_0$ crossing the y -axis between the numbers $f(a)$ and $f(b)$ will cross the curve $y = f(x)$ at least once over the interval $[a, b]$.

The proof of the Intermediate Value Theorem depends on the completeness property of the real number system (Appendix 6) and can be found in more advanced texts.

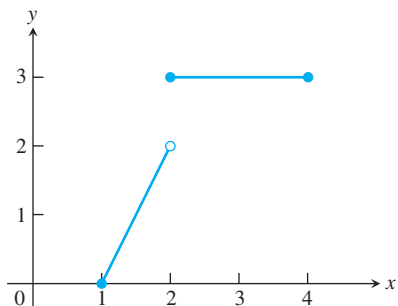


FIGURE 2.46 The function $f(x) = \begin{cases} 2x - 2, & 1 \leq x < 2 \\ 3, & 2 \leq x \leq 4 \end{cases}$ does not take on all values between $f(1) = 0$ and $f(4) = 3$; it misses all the values between 2 and 3.

The continuity of f on the interval is essential to Theorem 11. If f is discontinuous at even one point of the interval, the theorem's conclusion may fail, as it does for the function graphed in Figure 2.46 (choose y_0 as any number between 2 and 3).

A Consequence for Graphing: Connectedness Theorem 11 implies that the graph of a function continuous on an interval cannot have any breaks over the interval. It will be **connected**—a single, unbroken curve. It will not have jumps like the graph of the greatest integer function (Figure 2.39), or separate branches like the graph of $1/x$ (Figure 2.41).

A Consequence for Root Finding We call a solution of the equation $f(x) = 0$ a **root** of the equation or **zero** of the function f . The Intermediate Value Theorem tells us that if f is continuous, then any interval on which f changes sign contains a zero of the function.

In practical terms, when we see the graph of a continuous function cross the horizontal axis on a computer screen, we know it is not stepping across. There really is a point where the function's value is zero.

EXAMPLE 11 Show that there is a root of the equation $x^3 - x - 1 = 0$ between 1 and 2.

Solution Let $f(x) = x^3 - x - 1$. Since $f(1) = 1 - 1 - 1 = -1 < 0$ and $f(2) = 2^3 - 2 - 1 = 5 > 0$, we see that $y_0 = 0$ is a value between $f(1)$ and $f(2)$. Since f is continuous, the Intermediate Value Theorem says there is a zero of f between 1 and 2. Figure 2.47 shows the result of zooming in to locate the root near $x = 1.32$. ■

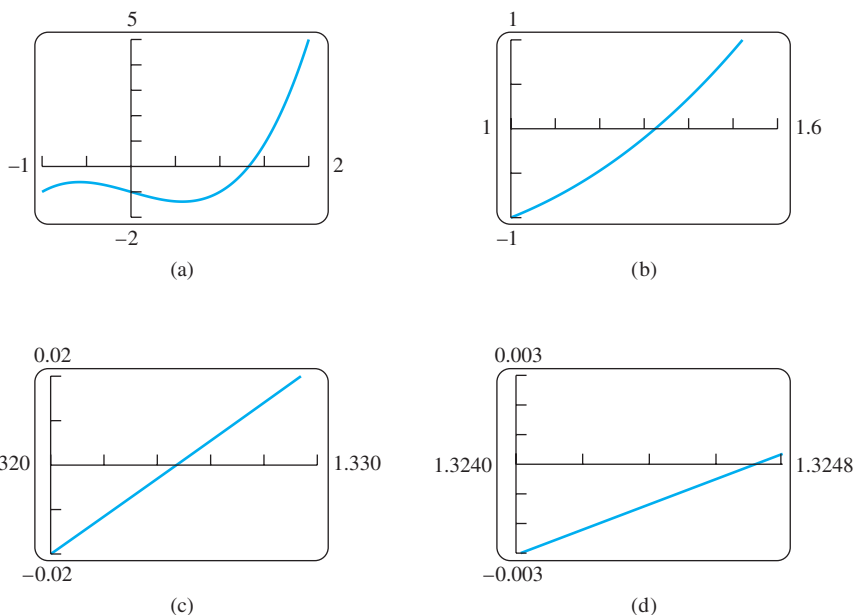


FIGURE 2.47 Zooming in on a zero of the function $f(x) = x^3 - x - 1$. The zero is near $x = 1.3247$ (Example 11).

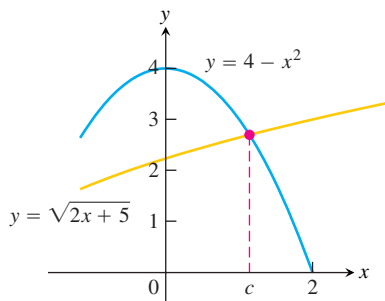


FIGURE 2.48 The curves $y = \sqrt{2x + 5}$ and $y = 4 - x^2$ have the same value at $x = c$ where $\sqrt{2x + 5} = 4 - x^2$ (Example 12).

EXAMPLE 12 Use the Intermediate Value Theorem to prove that the equation

$$\sqrt{2x + 5} = 4 - x^2$$

has a solution (Figure 2.48).

Solution We rewrite the equation as

$$\sqrt{2x + 5} + x^2 = 4,$$

and set $f(x) = \sqrt{2x + 5} + x^2$. Now $g(x) = \sqrt{2x + 5}$ is continuous on the interval $[-5/2, \infty)$ since it is the composite of the square root function with the nonnegative linear

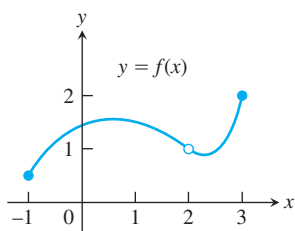
function $y = 2x + 5$. Then f is the sum of the function g and the quadratic function $y = x^2$, and the quadratic function is continuous for all values of x . It follows that $f(x) = \sqrt{2x + 5} + x^2$ is continuous on the interval $[-5/2, \infty)$. By trial and error, we find the function values $f(0) = \sqrt{5} \approx 2.24$ and $f(2) = \sqrt{9} + 4 = 7$, and note that f is also continuous on the finite closed interval $[0, 2] \subset [-5/2, \infty)$. Since the value $y_0 = 4$ is between the numbers 2.24 and 7, by the Intermediate Value Theorem there is a number $c \in [0, 2]$ such that $f(c) = 4$. That is, the number c solves the original equation. ■

Exercises 2.5

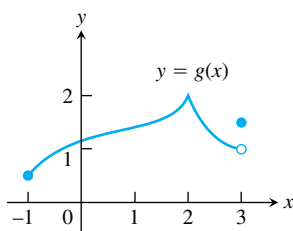
Continuity from Graphs

In Exercises 1–4, say whether the function graphed is continuous on $[-1, 3]$. If not, where does it fail to be continuous and why?

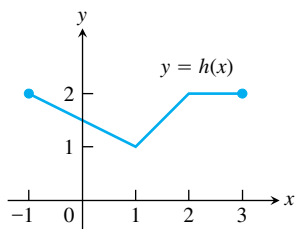
1.



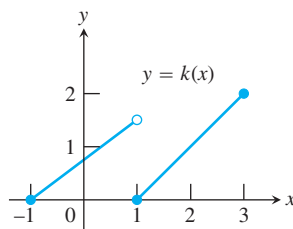
2.



3.



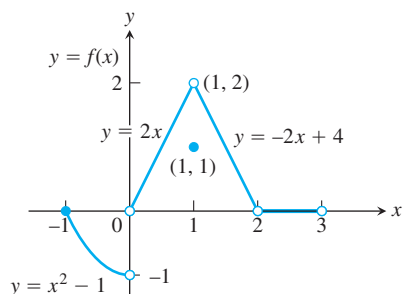
4.



Exercises 5–10 refer to the function

$$f(x) = \begin{cases} x^2 - 1, & -1 \leq x < 0 \\ 2x, & 0 < x < 1 \\ 1, & x = 1 \\ -2x + 4, & 1 < x < 2 \\ 0, & 2 < x < 3 \end{cases}$$

graphed in the accompanying figure.



The graph for Exercises 5–10.

5. a. Does $f(-1)$ exist?
b. Does $\lim_{x \rightarrow -1^+} f(x)$ exist?
c. Does $\lim_{x \rightarrow -1^+} f(x) = f(-1)$?
d. Is f continuous at $x = -1$?
6. a. Does $f(1)$ exist?
b. Does $\lim_{x \rightarrow 1} f(x)$ exist?
c. Does $\lim_{x \rightarrow 1} f(x) = f(1)$?
d. Is f continuous at $x = 1$?
7. a. Is f defined at $x = 2$? (Look at the definition of f .)
b. Is f continuous at $x = 2$?
8. At what values of x is f continuous?
9. What value should be assigned to $f(2)$ to make the extended function continuous at $x = 2$?
10. To what new value should $f(1)$ be changed to remove the discontinuity?

Applying the Continuity Test

At which points do the functions in Exercises 11 and 12 fail to be continuous? At which points, if any, are the discontinuities removable? Not removable? Give reasons for your answers.

11. Exercise 1, Section 2.4
12. Exercise 2, Section 2.4

At what points are the functions in Exercises 13–30 continuous?

13. $y = \frac{1}{x-2} - 3x$
14. $y = \frac{1}{(x+2)^2} + 4$
15. $y = \frac{x+1}{x^2-4x+3}$
16. $y = \frac{x+3}{x^2-3x-10}$
17. $y = |x-1| + \sin x$
18. $y = \frac{1}{|x|+1} - \frac{x^2}{2}$
19. $y = \frac{\cos x}{x}$
20. $y = \frac{x+2}{\cos x}$
21. $y = \csc 2x$
22. $y = \tan \frac{\pi x}{2}$
23. $y = \frac{x \tan x}{x^2+1}$
24. $y = \frac{\sqrt{x^4+1}}{1+\sin^2 x}$
25. $y = \sqrt{2x+3}$
26. $y = \sqrt[4]{3x-1}$
27. $y = (2x-1)^{1/3}$
28. $y = (2-x)^{1/5}$

$$29. g(x) = \begin{cases} \frac{x^2 - x - 6}{x - 3}, & x \neq 3 \\ 5, & x = 3 \end{cases}$$

$$30. f(x) = \begin{cases} \frac{x^3 - 8}{x^2 - 4}, & x \neq 2, x \neq -2 \\ 3, & x = 2 \\ 4, & x = -2 \end{cases}$$

Limits Involving Trigonometric Functions

Find the limits in Exercises 31–38. Are the functions continuous at the point being approached?

$$31. \lim_{x \rightarrow \pi} \sin(x - \sin x) \quad 32. \lim_{t \rightarrow 0} \sin\left(\frac{\pi}{2} \cos(\tan t)\right)$$

$$33. \lim_{y \rightarrow 1} \sec(y \sec^2 y - \tan^2 y - 1)$$

$$34. \lim_{x \rightarrow 0} \tan\left(\frac{\pi}{4} \cos(\sin x^{1/3})\right)$$

$$35. \lim_{t \rightarrow 0} \cos\left(\frac{\pi}{\sqrt{19 - 3 \sec 2t}}\right) \quad 36. \lim_{x \rightarrow \pi/6} \sqrt{\csc^2 x + 5\sqrt{3} \tan x}$$

$$37. \lim_{x \rightarrow 0^+} \sin\left(\frac{\pi}{2} e^{\sqrt{x}}\right) \quad 38. \lim_{x \rightarrow 1} \cos^{-1}(\ln \sqrt{x})$$

Continuous Extensions

39. Define $g(3)$ in a way that extends $g(x) = (x^2 - 9)/(x - 3)$ to be continuous at $x = 3$.

40. Define $h(2)$ in a way that extends $h(t) = (t^2 + 3t - 10)/(t - 2)$ to be continuous at $t = 2$.

41. Define $f(1)$ in a way that extends $f(s) = (s^3 - 1)/(s^2 - 1)$ to be continuous at $s = 1$.

42. Define $g(4)$ in a way that extends

$$g(x) = (x^2 - 16)/(x^2 - 3x - 4)$$

to be continuous at $x = 4$.

43. For what value of a is

$$f(x) = \begin{cases} x^2 - 1, & x < 3 \\ 2ax, & x \geq 3 \end{cases}$$

continuous at every x ?

44. For what value of b is

$$g(x) = \begin{cases} x, & x < -2 \\ bx^2, & x \geq -2 \end{cases}$$

continuous at every x ?

45. For what values of a is

$$f(x) = \begin{cases} a^2x - 2a, & x \geq 2 \\ 12, & x < 2 \end{cases}$$

continuous at every x ?

46. For what value of b is

$$g(x) = \begin{cases} \frac{x - b}{b + 1}, & x < 0 \\ x^2 + b, & x > 0 \end{cases}$$

continuous at every x ?

47. For what values of a and b is

$$f(x) = \begin{cases} -2, & x \leq -1 \\ ax - b, & -1 < x < 1 \\ 3, & x \geq 1 \end{cases}$$

continuous at every x ?

48. For what values of a and b is

$$g(x) = \begin{cases} ax + 2b, & x \leq 0 \\ x^2 + 3a - b, & 0 < x \leq 2 \\ 3x - 5, & x > 2 \end{cases}$$

continuous at every x ?

T In Exercises 49–52, graph the function f to see whether it appears to have a continuous extension to the origin. If it does, use Trace and Zoom to find a good candidate for the extended function's value at $x = 0$. If the function does not appear to have a continuous extension, can it be extended to be continuous at the origin from the right or from the left? If so, what do you think the extended function's value(s) should be?

$$49. f(x) = \frac{10^x - 1}{x}$$

$$50. f(x) = \frac{10^{|x|} - 1}{x}$$

$$51. f(x) = \frac{\sin x}{|x|}$$

$$52. f(x) = (1 + 2x)^{1/x}$$

Theory and Examples

53. A continuous function $y = f(x)$ is known to be negative at $x = 0$ and positive at $x = 1$. Why does the equation $f(x) = 0$ have at least one solution between $x = 0$ and $x = 1$? Illustrate with a sketch.

54. Explain why the equation $\cos x = x$ has at least one solution.

55. **Roots of a cubic** Show that the equation $x^3 - 15x + 1 = 0$ has three solutions in the interval $[-4, 4]$.

56. **A function value** Show that the function $F(x) = (x - a)^2 \cdot (x - b)^2 + x$ takes on the value $(a + b)/2$ for some value of x .

57. **Solving an equation** If $f(x) = x^3 - 8x + 10$, show that there are values c for which $f(c)$ equals (a) π ; (b) $-\sqrt{3}$; (c) 5,000,000.

58. Explain why the following five statements ask for the same information.

a. Find the roots of $f(x) = x^3 - 3x - 1$.

b. Find the x -coordinates of the points where the curve $y = x^3$ crosses the line $y = 3x + 1$.

c. Find all the values of x for which $x^3 - 3x = 1$.

d. Find the x -coordinates of the points where the cubic curve $y = x^3 - 3x$ crosses the line $y = 1$.

e. Solve the equation $x^3 - 3x - 1 = 0$.

59. **Removable discontinuity** Give an example of a function $f(x)$ that is continuous for all values of x except $x = 2$, where it has a removable discontinuity. Explain how you know that f is discontinuous at $x = 2$, and how you know the discontinuity is removable.

60. **Nonremovable discontinuity** Give an example of a function $g(x)$ that is continuous for all values of x except $x = -1$, where it has a nonremovable discontinuity. Explain how you know that g is discontinuous there and why the discontinuity is not removable.

61. A function discontinuous at every point

- a. Use the fact that every nonempty interval of real numbers contains both rational and irrational numbers to show that the function

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

is discontinuous at every point.

- b. Is f right-continuous or left-continuous at any point?

- 62.** If functions $f(x)$ and $g(x)$ are continuous for $0 \leq x \leq 1$, could $f(x)/g(x)$ possibly be discontinuous at a point of $[0, 1]$? Give reasons for your answer.
- 63.** If the product function $h(x) = f(x) \cdot g(x)$ is continuous at $x = 0$, must $f(x)$ and $g(x)$ be continuous at $x = 0$? Give reasons for your answer.
- 64. Discontinuous composite of continuous functions** Give an example of functions f and g , both continuous at $x = 0$, for which the composite $f \circ g$ is discontinuous at $x = 0$. Does this contradict Theorem 9? Give reasons for your answer.
- 65. Never-zero continuous functions** Is it true that a continuous function that is never zero on an interval never changes sign on that interval? Give reasons for your answer.
- 66. Stretching a rubber band** Is it true that if you stretch a rubber band by moving one end to the right and the other to the left, some point of the band will end up in its original position? Give reasons for your answer.
- 67. A fixed point theorem** Suppose that a function f is continuous on the closed interval $[0, 1]$ and that $0 \leq f(x) \leq 1$ for every x in $[0, 1]$. Show that there must exist a number c in $[0, 1]$ such that $f(c) = c$ (c is called a **fixed point** of f).

- 68. The sign-preserving property of continuous functions** Let f be defined on an interval (a, b) and suppose that $f(c) \neq 0$ at some c where f is continuous. Show that there is an interval $(c - \delta, c + \delta)$ about c where f has the same sign as $f(c)$.

- 69.** Prove that f is continuous at c if and only if

$$\lim_{h \rightarrow 0} f(c + h) = f(c).$$

- 70.** Use Exercise 69 together with the identities

$$\sin(h + c) = \sin h \cos c + \cos h \sin c,$$

$$\cos(h + c) = \cos h \cos c - \sin h \sin c$$

to prove that both $f(x) = \sin x$ and $g(x) = \cos x$ are continuous at every point $x = c$.

Solving Equations Graphically

T Use the Intermediate Value Theorem in Exercises 71–78 to prove that each equation has a solution. Then use a graphing calculator or computer grapher to solve the equations.

71. $x^3 - 3x - 1 = 0$

72. $2x^3 - 2x^2 - 2x + 1 = 0$

73. $x(x - 1)^2 = 1$ (one root)

74. $x^x = 2$

75. $\sqrt{x} + \sqrt{1 + x} = 4$

76. $x^3 - 15x + 1 = 0$ (three roots)

77. $\cos x = x$ (one root). Make sure you are using radian mode.

78. $2 \sin x = x$ (three roots). Make sure you are using radian mode.

2.6

Limits Involving Infinity; Asymptotes of Graphs

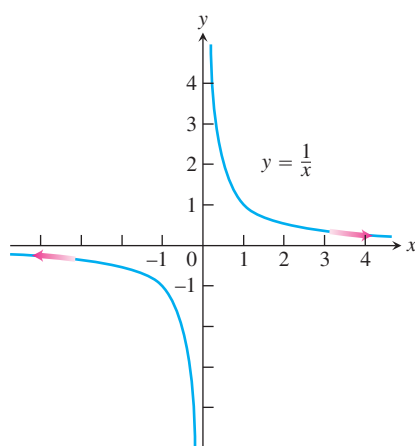


FIGURE 2.49 The graph of $y = 1/x$ approaches 0 as $x \rightarrow \infty$ or $x \rightarrow -\infty$.

In this section we investigate the behavior of a function when the magnitude of the independent variable x becomes increasingly large, or $x \rightarrow \pm\infty$. We further extend the concept of limit to *infinite limits*, which are not limits as before, but rather a new use of the term limit. Infinite limits provide useful symbols and language for describing the behavior of functions whose values become arbitrarily large in magnitude. We use these limit ideas to analyze the graphs of functions having *horizontal* or *vertical asymptotes*.

Finite Limits as $x \rightarrow \pm\infty$

The symbol for infinity (∞) does not represent a real number. We use ∞ to describe the behavior of a function when the values in its domain or range outgrow all finite bounds. For example, the function $f(x) = 1/x$ is defined for all $x \neq 0$ (Figure 2.49). When x is positive and becomes increasingly large, $1/x$ becomes increasingly small. When x is negative and its magnitude becomes increasingly large, $1/x$ again becomes small. We summarize these observations by saying that $f(x) = 1/x$ has limit 0 as $x \rightarrow \infty$ or $x \rightarrow -\infty$, or that 0 is a *limit of $f(x) = 1/x$ at infinity and negative infinity*. Here are precise definitions.

DEFINITIONS

1. We say that $f(x)$ has the **limit L as x approaches infinity** and write

$$\lim_{x \rightarrow \infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number M such that for all x

$$x > M \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

2. We say that $f(x)$ has the **limit L as x approaches minus infinity** and write

$$\lim_{x \rightarrow -\infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number N such that for all x

$$x < N \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

Intuitively, $\lim_{x \rightarrow \infty} f(x) = L$ if, as x moves increasingly far from the origin in the positive direction, $f(x)$ gets arbitrarily close to L . Similarly, $\lim_{x \rightarrow -\infty} f(x) = L$ if, as x moves increasingly far from the origin in the negative direction, $f(x)$ gets arbitrarily close to L .

The strategy for calculating limits of functions as $x \rightarrow \pm\infty$ is similar to the one for finite limits in Section 2.2. There we first found the limits of the constant and identity functions $y = k$ and $y = x$. We then extended these results to other functions by applying Theorem 1 on limits of algebraic combinations. Here we do the same thing, except that the starting functions are $y = k$ and $y = 1/x$ instead of $y = k$ and $y = x$.

The basic facts to be verified by applying the formal definition are

$$\lim_{x \rightarrow \pm\infty} k = k \quad \text{and} \quad \lim_{x \rightarrow \pm\infty} \frac{1}{x} = 0. \tag{1}$$

We prove the second result and leave the first to Exercises 87 and 88.

EXAMPLE 1 Show that

- (a) $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$
- (b) $\lim_{x \rightarrow -\infty} \frac{1}{x} = 0$.

Solution

(a) Let $\epsilon > 0$ be given. We must find a number M such that for all x

$$x > M \quad \Rightarrow \quad \left| \frac{1}{x} - 0 \right| = \left| \frac{1}{x} \right| < \epsilon.$$

The implication will hold if $M = 1/\epsilon$ or any larger positive number (Figure 2.50). This proves $\lim_{x \rightarrow \infty} (1/x) = 0$.

(b) Let $\epsilon > 0$ be given. We must find a number N such that for all x

$$x < N \quad \Rightarrow \quad \left| \frac{1}{x} - 0 \right| = \left| \frac{1}{x} \right| < \epsilon.$$

The implication will hold if $N = -1/\epsilon$ or any number less than $-1/\epsilon$ (Figure 2.50). This proves $\lim_{x \rightarrow -\infty} (1/x) = 0$. ■

Limits at infinity have properties similar to those of finite limits.

THEOREM 12 All the limit laws in Theorem 1 are true when we replace $\lim_{x \rightarrow c}$ by $\lim_{x \rightarrow \infty}$ or $\lim_{x \rightarrow -\infty}$. That is, the variable x may approach a finite number c or $\pm\infty$.

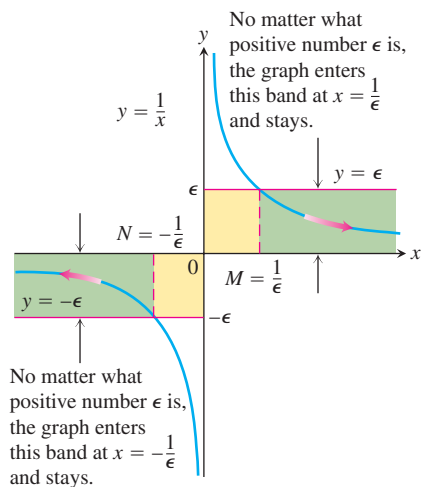


FIGURE 2.50 The geometry behind the argument in Example 1.

EXAMPLE 2 The properties in Theorem 12 are used to calculate limits in the same way as when x approaches a finite number c .

$$\begin{aligned} \text{(a)} \quad \lim_{x \rightarrow \infty} \left(5 + \frac{1}{x} \right) &= \lim_{x \rightarrow \infty} 5 + \lim_{x \rightarrow \infty} \frac{1}{x} && \text{Sum Rule} \\ &= 5 + 0 = 5 && \text{Known limits} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \lim_{x \rightarrow -\infty} \frac{\pi\sqrt{3}}{x^2} &= \lim_{x \rightarrow -\infty} \pi\sqrt{3} \cdot \frac{1}{x} \cdot \frac{1}{x} \\ &= \lim_{x \rightarrow -\infty} \pi\sqrt{3} \cdot \lim_{x \rightarrow -\infty} \frac{1}{x} \cdot \lim_{x \rightarrow -\infty} \frac{1}{x} && \text{Product Rule} \\ &= \pi\sqrt{3} \cdot 0 \cdot 0 = 0 && \text{Known limits} \end{aligned}$$

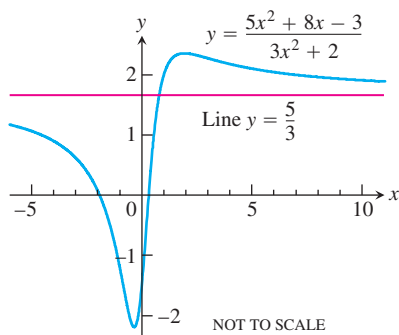


FIGURE 2.51 The graph of the function in Example 3a. The graph approaches the line $y = 5/3$ as $|x|$ increases.

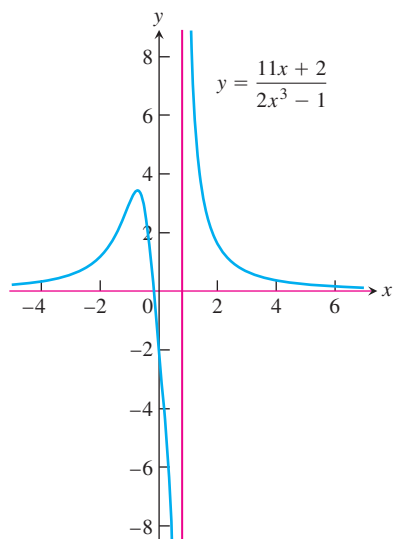


FIGURE 2.52 The graph of the function in Example 3b. The graph approaches the x -axis as $|x|$ increases.

Limits at Infinity of Rational Functions

To determine the limit of a rational function as $x \rightarrow \pm\infty$, we first divide the numerator and denominator by the highest power of x in the denominator. The result then depends on the degrees of the polynomials involved.

EXAMPLE 3 These examples illustrate what happens when the degree of the numerator is less than or equal to the degree of the denominator.

$$\begin{aligned} \text{(a)} \quad \lim_{x \rightarrow \infty} \frac{5x^2 + 8x - 3}{3x^2 + 2} &= \lim_{x \rightarrow \infty} \frac{5 + (8/x) - (3/x^2)}{3 + (2/x^2)} && \text{Divide numerator and denominator by } x^2. \\ &= \frac{5 + 0 - 0}{3 + 0} = \frac{5}{3} && \text{See Fig. 2.51.} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \lim_{x \rightarrow -\infty} \frac{11x + 2}{2x^3 - 1} &= \lim_{x \rightarrow -\infty} \frac{(11/x^2) + (2/x^3)}{2 - (1/x^3)} && \text{Divide numerator and denominator by } x^3. \\ &= \frac{0 + 0}{2 - 0} = 0 && \text{See Fig. 2.52.} \end{aligned}$$

A case for which the degree of the numerator is greater than the degree of the denominator is illustrated in Example 10.

Horizontal Asymptotes

If the distance between the graph of a function and some fixed line approaches zero as a point on the graph moves increasingly far from the origin, we say that the graph approaches the line asymptotically and that the line is an *asymptote* of the graph.

Looking at $f(x) = 1/x$ (see Figure 2.49), we observe that the x -axis is an asymptote of the curve on the right because

$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

and on the left because

$$\lim_{x \rightarrow -\infty} \frac{1}{x} = 0.$$

We say that the x -axis is a *horizontal asymptote* of the graph of $f(x) = 1/x$.

DEFINITION A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \rightarrow \infty} f(x) = b \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x) = b.$$

The graph of the function

$$f(x) = \frac{5x^2 + 8x - 3}{3x^2 + 2}$$

sketched in Figure 2.51 (Example 3a) has the line $y = 5/3$ as a horizontal asymptote on both the right and the left because

$$\lim_{x \rightarrow \infty} f(x) = \frac{5}{3} \quad \text{and} \quad \lim_{x \rightarrow -\infty} f(x) = \frac{5}{3}.$$

EXAMPLE 4 Find the horizontal asymptotes of the graph of

$$f(x) = \frac{x^3 - 2}{|x|^3 + 1}.$$

Solution We calculate the limits as $x \rightarrow \pm\infty$.

$$\text{For } x \geq 0: \quad \lim_{x \rightarrow \infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x \rightarrow \infty} \frac{x^3 - 2}{x^3 + 1} = \lim_{x \rightarrow \infty} \frac{1 - (2/x^3)}{1 + (1/x^3)} = 1.$$

$$\text{For } x < 0: \quad \lim_{x \rightarrow -\infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x \rightarrow -\infty} \frac{x^3 - 2}{(-x)^3 + 1} = \lim_{x \rightarrow -\infty} \frac{1 - (2/x^3)}{-1 + (1/x^3)} = -1.$$

The horizontal asymptotes are $y = -1$ and $y = 1$. The graph is displayed in Figure 2.53. Notice that the graph crosses the horizontal asymptote $y = -1$ for a positive value of x . ■

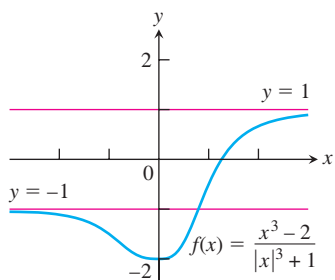


FIGURE 2.53 The graph of the function in Example 4 has two horizontal asymptotes.

EXAMPLE 5 The x -axis (the line $y = 0$) is a horizontal asymptote of the graph of $y = e^x$ because

$$\lim_{x \rightarrow -\infty} e^x = 0.$$

To see this, we use the definition of a limit as x approaches $-\infty$. So let $\epsilon > 0$ be given, but arbitrary. We must find a constant N such that for all x ,

$$x < N \quad \Rightarrow \quad |e^x - 0| < \epsilon.$$

Now $|e^x - 0| = e^x$, so the condition that needs to be satisfied whenever $x < N$ is

$$e^x < \epsilon.$$

Let $x = N$ be the number where $e^x = \epsilon$. Since e^x is an increasing function, if $x < N$, then $e^x < \epsilon$. We find N by taking the natural logarithm of both sides of the equation $e^N = \epsilon$, so $N = \ln \epsilon$ (see Figure 2.54). With this value of N the condition is satisfied, and we conclude that $\lim_{x \rightarrow -\infty} e^x = 0$. ■

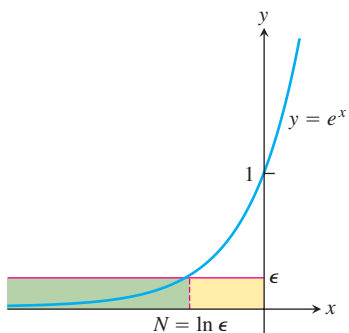


FIGURE 2.54 The graph of $y = e^x$ approaches the x -axis as $x \rightarrow -\infty$ (Example 5).

EXAMPLE 6 Find (a) $\lim_{x \rightarrow \infty} \sin(1/x)$ and (b) $\lim_{x \rightarrow \pm\infty} x \sin(1/x)$.

Solution

(a) We introduce the new variable $t = 1/x$. From Example 1, we know that $t \rightarrow 0^+$ as $x \rightarrow \infty$ (see Figure 2.49). Therefore,

$$\lim_{x \rightarrow \infty} \sin \frac{1}{x} = \lim_{t \rightarrow 0^+} \sin t = 0.$$

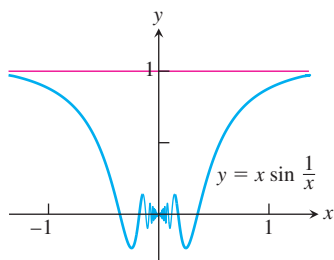


FIGURE 2.55 The line $y = 1$ is a horizontal asymptote of the function graphed here (Example 6b).

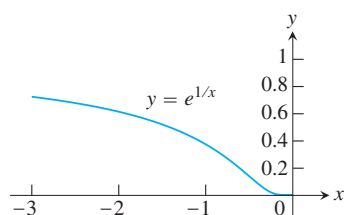


FIGURE 2.56 The graph of $y = e^{1/x}$ for $x < 0$ shows $\lim_{x \rightarrow 0^-} e^{1/x} = 0$ (Example 7).

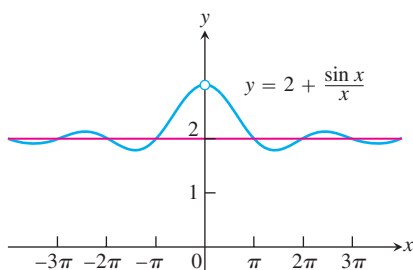


FIGURE 2.57 A curve may cross one of its asymptotes infinitely often (Example 8).

(b) We calculate the limits as $x \rightarrow \infty$ and $x \rightarrow -\infty$:

$$\lim_{x \rightarrow \infty} x \sin \frac{1}{x} = \lim_{t \rightarrow 0^+} \frac{\sin t}{t} = 1 \quad \text{and} \quad \lim_{x \rightarrow -\infty} x \sin \frac{1}{x} = \lim_{t \rightarrow 0^-} \frac{\sin t}{t} = 1.$$

The graph is shown in Figure 2.55, and we see that the line $y = 1$ is a horizontal asymptote. ■

Likewise, we can investigate the behavior of $y = f(1/x)$ as $x \rightarrow 0$ by investigating $y = f(t)$ as $t \rightarrow \pm\infty$, where $t = 1/x$.

EXAMPLE 7 Find $\lim_{x \rightarrow 0^-} e^{1/x}$.

Solution We let $t = 1/x$. From Figure 2.49, we can see that $t \rightarrow -\infty$ as $x \rightarrow 0^-$. (We make this idea more precise further on.) Therefore,

$$\lim_{x \rightarrow 0^-} e^{1/x} = \lim_{t \rightarrow -\infty} e^t = 0 \quad \text{Example 5}$$

(Figure 2.56). ■

The Sandwich Theorem also holds for limits as $x \rightarrow \pm\infty$. You must be sure, though, that the function whose limit you are trying to find stays between the bounding functions at very large values of x in magnitude consistent with whether $x \rightarrow \infty$ or $x \rightarrow -\infty$.

EXAMPLE 8 Using the Sandwich Theorem, find the horizontal asymptote of the curve

$$y = 2 + \frac{\sin x}{x}.$$

Solution We are interested in the behavior as $x \rightarrow \pm\infty$. Since

$$0 \leq \left| \frac{\sin x}{x} \right| \leq \left| \frac{1}{x} \right|$$

and $\lim_{x \rightarrow \pm\infty} |1/x| = 0$, we have $\lim_{x \rightarrow \pm\infty} (\sin x)/x = 0$ by the Sandwich Theorem. Hence,

$$\lim_{x \rightarrow \pm\infty} \left(2 + \frac{\sin x}{x} \right) = 2 + 0 = 2,$$

and the line $y = 2$ is a horizontal asymptote of the curve on both left and right (Figure 2.57).

This example illustrates that a curve may cross one of its horizontal asymptotes many times. ■

EXAMPLE 9 Find $\lim_{x \rightarrow \infty} (x - \sqrt{x^2 + 16})$.

Solution Both of the terms x and $\sqrt{x^2 + 16}$ approach infinity as $x \rightarrow \infty$, so what happens to the difference in the limit is unclear (we cannot subtract ∞ from ∞ because the symbol does not represent a real number). In this situation we can multiply the numerator and the denominator by the conjugate radical expression to obtain an equivalent algebraic result:

$$\begin{aligned} \lim_{x \rightarrow \infty} (x - \sqrt{x^2 + 16}) &= \lim_{x \rightarrow \infty} (x - \sqrt{x^2 + 16}) \frac{x + \sqrt{x^2 + 16}}{x + \sqrt{x^2 + 16}} \\ &= \lim_{x \rightarrow \infty} \frac{x^2 - (x^2 + 16)}{x + \sqrt{x^2 + 16}} = \lim_{x \rightarrow \infty} \frac{-16}{x + \sqrt{x^2 + 16}}. \end{aligned}$$

As $x \rightarrow \infty$, the denominator in this last expression becomes arbitrarily large, so we see that the limit is 0. We can also obtain this result by a direct calculation using the Limit Laws:

$$\lim_{x \rightarrow \infty} \frac{-16}{x + \sqrt{x^2 + 16}} = \lim_{x \rightarrow \infty} \frac{-\frac{16}{x}}{1 + \sqrt{\frac{x^2}{x^2} + \frac{16}{x^2}}} = \frac{0}{1 + \sqrt{1 + 0}} = 0.$$

Oblique Asymptotes

If the degree of the numerator of a rational function is 1 greater than the degree of the denominator, the graph has an **oblique** or **slant line asymptote**. We find an equation for the asymptote by dividing numerator by denominator to express f as a linear function plus a remainder that goes to zero as $x \rightarrow \pm\infty$.

EXAMPLE 10 Find the oblique asymptote of the graph of

$$f(x) = \frac{x^2 - 3}{2x - 4}$$

in Figure 2.58.

Solution We are interested in the behavior as $x \rightarrow \pm\infty$. We divide $(x^2 - 3)$ into $(2x - 4)$:

$$\begin{array}{r} \frac{x}{2} + 1 \\ 2x - 4 \overline{) x^2 - 3} \\ \underline{x^2 - 2x} \\ 2x - 3 \\ \underline{2x - 4} \\ 1 \end{array}$$

This tells us that

$$f(x) = \frac{x^2 - 3}{2x - 4} = \underbrace{\left(\frac{x}{2} + 1\right)}_{\text{linear } g(x)} + \underbrace{\left(\frac{1}{2x - 4}\right)}_{\text{remainder}}.$$

As $x \rightarrow \pm\infty$, the remainder, whose magnitude gives the vertical distance between the graphs of f and g , goes to zero, making the slanted line

$$g(x) = \frac{x}{2} + 1$$

an asymptote of the graph of f (Figure 2.58). The line $y = g(x)$ is an asymptote both to the right and to the left. The next subsection will confirm that the function $f(x)$ grows arbitrarily large in absolute value as $x \rightarrow 2$ (where the denominator is zero), as shown in the graph. ■

Notice in Example 10 that if the degree of the numerator in a rational function is greater than the degree of the denominator, then the limit as $|x|$ becomes large is $+\infty$ or $-\infty$, depending on the signs assumed by the numerator and denominator.

Infinite Limits

Let us look again at the function $f(x) = 1/x$. As $x \rightarrow 0^+$, the values of f grow without bound, eventually reaching and surpassing every positive real number. That is, given any positive real number B , however large, the values of f become larger still (Figure 2.59).

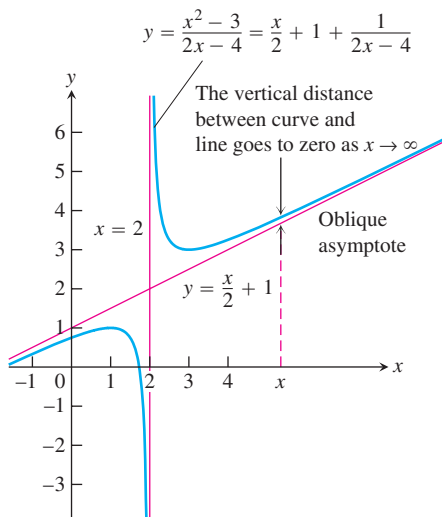


FIGURE 2.58 The graph of the function in Example 10 has an oblique asymptote.

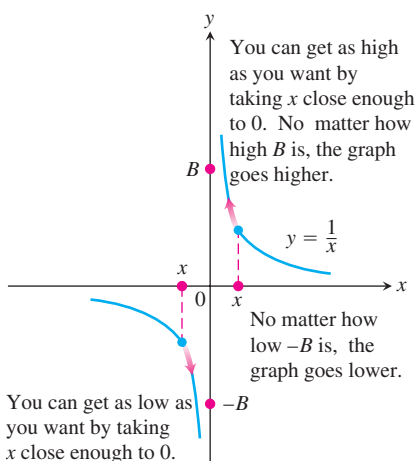


FIGURE 2.59 One-sided infinite limits: $\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty$ and $\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$.

Thus, f has no limit as $x \rightarrow 0^+$. It is nevertheless convenient to describe the behavior of f by saying that $f(x)$ approaches ∞ as $x \rightarrow 0^+$. We write

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{1}{x} = \infty.$$

In writing this equation, we are *not* saying that the limit exists. Nor are we saying that there is a real number ∞ , for there is no such number. Rather, we are saying that $\lim_{x \rightarrow 0^+} (1/x)$ *does not exist because $1/x$ becomes arbitrarily large and positive as $x \rightarrow 0^+$* .

As $x \rightarrow 0^-$, the values of $f(x) = 1/x$ become arbitrarily large and negative. Given any negative real number $-B$, the values of f eventually lie below $-B$. (See Figure 2.59.) We write

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty.$$

Again, we are not saying that the limit exists and equals the number $-\infty$. There *is* no real number $-\infty$. We are describing the behavior of a function whose limit as $x \rightarrow 0^-$ *does not exist because its values become arbitrarily large and negative*.

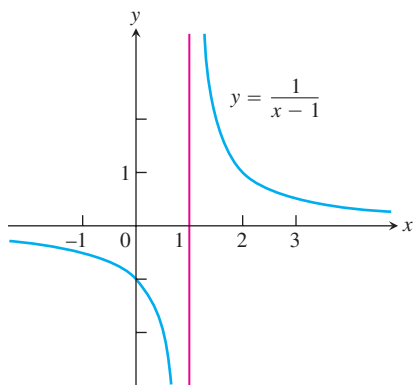


FIGURE 2.60 Near $x = 1$, the function $y = 1/(x - 1)$ behaves the way the function $y = 1/x$ behaves near $x = 0$. Its graph is the graph of $y = 1/x$ shifted 1 unit to the right (Example 11).

EXAMPLE 11 Find $\lim_{x \rightarrow 1^+} \frac{1}{x - 1}$ and $\lim_{x \rightarrow 1^-} \frac{1}{x - 1}$.

Geometric Solution The graph of $y = 1/(x - 1)$ is the graph of $y = 1/x$ shifted 1 unit to the right (Figure 2.60). Therefore, $y = 1/(x - 1)$ behaves near 1 exactly the way $y = 1/x$ behaves near 0:

$$\lim_{x \rightarrow 1^+} \frac{1}{x - 1} = \infty \quad \text{and} \quad \lim_{x \rightarrow 1^-} \frac{1}{x - 1} = -\infty.$$

Analytic Solution Think about the number $x - 1$ and its reciprocal. As $x \rightarrow 1^+$, we have $(x - 1) \rightarrow 0^+$ and $1/(x - 1) \rightarrow \infty$. As $x \rightarrow 1^-$, we have $(x - 1) \rightarrow 0^-$ and $1/(x - 1) \rightarrow -\infty$. ■

EXAMPLE 12 Discuss the behavior of

$$f(x) = \frac{1}{x^2} \quad \text{as} \quad x \rightarrow 0.$$

Solution As x approaches zero from either side, the values of $1/x^2$ are positive and become arbitrarily large (Figure 2.61). This means that

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{1}{x^2} = \infty. \quad \blacksquare$$

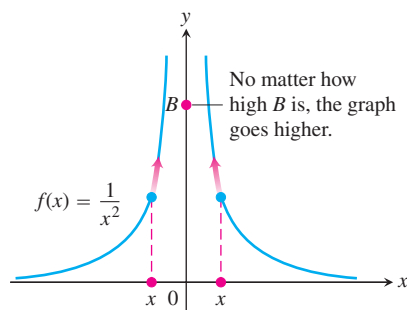


FIGURE 2.61 The graph of $f(x)$ in Example 12 approaches infinity as $x \rightarrow 0$.

The function $y = 1/x$ shows no consistent behavior as $x \rightarrow 0$. We have $1/x \rightarrow \infty$ if $x \rightarrow 0^+$, but $1/x \rightarrow -\infty$ if $x \rightarrow 0^-$. All we can say about $\lim_{x \rightarrow 0} (1/x)$ is that it does not exist. The function $y = 1/x^2$ is different. Its values approach infinity as x approaches zero from either side, so we can say that $\lim_{x \rightarrow 0} (1/x^2) = \infty$.

EXAMPLE 13 These examples illustrate that rational functions can behave in various ways near zeros of the denominator.

(a)
$$\lim_{x \rightarrow 2} \frac{(x - 2)^2}{x^2 - 4} = \lim_{x \rightarrow 2} \frac{(x - 2)^2}{(x - 2)(x + 2)} = \lim_{x \rightarrow 2} \frac{x - 2}{x + 2} = 0$$

(b)
$$\lim_{x \rightarrow 2} \frac{x - 2}{x^2 - 4} = \lim_{x \rightarrow 2} \frac{x - 2}{(x - 2)(x + 2)} = \lim_{x \rightarrow 2} \frac{1}{x + 2} = \frac{1}{4}$$

(c) $\lim_{x \rightarrow 2^+} \frac{x-3}{x^2-4} = \lim_{x \rightarrow 2^+} \frac{x-3}{(x-2)(x+2)} = -\infty$

The values are negative for $x > 2$, x near 2.

(d) $\lim_{x \rightarrow 2^-} \frac{x-3}{x^2-4} = \lim_{x \rightarrow 2^-} \frac{x-3}{(x-2)(x+2)} = \infty$

The values are positive for $x < 2$, x near 2.

(e) $\lim_{x \rightarrow 2} \frac{x-3}{x^2-4} = \lim_{x \rightarrow 2} \frac{x-3}{(x-2)(x+2)}$ does not exist.

See parts (c) and (d).

(f) $\lim_{x \rightarrow 2} \frac{2-x}{(x-2)^3} = \lim_{x \rightarrow 2} \frac{-(x-2)}{(x-2)^3} = \lim_{x \rightarrow 2} \frac{-1}{(x-2)^2} = -\infty$

In parts (a) and (b) the effect of the zero in the denominator at $x = 2$ is canceled because the numerator is zero there also. Thus a finite limit exists. This is not true in part (f), where cancellation still leaves a zero factor in the denominator. ■

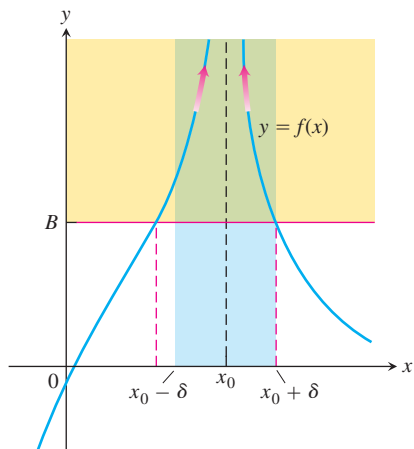


FIGURE 2.62 For $x_0 - \delta < x < x_0 + \delta$, the graph of $f(x)$ lies above the line $y = B$.

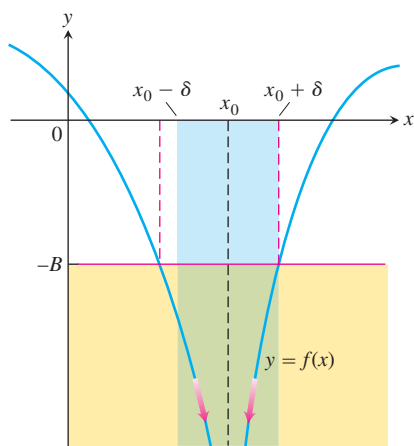


FIGURE 2.63 For $x_0 - \delta < x < x_0 + \delta$, the graph of $f(x)$ lies below the line $y = -B$.

Precise Definitions of Infinite Limits

Instead of requiring $f(x)$ to lie arbitrarily close to a finite number L for all x sufficiently close to x_0 , the definitions of infinite limits require $f(x)$ to lie arbitrarily far from zero. Except for this change, the language is very similar to what we have seen before. Figures 2.62 and 2.63 accompany these definitions.

DEFINITIONS

1. We say that $f(x)$ approaches infinity as x approaches x_0 , and write

$$\lim_{x \rightarrow x_0} f(x) = \infty,$$

if for every positive real number B there exists a corresponding $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \implies f(x) > B.$$

2. We say that $f(x)$ approaches minus infinity as x approaches x_0 , and write

$$\lim_{x \rightarrow x_0} f(x) = -\infty,$$

if for every negative real number $-B$ there exists a corresponding $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \implies f(x) < -B.$$

The precise definitions of one-sided infinite limits at x_0 are similar and are stated in the exercises.

EXAMPLE 14 Prove that $\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$.

Solution Given $B > 0$, we want to find $\delta > 0$ such that

$$0 < |x - 0| < \delta \implies \frac{1}{x^2} > B.$$

Now,

$$\frac{1}{x^2} > B \quad \text{if and only if} \quad x^2 < \frac{1}{B}$$

or, equivalently,

$$|x| < \frac{1}{\sqrt{B}}.$$

Thus, choosing $\delta = 1/\sqrt{B}$ (or any smaller positive number), we see that

$$|x| < \delta \quad \text{implies} \quad \frac{1}{x^2} > \frac{1}{\delta^2} \geq B.$$

Therefore, by definition,

$$\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty. \quad \blacksquare$$

Vertical Asymptotes

Notice that the distance between a point on the graph of $f(x) = 1/x$ and the y -axis approaches zero as the point moves vertically along the graph and away from the origin (Figure 2.64). The function $f(x) = 1/x$ is unbounded as x approaches 0 because

$$\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty \quad \text{and} \quad \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty.$$

We say that the line $x = 0$ (the y -axis) is a *vertical asymptote* of the graph of $f(x) = 1/x$. Observe that the denominator is zero at $x = 0$ and the function is undefined there.

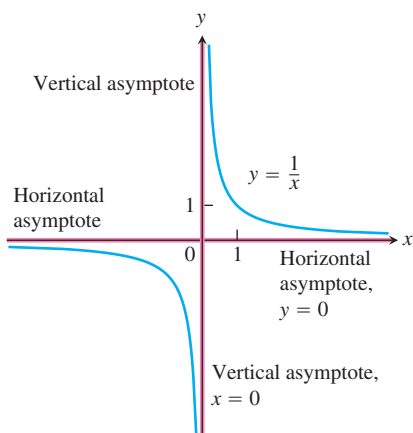


FIGURE 2.64 The coordinate axes are asymptotes of both branches of the hyperbola $y = 1/x$.

DEFINITION A line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \rightarrow a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \rightarrow a^-} f(x) = \pm\infty.$$

EXAMPLE 15 Find the horizontal and vertical asymptotes of the curve

$$y = \frac{x+3}{x+2}.$$

Solution We are interested in the behavior as $x \rightarrow \pm\infty$ and the behavior as $x \rightarrow -2$, where the denominator is zero.

The asymptotes are quickly revealed if we recast the rational function as a polynomial with a remainder, by dividing $(x+3)$ into $(x+2)$:

$$\begin{array}{r} 1 \\ x+2 \overline{) x+3} \\ \underline{x+2} \\ 1 \end{array}$$

This result enables us to rewrite y as:

$$y = 1 + \frac{1}{x+2}.$$

As $x \rightarrow \pm\infty$, the curve approaches the horizontal asymptote $y = 1$; as $x \rightarrow -2$, the curve approaches the vertical asymptote $x = -2$. We see that the curve in question is the graph of $f(x) = 1/x$ shifted 1 unit up and 2 units left (Figure 2.65). The asymptotes, instead of being the coordinate axes, are now the lines $y = 1$ and $x = -2$. ■

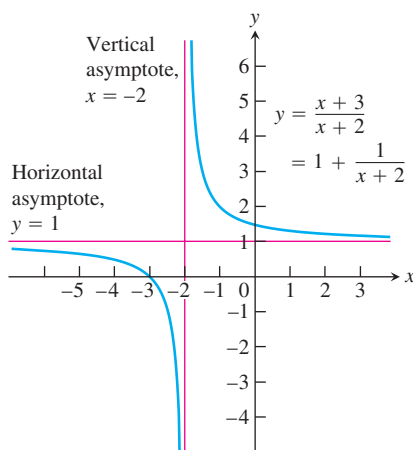


FIGURE 2.65 The lines $y = 1$ and $x = -2$ are asymptotes of the curve in Example 15.

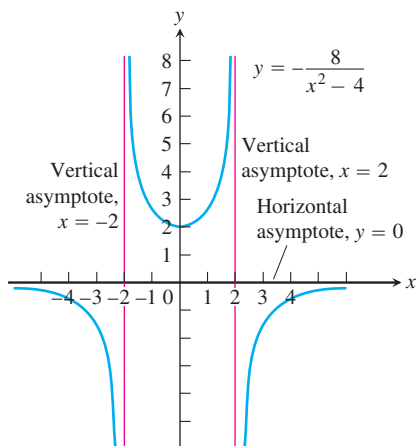


FIGURE 2.66 Graph of the function in Example 16. Notice that the curve approaches the x -axis from only one side. Asymptotes do not have to be two-sided.

EXAMPLE 16 Find the horizontal and vertical asymptotes of the graph of

$$f(x) = -\frac{8}{x^2 - 4}.$$

Solution We are interested in the behavior as $x \rightarrow \pm\infty$ and as $x \rightarrow \pm 2$, where the denominator is zero. Notice that f is an even function of x , so its graph is symmetric with respect to the y -axis.

(a) *The behavior as $x \rightarrow \pm\infty$.* Since $\lim_{x \rightarrow \infty} f(x) = 0$, the line $y = 0$ is a horizontal asymptote of the graph to the right. By symmetry it is an asymptote to the left as well (Figure 2.66). Notice that the curve approaches the x -axis from only the negative side (or from below). Also, $f(0) = 2$.

(b) *The behavior as $x \rightarrow \pm 2$.* Since

$$\lim_{x \rightarrow 2^+} f(x) = -\infty \quad \text{and} \quad \lim_{x \rightarrow 2^-} f(x) = \infty,$$

the line $x = 2$ is a vertical asymptote both from the right and from the left. By symmetry, the line $x = -2$ is also a vertical asymptote.

There are no other asymptotes because f has a finite limit at every other point. ■

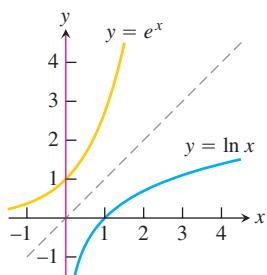


FIGURE 2.67 The line $x = 0$ is a vertical asymptote of the natural logarithm function (Example 17).

EXAMPLE 17 The graph of the natural logarithm function has the y -axis (the line $x = 0$) as a vertical asymptote. We see this from the graph sketched in Figure 2.67 (which is the reflection of the graph of the natural exponential function across the line $y = x$) and the fact that the x -axis is a horizontal asymptote of $y = e^x$ (Example 5). Thus,

$$\lim_{x \rightarrow 0^+} \ln x = -\infty.$$

The same result is true for $y = \log_a x$ whenever $a > 1$. ■

EXAMPLE 18 The curves

$$y = \sec x = \frac{1}{\cos x} \quad \text{and} \quad y = \tan x = \frac{\sin x}{\cos x}$$

both have vertical asymptotes at odd-integer multiples of $\pi/2$, where $\cos x = 0$ (Figure 2.68).

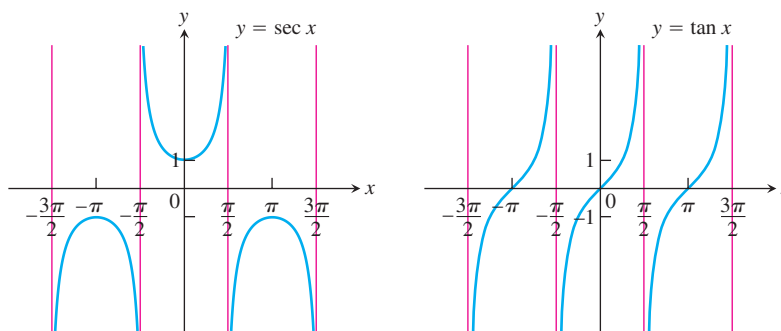


FIGURE 2.68 The graphs of $\sec x$ and $\tan x$ have infinitely many vertical asymptotes (Example 18). ■

Dominant Terms

In Example 10 we saw that by long division we could rewrite the function

$$f(x) = \frac{x^2 - 3}{2x - 4}$$

as a linear function plus a remainder term:

$$f(x) = \left(\frac{x}{2} + 1\right) + \left(\frac{1}{2x - 4}\right).$$

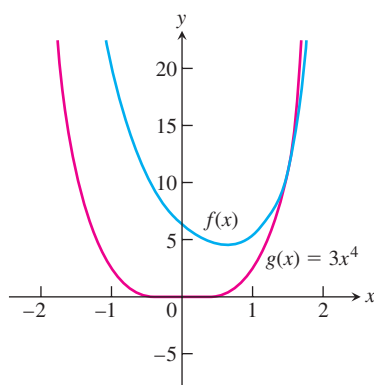
This tells us immediately that

$$f(x) \approx \frac{x}{2} + 1 \quad \text{For } x \text{ numerically large, } \frac{1}{2x - 4} \text{ is near 0.}$$

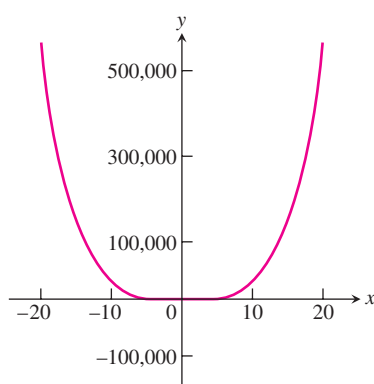
$$f(x) \approx \frac{1}{2x - 4} \quad \text{For } x \text{ near 2, this term is very large.}$$

If we want to know how f behaves, this is the way to find out. It behaves like $y = (x/2) + 1$ when x is numerically large and the contribution of $1/(2x - 4)$ to the total value of f is insignificant. It behaves like $1/(2x - 4)$ when x is so close to 2 that $1/(2x - 4)$ makes the dominant contribution.

We say that $(x/2) + 1$ **dominates** when x is numerically large, and we say that $1/(2x - 4)$ dominates when x is near 2. **Dominant terms** like these help us predict a function's behavior.



(a)



(b)

FIGURE 2.69 The graphs of f and g are (a) distinct for $|x|$ small, and (b) nearly identical for $|x|$ large (Example 19).

EXAMPLE 19 Let $f(x) = 3x^4 - 2x^3 + 3x^2 - 5x + 6$ and $g(x) = 3x^4$. Show that although f and g are quite different for numerically small values of x , they are virtually identical for $|x|$ very large, in the sense that their ratios approach 1 as $x \rightarrow \infty$ or $x \rightarrow -\infty$.

Solution The graphs of f and g behave quite differently near the origin (Figure 2.69a), but appear as virtually identical on a larger scale (Figure 2.69b).

We can test that the term $3x^4$ in f , represented graphically by g , dominates the polynomial f for numerically large values of x by examining the ratio of the two functions as $x \rightarrow \pm\infty$. We find that

$$\begin{aligned} \lim_{x \rightarrow \pm\infty} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow \pm\infty} \frac{3x^4 - 2x^3 + 3x^2 - 5x + 6}{3x^4} \\ &= \lim_{x \rightarrow \pm\infty} \left(1 - \frac{2}{3x} + \frac{1}{x^2} - \frac{5}{3x^3} + \frac{2}{x^4}\right) \\ &= 1, \end{aligned}$$

which means that f and g appear nearly identical for $|x|$ large. ■

Summary

In this chapter we presented several important calculus ideas that are made meaningful and precise by the concept of the limit. These include the three ideas of the exact rate of change of a function, the slope of the graph of a function at a point, and the continuity of a function. The primary methods used for calculating limits of many functions are captured in the algebraic limit laws of Theorem 1 and in the Sandwich Theorem, all of which are proved from the precise definition of the limit. We saw that these computational rules also apply to one-sided limits and to limits at infinity. Moreover, we can sometimes apply these rules to calculating limits of simple transcendental functions, as illustrated by our examples or in cases like the following:

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{e^{2x} - 1} = \lim_{x \rightarrow 0} \frac{e^x - 1}{(e^x - 1)(e^x + 1)} = \lim_{x \rightarrow 0} \frac{1}{e^x + 1} = \frac{1}{1 + 1} = \frac{1}{2}.$$

However, calculating more complicated limits involving transcendental functions such as

$$\lim_{x \rightarrow 0} \frac{x}{e^{2x} - 1}, \quad \lim_{x \rightarrow 0} \frac{\ln x}{x}, \quad \text{and} \quad \lim_{x \rightarrow 0} \left(1 + \frac{1}{x}\right)^x$$

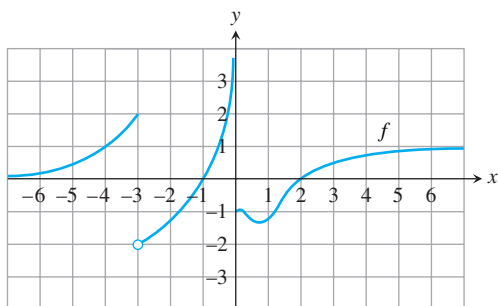
requires more than simple algebraic techniques. The *derivative* is exactly the tool we need to calculate limits in these kinds of cases (see Section 4.5), and this notion is the main subject of our next chapter.

Exercises 2.6

Finding Limits

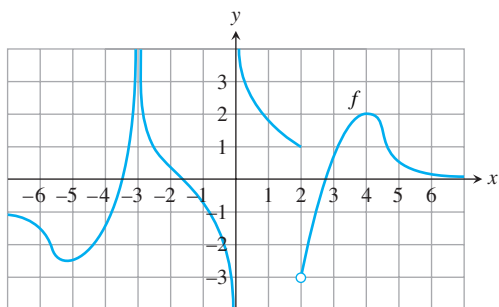
1. For the function f whose graph is given, determine the following limits.

a. $\lim_{x \rightarrow 2} f(x)$	b. $\lim_{x \rightarrow -3^+} f(x)$	c. $\lim_{x \rightarrow -3^-} f(x)$
d. $\lim_{x \rightarrow -3} f(x)$	e. $\lim_{x \rightarrow 0^+} f(x)$	f. $\lim_{x \rightarrow 0^-} f(x)$
g. $\lim_{x \rightarrow 0} f(x)$	h. $\lim_{x \rightarrow \infty} f(x)$	i. $\lim_{x \rightarrow -\infty} f(x)$



2. For the function f whose graph is given, determine the following limits.

a. $\lim_{x \rightarrow 4} f(x)$	b. $\lim_{x \rightarrow 2^+} f(x)$	c. $\lim_{x \rightarrow 2^-} f(x)$
d. $\lim_{x \rightarrow 2} f(x)$	e. $\lim_{x \rightarrow -3^+} f(x)$	f. $\lim_{x \rightarrow -3^-} f(x)$
g. $\lim_{x \rightarrow -3} f(x)$	h. $\lim_{x \rightarrow 0^+} f(x)$	i. $\lim_{x \rightarrow 0^-} f(x)$
j. $\lim_{x \rightarrow 0} f(x)$	k. $\lim_{x \rightarrow \infty} f(x)$	l. $\lim_{x \rightarrow -\infty} f(x)$



In Exercises 3–8, find the limit of each function (a) as $x \rightarrow \infty$ and (b) as $x \rightarrow -\infty$. (You may wish to visualize your answer with a graphing calculator or computer.)

3. $f(x) = \frac{2}{x} - 3$

4. $f(x) = \pi - \frac{2}{x^2}$

5. $g(x) = \frac{1}{2 + (1/x)}$

6. $g(x) = \frac{1}{8 - (5/x^2)}$

7. $h(x) = \frac{-5 + (7/x)}{3 - (1/x^2)}$

8. $h(x) = \frac{3 - (2/x)}{4 + (\sqrt{2}/x^2)}$

Find the limits in Exercises 9–12.

9. $\lim_{x \rightarrow \infty} \frac{\sin 2x}{x}$

10. $\lim_{\theta \rightarrow -\infty} \frac{\cos \theta}{3\theta}$

11. $\lim_{t \rightarrow -\infty} \frac{2 - t + \sin t}{t + \cos t}$

12. $\lim_{r \rightarrow \infty} \frac{r + \sin r}{2r + 7 - 5 \sin r}$

Limits of Rational Functions

In Exercises 13–22, find the limit of each rational function (a) as $x \rightarrow \infty$ and (b) as $x \rightarrow -\infty$.

13. $f(x) = \frac{2x + 3}{5x + 7}$

14. $f(x) = \frac{2x^3 + 7}{x^3 - x^2 + x + 7}$

15. $f(x) = \frac{x + 1}{x^2 + 3}$

16. $f(x) = \frac{3x + 7}{x^2 - 2}$

17. $h(x) = \frac{7x^3}{x^3 - 3x^2 + 6x}$

18. $g(x) = \frac{1}{x^3 - 4x + 1}$

19. $g(x) = \frac{10x^5 + x^4 + 31}{x^6}$

20. $h(x) = \frac{9x^4 + x}{2x^4 + 5x^2 - x + 6}$

21. $h(x) = \frac{-2x^3 - 2x + 3}{3x^3 + 3x^2 - 5x}$

22. $h(x) = \frac{-x^4}{x^4 - 7x^3 + 7x^2 + 9}$

Limits as $x \rightarrow \infty$ or $x \rightarrow -\infty$

The process by which we determine limits of rational functions applies equally well to ratios containing noninteger or negative powers of x : divide numerator and denominator by the highest power of x in the denominator and proceed from there. Find the limits in Exercises 23–36.

23. $\lim_{x \rightarrow \infty} \sqrt{\frac{8x^2 - 3}{2x^2 + x}}$

24. $\lim_{x \rightarrow -\infty} \left(\frac{x^2 + x - 1}{8x^2 - 3}\right)^{1/3}$

25. $\lim_{x \rightarrow -\infty} \left(\frac{1 - x^3}{x^2 + 7x}\right)^5$

26. $\lim_{x \rightarrow \infty} \sqrt{\frac{x^2 - 5x}{x^3 + x - 2}}$

27. $\lim_{x \rightarrow \infty} \frac{2\sqrt{x} + x^{-1}}{3x - 7}$

28. $\lim_{x \rightarrow \infty} \frac{2 + \sqrt{x}}{2 - \sqrt{x}}$

29. $\lim_{x \rightarrow -\infty} \frac{\sqrt[3]{x} - \sqrt[5]{x}}{\sqrt[3]{x} + \sqrt[5]{x}}$

30. $\lim_{x \rightarrow \infty} \frac{x^{-1} + x^{-4}}{x^{-2} - x^{-3}}$

31. $\lim_{x \rightarrow \infty} \frac{2x^{5/3} - x^{1/3} + 7}{x^{8/5} + 3x + \sqrt{x}}$

32. $\lim_{x \rightarrow -\infty} \frac{\sqrt[3]{x} - 5x + 3}{2x + x^{2/3} - 4}$

$$33. \lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 1}}{x + 1} \qquad 34. \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2 + 1}}{x + 1}$$

$$35. \lim_{x \rightarrow \infty} \frac{x - 3}{\sqrt{4x^2 + 25}} \qquad 36. \lim_{x \rightarrow -\infty} \frac{4 - 3x^3}{\sqrt{x^6 + 9}}$$

Infinite Limits

Find the limits in Exercises 37–48.

$$37. \lim_{x \rightarrow 0^+} \frac{1}{3x} \qquad 38. \lim_{x \rightarrow 0^-} \frac{5}{2x}$$

$$39. \lim_{x \rightarrow 2^-} \frac{3}{x - 2} \qquad 40. \lim_{x \rightarrow 3^+} \frac{1}{x - 3}$$

$$41. \lim_{x \rightarrow -8^+} \frac{2x}{x + 8} \qquad 42. \lim_{x \rightarrow -5^-} \frac{3x}{2x + 10}$$

$$43. \lim_{x \rightarrow 7} \frac{4}{(x - 7)^2} \qquad 44. \lim_{x \rightarrow 0} \frac{-1}{x^2(x + 1)}$$

$$45. \text{ a. } \lim_{x \rightarrow 0^+} \frac{2}{3x^{1/3}} \qquad \text{ b. } \lim_{x \rightarrow 0^-} \frac{2}{3x^{1/3}}$$

$$46. \text{ a. } \lim_{x \rightarrow 0^+} \frac{2}{x^{1/5}} \qquad \text{ b. } \lim_{x \rightarrow 0^-} \frac{2}{x^{1/5}}$$

$$47. \lim_{x \rightarrow 0} \frac{4}{x^{2/5}} \qquad 48. \lim_{x \rightarrow 0} \frac{1}{x^{2/3}}$$

Find the limits in Exercises 49–52.

$$49. \lim_{x \rightarrow (\pi/2)^-} \tan x \qquad 50. \lim_{x \rightarrow (-\pi/2)^+} \sec x$$

$$51. \lim_{\theta \rightarrow 0^-} (1 + \csc \theta) \qquad 52. \lim_{\theta \rightarrow 0} (2 - \cot \theta)$$

Find the limits in Exercises 53–58.

$$53. \lim_{x \rightarrow 2} \frac{1}{x^2 - 4} \text{ as}$$

$$\text{ a. } x \rightarrow 2^+ \qquad \text{ b. } x \rightarrow 2^-$$

$$\text{ c. } x \rightarrow -2^+ \qquad \text{ d. } x \rightarrow -2^-$$

$$54. \lim_{x \rightarrow 1} \frac{x}{x^2 - 1} \text{ as}$$

$$\text{ a. } x \rightarrow 1^+ \qquad \text{ b. } x \rightarrow 1^-$$

$$\text{ c. } x \rightarrow -1^+ \qquad \text{ d. } x \rightarrow -1^-$$

$$55. \lim_{x \rightarrow 0} \left(\frac{x^2}{2} - \frac{1}{x} \right) \text{ as}$$

$$\text{ a. } x \rightarrow 0^+ \qquad \text{ b. } x \rightarrow 0^-$$

$$\text{ c. } x \rightarrow \sqrt[3]{2} \qquad \text{ d. } x \rightarrow -1$$

$$56. \lim_{x \rightarrow -4} \frac{x^2 - 1}{2x + 4} \text{ as}$$

$$\text{ a. } x \rightarrow -2^+ \qquad \text{ b. } x \rightarrow -2^-$$

$$\text{ c. } x \rightarrow 1^+ \qquad \text{ d. } x \rightarrow 0^-$$

$$57. \lim_{x \rightarrow 2} \frac{x^2 - 3x + 2}{x^3 - 2x^2} \text{ as}$$

$$\text{ a. } x \rightarrow 0^+ \qquad \text{ b. } x \rightarrow 2^+$$

$$\text{ c. } x \rightarrow 2^- \qquad \text{ d. } x \rightarrow 2$$

$$\text{ e. What, if anything, can be said about the limit as } x \rightarrow 0?$$

$$58. \lim_{x \rightarrow 0} \frac{x^2 - 3x + 2}{x^3 - 4x} \text{ as}$$

$$\text{ a. } x \rightarrow 2^+ \qquad \text{ b. } x \rightarrow -2^+$$

$$\text{ c. } x \rightarrow 0^- \qquad \text{ d. } x \rightarrow 1^+$$

$$\text{ e. What, if anything, can be said about the limit as } x \rightarrow 0?$$

Find the limits in Exercises 59–62.

$$59. \lim_{t \rightarrow 0} \left(2 - \frac{3}{t^{1/3}} \right) \text{ as}$$

$$\text{ a. } t \rightarrow 0^+ \qquad \text{ b. } t \rightarrow 0^-$$

$$60. \lim_{t \rightarrow 0} \left(\frac{1}{t^{3/5}} + 7 \right) \text{ as}$$

$$\text{ a. } t \rightarrow 0^+ \qquad \text{ b. } t \rightarrow 0^-$$

$$61. \lim_{x \rightarrow 1} \left(\frac{1}{x^{2/3}} + \frac{2}{(x - 1)^{2/3}} \right) \text{ as}$$

$$\text{ a. } x \rightarrow 0^+ \qquad \text{ b. } x \rightarrow 0^-$$

$$\text{ c. } x \rightarrow 1^+ \qquad \text{ d. } x \rightarrow 1^-$$

$$62. \lim_{x \rightarrow 1} \left(\frac{1}{x^{1/3}} - \frac{1}{(x - 1)^{4/3}} \right) \text{ as}$$

$$\text{ a. } x \rightarrow 0^+ \qquad \text{ b. } x \rightarrow 0^-$$

$$\text{ c. } x \rightarrow 1^+ \qquad \text{ d. } x \rightarrow 1^-$$

Graphing Simple Rational Functions

Graph the rational functions in Exercises 63–68. Include the graphs and equations of the asymptotes and dominant terms.

$$63. y = \frac{1}{x - 1} \qquad 64. y = \frac{1}{x + 1}$$

$$65. y = \frac{1}{2x + 4} \qquad 66. y = \frac{-3}{x - 3}$$

$$67. y = \frac{x + 3}{x + 2} \qquad 68. y = \frac{2x}{x + 1}$$

Inventing Graphs and FunctionsIn Exercises 69–72, sketch the graph of a function $y = f(x)$ that satisfies the given conditions. No formulas are required—just label the coordinate axes and sketch an appropriate graph. (The answers are not unique, so your graphs may not be exactly like those in the answer section.)

$$69. f(0) = 0, f(1) = 2, f(-1) = -2, \lim_{x \rightarrow -\infty} f(x) = -1, \text{ and}$$

$$\lim_{x \rightarrow \infty} f(x) = 1$$

$$70. f(0) = 0, \lim_{x \rightarrow \pm\infty} f(x) = 0, \lim_{x \rightarrow 0^+} f(x) = 2, \text{ and}$$

$$\lim_{x \rightarrow 0^-} f(x) = -2$$

$$71. f(0) = 0, \lim_{x \rightarrow \pm\infty} f(x) = 0, \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow -1^+} f(x) = \infty,$$

$$\lim_{x \rightarrow 1^+} f(x) = -\infty, \text{ and } \lim_{x \rightarrow -1^-} f(x) = -\infty$$

$$72. f(2) = 1, f(-1) = 0, \lim_{x \rightarrow \infty} f(x) = 0, \lim_{x \rightarrow 0^+} f(x) = \infty,$$

$$\lim_{x \rightarrow 0^-} f(x) = -\infty, \text{ and } \lim_{x \rightarrow -\infty} f(x) = 1$$

In Exercises 73–76, find a function that satisfies the given conditions and sketch its graph. (The answers here are not unique. Any function that satisfies the conditions is acceptable. Feel free to use formulas defined in pieces if that will help.)

$$73. \lim_{x \rightarrow \pm\infty} f(x) = 0, \lim_{x \rightarrow 2^-} f(x) = \infty, \text{ and } \lim_{x \rightarrow 2^+} f(x) = \infty$$

$$74. \lim_{x \rightarrow \pm\infty} g(x) = 0, \lim_{x \rightarrow 3^-} g(x) = -\infty, \text{ and } \lim_{x \rightarrow 3^+} g(x) = \infty$$

$$75. \lim_{x \rightarrow -\infty} h(x) = -1, \lim_{x \rightarrow \infty} h(x) = 1, \lim_{x \rightarrow 0^-} h(x) = -1, \text{ and}$$

$$\lim_{x \rightarrow 0^+} h(x) = 1$$

$$76. \lim_{x \rightarrow \pm\infty} k(x) = 1, \lim_{x \rightarrow 1^-} k(x) = \infty, \text{ and } \lim_{x \rightarrow 1^+} k(x) = -\infty$$

77. Suppose that $f(x)$ and $g(x)$ are polynomials in x and that $\lim_{x \rightarrow \infty} (f(x)/g(x)) = 2$. Can you conclude anything about $\lim_{x \rightarrow -\infty} (f(x)/g(x))$? Give reasons for your answer.
78. Suppose that $f(x)$ and $g(x)$ are polynomials in x . Can the graph of $f(x)/g(x)$ have an asymptote if $g(x)$ is never zero? Give reasons for your answer.
79. How many horizontal asymptotes can the graph of a given rational function have? Give reasons for your answer.

Finding Limits of Differences when $x \rightarrow \pm\infty$

Find the limits in Exercises 80–86.

80. $\lim_{x \rightarrow \infty} (\sqrt{x+9} - \sqrt{x+4})$
81. $\lim_{x \rightarrow \infty} (\sqrt{x^2+25} - \sqrt{x^2-1})$
82. $\lim_{x \rightarrow -\infty} (\sqrt{x^2+3} + x)$
83. $\lim_{x \rightarrow -\infty} (2x + \sqrt{4x^2+3x-2})$
84. $\lim_{x \rightarrow \infty} (\sqrt{9x^2-x-3x})$
85. $\lim_{x \rightarrow \infty} (\sqrt{x^2+3x} - \sqrt{x^2-2x})$
86. $\lim_{x \rightarrow \infty} (\sqrt{x^2+x} - \sqrt{x^2-x})$

Using the Formal Definitions

Use the formal definitions of limits as $x \rightarrow \pm\infty$ to establish the limits in Exercises 87 and 88.

87. If f has the constant value $f(x) = k$, then $\lim_{x \rightarrow \infty} f(x) = k$.
88. If f has the constant value $f(x) = k$, then $\lim_{x \rightarrow -\infty} f(x) = k$.

Use formal definitions to prove the limit statements in Exercises 89–92.

89. $\lim_{x \rightarrow 0} \frac{-1}{x^2} = -\infty$
90. $\lim_{x \rightarrow 0} \frac{1}{|x|} = \infty$
91. $\lim_{x \rightarrow 3} \frac{-2}{(x-3)^2} = -\infty$
92. $\lim_{x \rightarrow -5} \frac{1}{(x+5)^2} = \infty$

93. Here is the definition of **infinite right-hand limit**.

We say that $f(x)$ approaches infinity as x approaches x_0 from the right, and write

$$\lim_{x \rightarrow x_0^+} f(x) = \infty,$$

if, for every positive real number B , there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 < x < x_0 + \delta \quad \Rightarrow \quad f(x) > B.$$

Modify the definition to cover the following cases.

- a. $\lim_{x \rightarrow x_0^-} f(x) = \infty$
- b. $\lim_{x \rightarrow x_0^+} f(x) = -\infty$
- c. $\lim_{x \rightarrow x_0^-} f(x) = -\infty$

Use the formal definitions from Exercise 93 to prove the limit statements in Exercises 94–98.

94. $\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty$
95. $\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$
96. $\lim_{x \rightarrow 2^-} \frac{1}{x-2} = -\infty$
97. $\lim_{x \rightarrow 2^+} \frac{1}{x-2} = \infty$
98. $\lim_{x \rightarrow 1^-} \frac{1}{1-x^2} = \infty$

Oblique Asymptotes

Graph the rational functions in Exercises 99–104. Include the graphs and equations of the asymptotes.

99. $y = \frac{x^2}{x-1}$
100. $y = \frac{x^2+1}{x-1}$
101. $y = \frac{x^2-4}{x-1}$
102. $y = \frac{x^2-1}{2x+4}$
103. $y = \frac{x^2-1}{x}$
104. $y = \frac{x^3+1}{x^2}$

Additional Graphing Exercises

- T** Graph the curves in Exercises 105–108. Explain the relationship between the curve's formula and what you see.

105. $y = \frac{x}{\sqrt{4-x^2}}$
106. $y = \frac{-1}{\sqrt{4-x^2}}$
107. $y = x^{2/3} + \frac{1}{x^{1/3}}$
108. $y = \sin\left(\frac{\pi}{x^2+1}\right)$

- T** Graph the functions in Exercises 109 and 110. Then answer the following questions.

- a. How does the graph behave as $x \rightarrow 0^+$?
- b. How does the graph behave as $x \rightarrow \pm\infty$?
- c. How does the graph behave near $x = 1$ and $x = -1$?

Give reasons for your answers.

109. $y = \frac{3}{2} \left(x - \frac{1}{x}\right)^{2/3}$
110. $y = \frac{3}{2} \left(\frac{x}{x-1}\right)^{2/3}$

Chapter 2 Questions to Guide Your Review

- What is the average rate of change of the function $g(t)$ over the interval from $t = a$ to $t = b$? How is it related to a secant line?
- What limit must be calculated to find the rate of change of a function $g(t)$ at $t = t_0$?

- Give an informal or intuitive definition of the limit

$$\lim_{x \rightarrow x_0} f(x) = L.$$

Why is the definition “informal”? Give examples.

- Does the existence and value of the limit of a function $f(x)$ as x approaches x_0 ever depend on what happens at $x = x_0$? Explain and give examples.
- What function behaviors might occur for which the limit may fail to exist? Give examples.
- What theorems are available for calculating limits? Give examples of how the theorems are used.
- How are one-sided limits related to limits? How can this relationship sometimes be used to calculate a limit or prove it does not exist? Give examples.
- What is the value of $\lim_{\theta \rightarrow 0} ((\sin \theta)/\theta)$? Does it matter whether θ is measured in degrees or radians? Explain.
- What exactly does $\lim_{x \rightarrow x_0} f(x) = L$ mean? Give an example in which you find a $\delta > 0$ for a given f , L , x_0 , and $\epsilon > 0$ in the precise definition of limit.
- Give precise definitions of the following statements.
 - $\lim_{x \rightarrow 2^-} f(x) = 5$
 - $\lim_{x \rightarrow 2^+} f(x) = 5$
 - $\lim_{x \rightarrow 2} f(x) = \infty$
 - $\lim_{x \rightarrow 2} f(x) = -\infty$
- What conditions must be satisfied by a function if it is to be continuous at an interior point of its domain? At an endpoint?
- How can looking at the graph of a function help you tell where the function is continuous?
- What does it mean for a function to be right-continuous at a point? Left-continuous? How are continuity and one-sided continuity related?
- What does it mean for a function to be continuous on an interval? Give examples to illustrate the fact that a function that is not continuous on its entire domain may still be continuous on selected intervals within the domain.
- What are the basic types of discontinuity? Give an example of each. What is a removable discontinuity? Give an example.
- What does it mean for a function to have the Intermediate Value Property? What conditions guarantee that a function has this property over an interval? What are the consequences for graphing and solving the equation $f(x) = 0$?
- Under what circumstances can you extend a function $f(x)$ to be continuous at a point $x = c$? Give an example.
- What exactly do $\lim_{x \rightarrow \infty} f(x) = L$ and $\lim_{x \rightarrow -\infty} f(x) = L$ mean? Give examples.
- What are $\lim_{x \rightarrow \pm\infty} k$ (k a constant) and $\lim_{x \rightarrow \pm\infty} (1/x)$? How do you extend these results to other functions? Give examples.
- How do you find the limit of a rational function as $x \rightarrow \pm\infty$? Give examples.
- What are horizontal and vertical asymptotes? Give examples.

Chapter 2 Practice Exercises

Limits and Continuity

- Graph the function

$$f(x) = \begin{cases} 1, & x \leq -1 \\ -x, & -1 < x < 0 \\ 1, & x = 0 \\ -x, & 0 < x < 1 \\ 1, & x \geq 1. \end{cases}$$

Then discuss, in detail, limits, one-sided limits, continuity, and one-sided continuity of f at $x = -1$, 0 , and 1 . Are any of the discontinuities removable? Explain.

- Repeat the instructions of Exercise 1 for

$$f(x) = \begin{cases} 0, & x \leq -1 \\ 1/x, & 0 < |x| < 1 \\ 0, & x = 1 \\ 1, & x > 1. \end{cases}$$

- Suppose that $f(t)$ and $g(t)$ are defined for all t and that $\lim_{t \rightarrow t_0} f(t) = -7$ and $\lim_{t \rightarrow t_0} g(t) = 0$. Find the limit as $t \rightarrow t_0$ of the following functions.

- $3f(t)$
- $(f(t))^2$
- $f(t) \cdot g(t)$
- $\frac{f(t)}{g(t) - 7}$

- $\cos(g(t))$
- $|f(t)|$
- $f(t) + g(t)$
- $1/f(t)$

- Suppose the functions $f(x)$ and $g(x)$ are defined for all x and that $\lim_{x \rightarrow 0} f(x) = 1/2$ and $\lim_{x \rightarrow 0} g(x) = \sqrt{2}$. Find the limits as $x \rightarrow 0$ of the following functions.

- $-g(x)$
- $g(x) \cdot f(x)$
- $f(x) + g(x)$
- $1/f(x)$
- $x + f(x)$
- $\frac{f(x) \cdot \cos x}{x - 1}$

In Exercises 5 and 6, find the value that $\lim_{x \rightarrow 0} g(x)$ must have if the given limit statements hold.

$$5. \lim_{x \rightarrow 0} \left(\frac{4 - g(x)}{x} \right) = 1$$

$$6. \lim_{x \rightarrow -4} \left(x \lim_{x \rightarrow 0} g(x) \right) = 2$$

- On what intervals are the following functions continuous?

- $f(x) = x^{1/3}$
- $g(x) = x^{3/4}$
- $h(x) = x^{-2/3}$
- $k(x) = x^{-1/6}$

- On what intervals are the following functions continuous?

- $f(x) = \tan x$
- $g(x) = \csc x$
- $h(x) = \frac{\cos x}{x - \pi}$
- $k(x) = \frac{\sin x}{x}$

Finding Limits

In Exercises 9–28, find the limit or explain why it does not exist.

9. $\lim_{x \rightarrow 0} \frac{x^2 - 4x + 4}{x^3 + 5x^2 - 14x}$
 a. as $x \rightarrow 0$ b. as $x \rightarrow 2$
10. $\lim_{x \rightarrow 0} \frac{x^2 + x}{x^5 + 2x^4 + x^3}$
 a. as $x \rightarrow 0$ b. as $x \rightarrow -1$
11. $\lim_{x \rightarrow 1} \frac{1 - \sqrt{x}}{1 - x}$ 12. $\lim_{x \rightarrow a} \frac{x^2 - a^2}{x^4 - a^4}$
13. $\lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h}$ 14. $\lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h}$
15. $\lim_{x \rightarrow 0} \frac{\frac{1}{2+x} - \frac{1}{2}}{x}$ 16. $\lim_{x \rightarrow 0} \frac{(2+x)^3 - 8}{x}$
17. $\lim_{x \rightarrow 1} \frac{x^{1/3} - 1}{\sqrt{x} - 1}$ 18. $\lim_{x \rightarrow 64} \frac{x^{2/3} - 16}{\sqrt{x} - 8}$
19. $\lim_{x \rightarrow 0} \frac{\tan(2x)}{\tan(\pi x)}$ 20. $\lim_{x \rightarrow \pi^-} \csc x$
21. $\lim_{x \rightarrow \pi} \sin\left(\frac{x}{2} + \sin x\right)$ 22. $\lim_{x \rightarrow \pi} \cos^2(x - \tan x)$
23. $\lim_{x \rightarrow 0} \frac{8x}{3 \sin x - x}$ 24. $\lim_{x \rightarrow 0} \frac{\cos 2x - 1}{\sin x}$
25. $\lim_{t \rightarrow 3^+} \ln(t - 3)$ 26. $\lim_{t \rightarrow 1} t^2 \ln(2 - \sqrt{t})$
27. $\lim_{\theta \rightarrow 0^+} \sqrt{\theta} e^{\cos(\pi/\theta)}$ 28. $\lim_{z \rightarrow 0^+} \frac{2e^{1/z}}{e^{1/z} + 1}$

In Exercises 29–32, find the limit of $g(x)$ as x approaches the indicated value.

29. $\lim_{x \rightarrow 0^+} (4g(x))^{1/3} = 2$
30. $\lim_{x \rightarrow \sqrt{5}} \frac{1}{x + g(x)} = 2$
31. $\lim_{x \rightarrow 1} \frac{3x^2 + 1}{g(x)} = \infty$
32. $\lim_{x \rightarrow -2} \frac{5 - x^2}{\sqrt{g(x)}} = 0$

Continuous Extension

33. Can $f(x) = x(x^2 - 1)/|x^2 - 1|$ be extended to be continuous at $x = 1$ or -1 ? Give reasons for your answers. (Graph the function—you will find the graph interesting.)
34. Explain why the function $f(x) = \sin(1/x)$ has no continuous extension to $x = 0$.

T In Exercises 35–38, graph the function to see whether it appears to have a continuous extension to the given point a . If it does, use Trace and Zoom to find a good candidate for the extended function's value at a . If the function does not appear to have a continuous extension, can it be extended to be continuous from the right or left? If so, what do you think the extended function's value should be?

35. $f(x) = \frac{x - 1}{x - \sqrt[4]{x}}$, $a = 1$
36. $g(\theta) = \frac{5 \cos \theta}{4\theta - 2\pi}$, $a = \pi/2$
37. $h(t) = (1 + |t|)^{1/t}$, $a = 0$
38. $k(x) = \frac{x}{1 - 2^{|x|}}$, $a = 0$

Roots

- T** 39. Let $f(x) = x^3 - x - 1$.
- a. Use the Intermediate Value Theorem to show that f has a zero between -1 and 2 .
- b. Solve the equation $f(x) = 0$ graphically with an error of magnitude at most 10^{-8} .
- c. It can be shown that the exact value of the solution in part (b) is

$$\left(\frac{1}{2} + \frac{\sqrt{69}}{18}\right)^{1/3} + \left(\frac{1}{2} - \frac{\sqrt{69}}{18}\right)^{1/3}.$$

Evaluate this exact answer and compare it with the value you found in part (b).

- T** 40. Let $f(\theta) = \theta^3 - 2\theta + 2$.
- a. Use the Intermediate Value Theorem to show that f has a zero between -2 and 0 .
- b. Solve the equation $f(\theta) = 0$ graphically with an error of magnitude at most 10^{-4} .
- c. It can be shown that the exact value of the solution in part (b) is

$$\left(\sqrt{\frac{19}{27}} - 1\right)^{1/3} - \left(\sqrt{\frac{19}{27}} + 1\right)^{1/3}.$$

Evaluate this exact answer and compare it with the value you found in part (b).

Limits at Infinity

Find the limits in Exercises 41–54.

41. $\lim_{x \rightarrow \infty} \frac{2x + 3}{5x + 7}$ 42. $\lim_{x \rightarrow \infty} \frac{2x^2 + 3}{5x^2 + 7}$
43. $\lim_{x \rightarrow \infty} \frac{x^2 - 4x + 8}{3x^3}$ 44. $\lim_{x \rightarrow \infty} \frac{1}{x^2 - 7x + 1}$
45. $\lim_{x \rightarrow \infty} \frac{x^2 - 7x}{x + 1}$ 46. $\lim_{x \rightarrow \infty} \frac{x^4 + x^3}{12x^3 + 128}$
47. $\lim_{x \rightarrow \infty} \frac{\sin x}{\lfloor x \rfloor}$ (If you have a grapher, try graphing the function for $-5 \leq x \leq 5$.)
48. $\lim_{\theta \rightarrow \infty} \frac{\cos \theta - 1}{\theta}$ (If you have a grapher, try graphing $f(x) = x(\cos(1/x) - 1)$ near the origin to “see” the limit at infinity.)
49. $\lim_{x \rightarrow \infty} \frac{x + \sin x + 2\sqrt{x}}{x + \sin x}$ 50. $\lim_{x \rightarrow \infty} \frac{x^{2/3} + x^{-1}}{x^{2/3} + \cos^2 x}$
51. $\lim_{x \rightarrow \infty} e^{1/x} \cos \frac{1}{x}$ 52. $\lim_{t \rightarrow \infty} \ln\left(1 + \frac{1}{t}\right)$
53. $\lim_{x \rightarrow -\infty} \tan^{-1} x$ 54. $\lim_{t \rightarrow -\infty} e^{3t} \sin^{-1} \frac{1}{t}$

Horizontal and Vertical Asymptotes

55. Use limits to determine the equations for all vertical asymptotes.

a. $y = \frac{x^2 + 4}{x - 3}$ b. $f(x) = \frac{x^2 - x - 2}{x^2 - 2x + 1}$

c. $y = \frac{x^2 + x - 6}{x^2 + 2x - 8}$

56. Use limits to determine the equations for all horizontal asymptotes.

a. $y = \frac{1 - x^2}{x^2 + 1}$ b. $f(x) = \frac{\sqrt{x + 4}}{\sqrt{x + 4}}$

c. $g(x) = \frac{\sqrt{x^2 + 4}}{x}$ d. $y = \sqrt{\frac{x^2 + 9}{9x^2 + 1}}$

Chapter 2 Additional and Advanced Exercises

T 1. **Assigning a value to 0^0** The rules of exponents tell us that $a^0 = 1$ if a is any number different from zero. They also tell us that $0^n = 0$ if n is any positive number.

If we tried to extend these rules to include the case 0^0 , we would get conflicting results. The first rule would say $0^0 = 1$, whereas the second would say $0^0 = 0$.

We are not dealing with a question of right or wrong here. Neither rule applies as it stands, so there is no contradiction. We could, in fact, define 0^0 to have any value we wanted as long as we could persuade others to agree.

What value would you like 0^0 to have? Here is an example that might help you to decide. (See Exercise 2 below for another example.)

- a. Calculate x^x for $x = 0.1, 0.01, 0.001$, and so on as far as your calculator can go. Record the values you get. What pattern do you see?
- b. Graph the function $y = x^x$ for $0 < x \leq 1$. Even though the function is not defined for $x \leq 0$, the graph will approach the y -axis from the right. Toward what y -value does it seem to be headed? Zoom in to further support your idea.

T 2. **A reason you might want 0^0 to be something other than 0 or 1** As the number x increases through positive values, the numbers $1/x$ and $1/(\ln x)$ both approach zero. What happens to the number

$$f(x) = \left(\frac{1}{x}\right)^{1/(\ln x)}$$

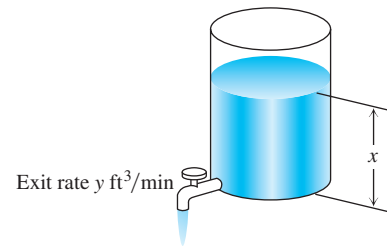
as x increases? Here are two ways to find out.

- a. Evaluate f for $x = 10, 100, 1000$, and so on as far as your calculator can reasonably go. What pattern do you see?
- b. Graph f in a variety of graphing windows, including windows that contain the origin. What do you see? Trace the y -values along the graph. What do you find?
3. **Lorentz contraction** In relativity theory, the length of an object, say a rocket, appears to an observer to depend on the speed at which the object is traveling with respect to the observer. If the observer measures the rocket's length as L_0 at rest, then at speed v the length will appear to be

$$L = L_0 \sqrt{1 - \frac{v^2}{c^2}}$$

This equation is the Lorentz contraction formula. Here, c is the speed of light in a vacuum, about 3×10^8 m/sec. What happens to L as v increases? Find $\lim_{v \rightarrow c^-} L$. Why was the left-hand limit needed?

4. **Controlling the flow from a draining tank** Torricelli's law says that if you drain a tank like the one in the figure shown, the rate y at which water runs out is a constant times the square root of the water's depth x . The constant depends on the size and shape of the exit valve.



Suppose that $y = \sqrt{x}/2$ for a certain tank. You are trying to maintain a fairly constant exit rate by adding water to the tank with a hose from time to time. How deep must you keep the water if you want to maintain the exit rate

- a. within 0.2 ft³/min of the rate $y_0 = 1$ ft³/min?
- b. within 0.1 ft³/min of the rate $y_0 = 1$ ft³/min?

5. **Thermal expansion in precise equipment** As you may know, most metals expand when heated and contract when cooled. The dimensions of a piece of laboratory equipment are sometimes so critical that the shop where the equipment is made must be held at the same temperature as the laboratory where the equipment is to be used. A typical aluminum bar that is 10 cm wide at 70°F will be

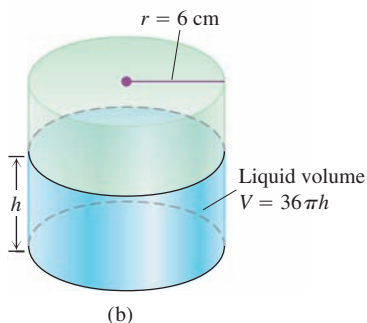
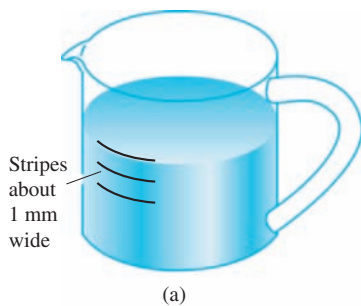
$$y = 10 + (t - 70) \times 10^{-4}$$

centimeters wide at a nearby temperature t . Suppose that you are using a bar like this in a gravity wave detector, where its width must stay within 0.0005 cm of the ideal 10 cm. How close to $t_0 = 70^\circ\text{F}$ must you maintain the temperature to ensure that this tolerance is not exceeded?

6. **Stripes on a measuring cup** The interior of a typical 1-L measuring cup is a right circular cylinder of radius 6 cm (see accompanying figure). The volume of water we put in the cup is therefore a function of the level h to which the cup is filled, the formula being

$$V = \pi 6^2 h = 36\pi h.$$

How closely must we measure h to measure out 1 L of water (1000 cm³) with an error of no more than 1% (10 cm³)?



A 1-L measuring cup (a), modeled as a right circular cylinder (b) of radius $r = 6$ cm

Precise Definition of Limit

In Exercises 7–10, use the formal definition of limit to prove that the function is continuous at x_0 .

7. $f(x) = x^2 - 7, x_0 = 1$ 8. $g(x) = 1/(2x), x_0 = 1/4$

9. $h(x) = \sqrt{2x - 3}, x_0 = 2$ 10. $F(x) = \sqrt{9 - x}, x_0 = 5$

11. **Uniqueness of limits** Show that a function cannot have two different limits at the same point. That is, if $\lim_{x \rightarrow x_0} f(x) = L_1$ and $\lim_{x \rightarrow x_0} f(x) = L_2$, then $L_1 = L_2$.

12. Prove the limit Constant Multiple Rule:

$$\lim_{x \rightarrow c} kf(x) = k \lim_{x \rightarrow c} f(x) \quad \text{for any constant } k.$$

13. **One-sided limits** If $\lim_{x \rightarrow 0^+} f(x) = A$ and $\lim_{x \rightarrow 0^-} f(x) = B$, find

- a. $\lim_{x \rightarrow 0^+} f(x^3 - x)$ b. $\lim_{x \rightarrow 0^-} f(x^3 - x)$
- c. $\lim_{x \rightarrow 0^+} f(x^2 - x^4)$ d. $\lim_{x \rightarrow 0^-} f(x^2 - x^4)$

14. **Limits and continuity** Which of the following statements are true, and which are false? If true, say why; if false, give a counterexample (that is, an example confirming the falsehood).

- a. If $\lim_{x \rightarrow a} f(x)$ exists but $\lim_{x \rightarrow a} g(x)$ does not exist, then $\lim_{x \rightarrow a} (f(x) + g(x))$ does not exist.
- b. If neither $\lim_{x \rightarrow a} f(x)$ nor $\lim_{x \rightarrow a} g(x)$ exists, then $\lim_{x \rightarrow a} (f(x) + g(x))$ does not exist.
- c. If f is continuous at x , then so is $|f|$.
- d. If $|f|$ is continuous at a , then so is f .

In Exercises 15 and 16, use the formal definition of limit to prove that the function has a continuous extension to the given value of x .

15. $f(x) = \frac{x^2 - 1}{x + 1}, x = -1$ 16. $g(x) = \frac{x^2 - 2x - 3}{2x - 6}, x = 3$

17. **A function continuous at only one point** Let

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

- a. Show that f is continuous at $x = 0$.
- b. Use the fact that every nonempty open interval of real numbers contains both rational and irrational numbers to show that f is not continuous at any nonzero value of x .

18. **The Dirichlet ruler function** If x is a rational number, then x can be written in a unique way as a quotient of integers m/n where $n > 0$ and m and n have no common factors greater than 1. (We say that such a fraction is in *lowest terms*. For example, $6/4$ written in lowest terms is $3/2$.) Let $f(x)$ be defined for all x in the interval $[0, 1]$ by

$$f(x) = \begin{cases} 1/n, & \text{if } x = m/n \text{ is a rational number in lowest terms} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

For instance, $f(0) = f(1) = 1, f(1/2) = 1/2, f(1/3) = f(2/3) = 1/3, f(1/4) = f(3/4) = 1/4$, and so on.

- a. Show that f is discontinuous at every rational number in $[0, 1]$.
- b. Show that f is continuous at every irrational number in $[0, 1]$. (*Hint:* If ϵ is a given positive number, show that there are only finitely many rational numbers r in $[0, 1]$ such that $f(r) \geq \epsilon$.)
- c. Sketch the graph of f . Why do you think f is called the “ruler function”?

19. **Antipodal points** Is there any reason to believe that there is always a pair of antipodal (diametrically opposite) points on Earth’s equator where the temperatures are the same? Explain.

20. If $\lim_{x \rightarrow c} (f(x) + g(x)) = 3$ and $\lim_{x \rightarrow c} (f(x) - g(x)) = -1$, find $\lim_{x \rightarrow c} f(x)g(x)$.

21. **Roots of a quadratic equation that is almost linear** The equation $ax^2 + 2x - 1 = 0$, where a is a constant, has two roots if $a > -1$ and $a \neq 0$, one positive and one negative:

$$r_+(a) = \frac{-1 + \sqrt{1 + a}}{a}, \quad r_-(a) = \frac{-1 - \sqrt{1 + a}}{a}.$$

- a. What happens to $r_+(a)$ as $a \rightarrow 0$? As $a \rightarrow -1^+$?
- b. What happens to $r_-(a)$ as $a \rightarrow 0$? As $a \rightarrow -1^+$?
- c. Support your conclusions by graphing $r_+(a)$ and $r_-(a)$ as functions of a . Describe what you see.
- d. For added support, graph $f(x) = ax^2 + 2x - 1$ simultaneously for $a = 1, 0.5, 0.2, 0.1$, and 0.05 .

22. **Root of an equation** Show that the equation $x + 2 \cos x = 0$ has at least one solution.

23. **Bounded functions** A real-valued function f is **bounded from above** on a set D if there exists a number N such that $f(x) \leq N$ for all x in D . We call N , when it exists, an **upper bound** for f on D and say that f is bounded from above by N . In a similar manner, we say that f is **bounded from below** on D if there exists a number M such that $f(x) \geq M$ for all x in D . We call M , when it exists, a **lower bound** for f on D and say that f is bounded from below by M . We say that f is **bounded** on D if it is bounded from both above and below.

- a. Show that f is bounded on D if and only if there exists a number B such that $|f(x)| \leq B$ for all x in D .

b. Suppose that f is bounded from above by N . Show that if $\lim_{x \rightarrow x_0} f(x) = L$, then $L \leq N$.

c. Suppose that f is bounded from below by M . Show that if $\lim_{x \rightarrow x_0} f(x) = L$, then $L \geq M$.

24. Max $\{a, b\}$ and min $\{a, b\}$

a. Show that the expression

$$\max\{a, b\} = \frac{a+b}{2} + \frac{|a-b|}{2}$$

equals a if $a \geq b$ and equals b if $b \geq a$. In other words, $\max\{a, b\}$ gives the larger of the two numbers a and b .

b. Find a similar expression for $\min\{a, b\}$, the smaller of a and b .

Generalized Limits Involving $\frac{\sin \theta}{\theta}$

The formula $\lim_{\theta \rightarrow 0} (\sin \theta)/\theta = 1$ can be generalized. If $\lim_{x \rightarrow c} f(x) = 0$ and $f(x)$ is never zero in an open interval containing the point $x = c$, except possibly c itself, then

$$\lim_{x \rightarrow c} \frac{\sin f(x)}{f(x)} = 1.$$

Here are several examples.

a. $\lim_{x \rightarrow 0} \frac{\sin x^2}{x^2} = 1$

b. $\lim_{x \rightarrow 0} \frac{\sin x^2}{x} = \lim_{x \rightarrow 0} \frac{\sin x^2}{x^2} \lim_{x \rightarrow 0} \frac{x^2}{x} = 1 \cdot 0 = 0$

c. $\lim_{x \rightarrow -1} \frac{\sin(x^2 - x - 2)}{x + 1} = \lim_{x \rightarrow -1} \frac{\sin(x^2 - x - 2)}{(x^2 - x - 2)}$
 $\lim_{x \rightarrow -1} \frac{(x^2 - x - 2)}{x + 1} = 1 \cdot \lim_{x \rightarrow -1} \frac{(x + 1)(x - 2)}{x + 1} = -3$

d. $\lim_{x \rightarrow 1} \frac{\sin(1 - \sqrt{x})}{x - 1} = \lim_{x \rightarrow 1} \frac{\sin(1 - \sqrt{x})}{1 - \sqrt{x}} \frac{1 - \sqrt{x}}{x - 1} =$
 $1 \cdot \lim_{x \rightarrow 1} \frac{(1 - \sqrt{x})(1 + \sqrt{x})}{(x - 1)(1 + \sqrt{x})} = \lim_{x \rightarrow 1} \frac{1 - x}{(x - 1)(1 + \sqrt{x})} = -\frac{1}{2}$

Find the limits in Exercises 25–30.

25. $\lim_{x \rightarrow 0} \frac{\sin(1 - \cos x)}{x}$

26. $\lim_{x \rightarrow 0^+} \frac{\sin x}{\sin \sqrt{x}}$

27. $\lim_{x \rightarrow 0} \frac{\sin(\sin x)}{x}$

28. $\lim_{x \rightarrow 0} \frac{\sin(x^2 + x)}{x}$

29. $\lim_{x \rightarrow 2} \frac{\sin(x^2 - 4)}{x - 2}$

30. $\lim_{x \rightarrow 9} \frac{\sin(\sqrt{x} - 3)}{x - 9}$

Oblique Asymptotes

Find all possible oblique asymptotes in Exercises 31–34.

31. $y = \frac{2x^{3/2} + 2x - 3}{\sqrt{x} + 1}$

32. $y = x + x \sin(1/x)$

33. $y = \sqrt{x^2 + 1}$

34. $y = \sqrt{x^2 + 2x}$

Chapter 2 Technology Application Projects

Mathematica/Maple Modules:

Take It to the Limit

Part I

Part II (Zero Raised to the Power Zero: What Does it Mean?)

Part III (One-Sided Limits)

Visualize and interpret the limit concept through graphical and numerical explorations.

Part IV (What a Difference a Power Makes)

See how sensitive limits can be with various powers of x .

Going to Infinity

Part I (Exploring Function Behavior as $x \rightarrow \infty$ or $x \rightarrow -\infty$)

This module provides four examples to explore the behavior of a function as $x \rightarrow \infty$ or $x \rightarrow -\infty$.

Part II (Rates of Growth)

Observe graphs that appear to be continuous, yet the function is not continuous. Several issues of continuity are explored to obtain results that you may find surprising.



3

DIFFERENTIATION

OVERVIEW In the beginning of Chapter 2 we discussed how to determine the slope of a curve at a point and how to measure the rate at which a function changes. Now that we have studied limits, we can define these ideas precisely and see that both are interpretations of the *derivative* of a function at a point. We then extend this concept from a single point to the *derivative function*, and we develop rules for finding this derivative function easily, without having to calculate any limits directly. These rules are used to find derivatives of most of the common functions reviewed in Chapter 1, as well as various combinations of them. The derivative is one of the key ideas in calculus, and we use it to solve a wide range of problems involving tangents and rates of change.

3.1

Tangents and the Derivative at a Point

In this section we define the slope and tangent to a curve at a point, and the derivative of a function at a point. Later in the chapter we interpret the derivative as the instantaneous rate of change of a function, and apply this interpretation to the study of certain types of motion.

Finding a Tangent to the Graph of a Function

To find a tangent to an arbitrary curve $y = f(x)$ at a point $P(x_0, f(x_0))$, we use the procedure introduced in Section 2.1. We calculate the slope of the secant through P and a nearby point $Q(x_0 + h, f(x_0 + h))$. We then investigate the limit of the slope as $h \rightarrow 0$ (Figure 3.1). If the limit exists, we call it the slope of the curve at P and define the tangent at P to be the line through P having this slope.

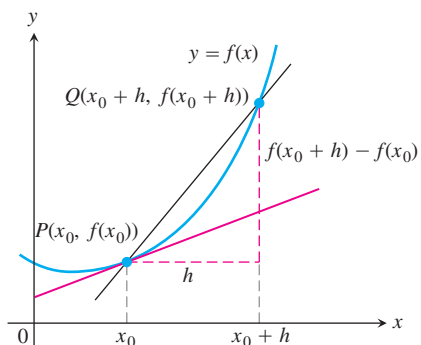


FIGURE 3.1 The slope of the tangent line at P is $\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$.

DEFINITIONS The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$m = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} \quad (\text{provided the limit exists}).$$

The **tangent line** to the curve at P is the line through P with this slope.

In Section 2.1, Example 3, we applied these definitions to find the slope of the parabola $f(x) = x^2$ at the point $P(2, 4)$ and the tangent line to the parabola at P . Let's look at another example.

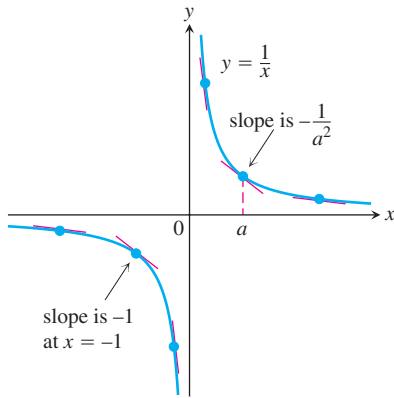


FIGURE 3.2 The tangent slopes, steep near the origin, become more gradual as the point of tangency moves away (Example 1).

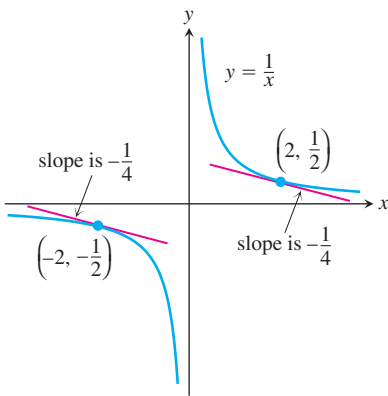


FIGURE 3.3 The two tangent lines to $y = 1/x$ having slope $-1/4$ (Example 1).

EXAMPLE 1

- (a) Find the slope of the curve $y = 1/x$ at any point $x = a \neq 0$. What is the slope at the point $x = -1$?
- (b) Where does the slope equal $-1/4$?
- (c) What happens to the tangent to the curve at the point $(a, 1/a)$ as a changes?

Solution

- (a) Here $f(x) = 1/x$. The slope at $(a, 1/a)$ is

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} &= \lim_{h \rightarrow 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h} = \lim_{h \rightarrow 0} \frac{\frac{a - (a+h)}{h a(a+h)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{-h}{h a(a+h)} = \lim_{h \rightarrow 0} \frac{-1}{a(a+h)} = -\frac{1}{a^2}. \end{aligned}$$

Notice how we had to keep writing “ $\lim_{h \rightarrow 0}$ ” before each fraction until the stage where we could evaluate the limit by substituting $h = 0$. The number a may be positive or negative, but not 0. When $a = -1$, the slope is $-1/(-1)^2 = -1$ (Figure 3.2).

- (b) The slope of $y = 1/x$ at the point where $x = a$ is $-1/a^2$. It will be $-1/4$ provided that

$$-\frac{1}{a^2} = -\frac{1}{4}.$$

This equation is equivalent to $a^2 = 4$, so $a = 2$ or $a = -2$. The curve has slope $-1/4$ at the two points $(2, 1/2)$ and $(-2, -1/2)$ (Figure 3.3).

- (c) The slope $-1/a^2$ is always negative if $a \neq 0$. As $a \rightarrow 0^+$, the slope approaches $-\infty$ and the tangent becomes increasingly steep (Figure 3.2). We see this situation again as $a \rightarrow 0^-$. As a moves away from the origin in either direction, the slope approaches 0 and the tangent levels off to become horizontal. ■

Rates of Change: Derivative at a Point

The expression

$$\frac{f(x_0 + h) - f(x_0)}{h}, \quad h \neq 0$$

is called the **difference quotient of f at x_0 with increment h** . If the difference quotient has a limit as h approaches zero, that limit is given a special name and notation.

DEFINITION The **derivative of a function f at a point x_0** , denoted $f'(x_0)$, is

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided this limit exists.

If we interpret the difference quotient as the slope of a secant line, then the derivative gives the slope of the curve $y = f(x)$ at the point $P(x_0, f(x_0))$. Exercise 31 shows

that the derivative of the linear function $f(x) = mx + b$ at any point x_0 is simply the slope of the line, so

$$f'(x_0) = m,$$

which is consistent with our definition of slope.

If we interpret the difference quotient as an average rate of change (Section 2.1), the derivative gives the function's instantaneous rate of change with respect to x at the point $x = x_0$. We study this interpretation in Section 3.4.

EXAMPLE 2 In Examples 1 and 2 in Section 2.1, we studied the speed of a rock falling freely from rest near the surface of the earth. We knew that the rock fell $y = 16t^2$ feet during the first t sec, and we used a sequence of average rates over increasingly short intervals to estimate the rock's speed at the instant $t = 1$. What was the rock's *exact* speed at this time?

Solution We let $f(t) = 16t^2$. The average speed of the rock over the interval between $t = 1$ and $t = 1 + h$ seconds, for $h > 0$, was found to be

$$\frac{f(1+h) - f(1)}{h} = \frac{16(1+h)^2 - 16(1)^2}{h} = \frac{16(h^2 + 2h)}{h} = 16(h + 2).$$

The rock's speed at the instant $t = 1$ is then

$$\lim_{h \rightarrow 0} 16(h + 2) = 16(0 + 2) = 32 \text{ ft/sec.}$$

Our original estimate of 32 ft/sec in Section 2.1 was right. ■

Summary

We have been discussing slopes of curves, lines tangent to a curve, the rate of change of a function, and the derivative of a function at a point. All of these ideas refer to the same limit.

The following are all interpretations for the limit of the difference quotient,

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

1. The slope of the graph of $y = f(x)$ at $x = x_0$
2. The slope of the tangent to the curve $y = f(x)$ at $x = x_0$
3. The rate of change of $f(x)$ with respect to x at $x = x_0$
4. The derivative $f'(x_0)$ at a point

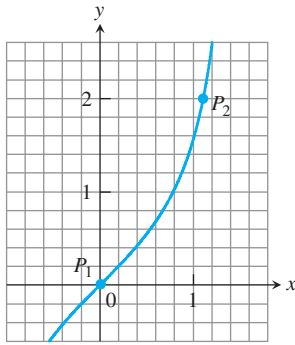
In the next sections, we allow the point x_0 to vary across the domain of the function f .

Exercises 3.1

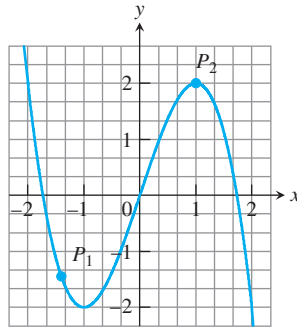
Slopes and Tangent Lines

In Exercises 1–4, use the grid and a straight edge to make a rough estimate of the slope of the curve (in y -units per x -unit) at the points P_1 and P_2 .

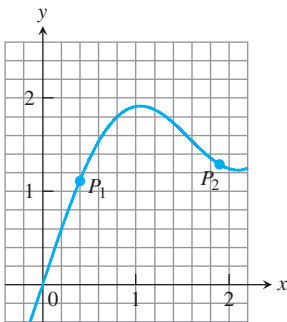
1.



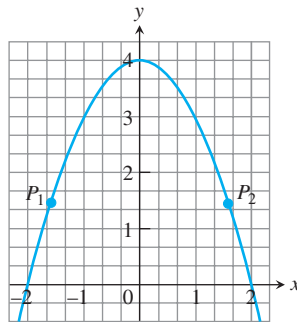
2.



3.



4.



In Exercises 5–10, find an equation for the tangent to the curve at the given point. Then sketch the curve and tangent together.

5. $y = 4 - x^2$, $(-1, 3)$

6. $y = (x - 1)^2 + 1$, $(1, 1)$

7. $y = 2\sqrt{x}$, $(1, 2)$

8. $y = \frac{1}{x^2}$, $(-1, 1)$

9. $y = x^3$, $(-2, -8)$

10. $y = \frac{1}{x^3}$, $(-2, -\frac{1}{8})$

In Exercises 11–18, find the slope of the function's graph at the given point. Then find an equation for the line tangent to the graph there.

11. $f(x) = x^2 + 1$, $(2, 5)$

12. $f(x) = x - 2x^2$, $(1, -1)$

13. $g(x) = \frac{x}{x-2}$, $(3, 3)$

14. $g(x) = \frac{8}{x^2}$, $(2, 2)$

15. $h(t) = t^3$, $(2, 8)$

16. $h(t) = t^3 + 3t$, $(1, 4)$

17. $f(x) = \sqrt{x}$, $(4, 2)$

18. $f(x) = \sqrt{x+1}$, $(8, 3)$

In Exercises 19–22, find the slope of the curve at the point indicated.

19. $y = 5x^2$, $x = -1$

20. $y = 1 - x^2$, $x = 2$

21. $y = \frac{1}{x-1}$, $x = 3$

22. $y = \frac{x-1}{x+1}$, $x = 0$

Tangent Lines with Specified Slopes

At what points do the graphs of the functions in Exercises 23 and 24 have horizontal tangents?

23. $f(x) = x^2 + 4x - 1$

24. $g(x) = x^3 - 3x$

25. Find equations of all lines having slope -1 that are tangent to the curve $y = 1/(x-1)$.

26. Find an equation of the straight line having slope $1/4$ that is tangent to the curve $y = \sqrt{x}$.

Rates of Change

27. **Object dropped from a tower** An object is dropped from the top of a 100-m-high tower. Its height above ground after t sec is $100 - 4.9t^2$ m. How fast is it falling 2 sec after it is dropped?

28. **Speed of a rocket** At t sec after liftoff, the height of a rocket is $3t^2$ ft. How fast is the rocket climbing 10 sec after liftoff?

29. **Circle's changing area** What is the rate of change of the area of a circle ($A = \pi r^2$) with respect to the radius when the radius is $r = 3$?

30. **Ball's changing volume** What is the rate of change of the volume of a ball ($V = (4/3)\pi r^3$) with respect to the radius when the radius is $r = 2$?

31. Show that the line $y = mx + b$ is its own tangent line at any point $(x_0, mx_0 + b)$.

32. Find the slope of the tangent to the curve $y = 1/\sqrt{x}$ at the point where $x = 4$.

Testing for Tangents

33. Does the graph of

$$f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

have a tangent at the origin? Give reasons for your answer.

34. Does the graph of

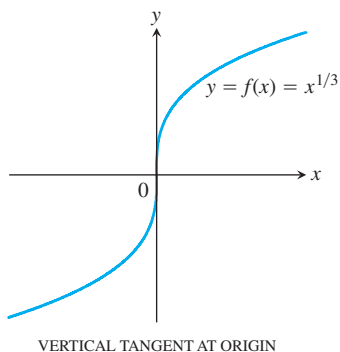
$$g(x) = \begin{cases} x \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

have a tangent at the origin? Give reasons for your answer.

Vertical Tangents

We say that a continuous curve $y = f(x)$ has a **vertical tangent** at the point where $x = x_0$ if $\lim_{h \rightarrow 0} (f(x_0 + h) - f(x_0))/h = \infty$ or $-\infty$. For example, $y = x^{1/3}$ has a vertical tangent at $x = 0$ (see accompanying figure):

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} &= \lim_{h \rightarrow 0} \frac{h^{1/3} - 0}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h^{2/3}} = \infty. \end{aligned}$$

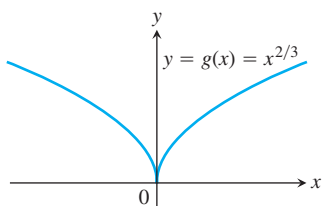


VERTICAL TANGENT AT ORIGIN

However, $y = x^{2/3}$ has *no* vertical tangent at $x = 0$ (see next figure):

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{g(0+h) - g(0)}{h} &= \lim_{h \rightarrow 0} \frac{h^{2/3} - 0}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h^{1/3}} \end{aligned}$$

does not exist, because the limit is ∞ from the right and $-\infty$ from the left.



NO VERTICAL TANGENT AT ORIGIN

35. Does the graph of

$$f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

have a vertical tangent at the origin? Give reasons for your answer.

36. Does the graph of

$$U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

have a vertical tangent at the point $(0, 1)$? Give reasons for your answer.

T Graph the curves in Exercises 37–46.

- Where do the graphs appear to have vertical tangents?
- Confirm your findings in part (a) with limit calculations. But before you do, read the introduction to Exercises 35 and 36.

- | | |
|--|-----------------------------------|
| 37. $y = x^{2/5}$ | 38. $y = x^{4/5}$ |
| 39. $y = x^{1/5}$ | 40. $y = x^{3/5}$ |
| 41. $y = 4x^{2/5} - 2x$ | 42. $y = x^{5/3} - 5x^{2/3}$ |
| 43. $y = x^{2/3} - (x - 1)^{1/3}$ | 44. $y = x^{1/3} + (x - 1)^{1/3}$ |
| 45. $y = \begin{cases} -\sqrt{ x }, & x \leq 0 \\ \sqrt{x}, & x > 0 \end{cases}$ | 46. $y = \sqrt{ 4 - x }$ |

COMPUTER EXPLORATIONS

Use a CAS to perform the following steps for the functions in Exercises 47–50:

- Plot $y = f(x)$ over the interval $(x_0 - 1/2) \leq x \leq (x_0 + 3)$.
- Holding x_0 fixed, the difference quotient

$$q(h) = \frac{f(x_0 + h) - f(x_0)}{h}$$

at x_0 becomes a function of the step size h . Enter this function into your CAS workspace.

- Find the limit of q as $h \rightarrow 0$.
 - Define the secant lines $y = f(x_0) + q \cdot (x - x_0)$ for $h = 3, 2,$ and 1 . Graph them together with f and the tangent line over the interval in part (a).
47. $f(x) = x^3 + 2x, \quad x_0 = 0$ 48. $f(x) = x + \frac{5}{x}, \quad x_0 = 1$
 49. $f(x) = x + \sin(2x), \quad x_0 = \pi/2$
 50. $f(x) = \cos x + 4 \sin(2x), \quad x_0 = \pi$

3.2 | The Derivative as a Function

In the last section we defined the derivative of $y = f(x)$ at the point $x = x_0$ to be the limit

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

We now investigate the derivative as a *function* derived from f by considering the limit at each point x in the domain of f .

DEFINITION The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h},$$

provided the limit exists.

HISTORICAL ESSAY

The Derivative

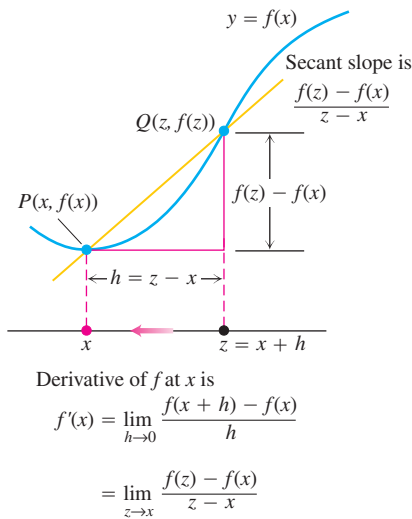


FIGURE 3.4 Two forms for the difference quotient.

We use the notation $f(x)$ in the definition to emphasize the independent variable x with respect to which the derivative function $f'(x)$ is being defined. The domain of f' is the set of points in the domain of f for which the limit exists, which means that the domain may be the same as or smaller than the domain of f . If f' exists at a particular x , we say that f is **differentiable (has a derivative) at x** . If f' exists at every point in the domain of f , we call f **differentiable**.

If we write $z = x + h$, then $h = z - x$ and h approaches 0 if and only if z approaches x . Therefore, an equivalent definition of the derivative is as follows (see Figure 3.4). This formula is sometimes more convenient to use when finding a derivative function.

Alternative Formula for the Derivative

$$f'(x) = \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x}.$$

Calculating Derivatives from the Definition

The process of calculating a derivative is called **differentiation**. To emphasize the idea that differentiation is an operation performed on a function $y = f(x)$, we use the notation

$$\frac{d}{dx} f(x)$$

as another way to denote the derivative $f'(x)$. Example 1 of Section 3.1 illustrated the differentiation process for the function $y = 1/x$ when $x = a$. For x representing any point in the domain, we get the formula

$$\frac{d}{dx} \left(\frac{1}{x} \right) = -\frac{1}{x^2}.$$

Here are two more examples in which we allow x to be any point in the domain of f .

EXAMPLE 1 Differentiate $f(x) = \frac{x}{x-1}$.

Solution We use the definition of derivative, which requires us to calculate $f(x+h)$ and then subtract $f(x)$ to obtain the numerator in the difference quotient. We have

$$f(x) = \frac{x}{x-1} \quad \text{and} \quad f(x+h) = \frac{(x+h)}{(x+h)-1}, \text{ so}$$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} && \text{Definition} \\ &= \lim_{h \rightarrow 0} \frac{\frac{x+h}{x+h-1} - \frac{x}{x-1}}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \cdot \frac{(x+h)(x-1) - x(x+h-1)}{(x+h-1)(x-1)} && \frac{a}{b} - \frac{c}{d} = \frac{ad - cb}{bd} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \cdot \frac{-h}{(x+h-1)(x-1)} && \text{Simplify.} \\ &= \lim_{h \rightarrow 0} \frac{-1}{(x+h-1)(x-1)} = \frac{-1}{(x-1)^2}. && \text{Cancel } h \neq 0. \end{aligned}$$

Derivative of the Reciprocal Function

$$\frac{d}{dx} \left(\frac{1}{x} \right) = -\frac{1}{x^2}, \quad x \neq 0$$

Derivative of the Square Root Function

$$\frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}}, \quad x > 0$$

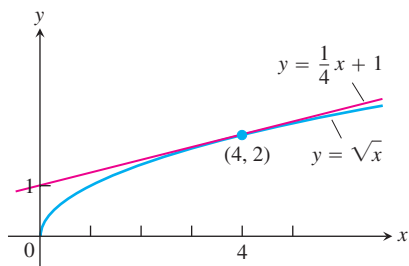


FIGURE 3.5 The curve $y = \sqrt{x}$ and its tangent at $(4, 2)$. The tangent's slope is found by evaluating the derivative at $x = 4$ (Example 2).

EXAMPLE 2

- (a) Find the derivative of $f(x) = \sqrt{x}$ for $x > 0$.
 (b) Find the tangent line to the curve $y = \sqrt{x}$ at $x = 4$.

Solution

- (a) We use the alternative formula to calculate f' :

$$\begin{aligned} f'(x) &= \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x} \\ &= \lim_{z \rightarrow x} \frac{\sqrt{z} - \sqrt{x}}{z - x} \\ &= \lim_{z \rightarrow x} \frac{\sqrt{z} - \sqrt{x}}{(\sqrt{z} - \sqrt{x})(\sqrt{z} + \sqrt{x})} \\ &= \lim_{z \rightarrow x} \frac{1}{\sqrt{z} + \sqrt{x}} = \frac{1}{2\sqrt{x}}. \end{aligned}$$

- (b) The slope of the curve at $x = 4$ is

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

The tangent is the line through the point $(4, 2)$ with slope $1/4$ (Figure 3.5):

$$y = 2 + \frac{1}{4}(x - 4)$$

$$y = \frac{1}{4}x + 1. \quad \blacksquare$$

Notations

There are many ways to denote the derivative of a function $y = f(x)$, where the independent variable is x and the dependent variable is y . Some common alternative notations for the derivative are

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = D(f)(x) = D_x f(x).$$

The symbols d/dx and D indicate the operation of differentiation. We read dy/dx as “the derivative of y with respect to x ,” and df/dx and $(d/dx)f(x)$ as “the derivative of f with respect to x .” The “prime” notations y' and f' come from notations that Newton used for derivatives. The d/dx notations are similar to those used by Leibniz. The symbol dy/dx should not be regarded as a ratio (until we introduce the idea of “differentials” in Section 3.11).

To indicate the value of a derivative at a specified number $x = a$, we use the notation

$$f'(a) = \left. \frac{dy}{dx} \right|_{x=a} = \left. \frac{df}{dx} \right|_{x=a} = \left. \frac{d}{dx} f(x) \right|_{x=a}.$$

For instance, in Example 2

$$f'(4) = \left. \frac{d}{dx} \sqrt{x} \right|_{x=4} = \left. \frac{1}{2\sqrt{x}} \right|_{x=4} = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

Graphing the Derivative

We can often make a reasonable plot of the derivative of $y = f(x)$ by estimating the slopes on the graph of f . That is, we plot the points $(x, f'(x))$ in the xy -plane and connect them with a smooth curve, which represents $y = f'(x)$.

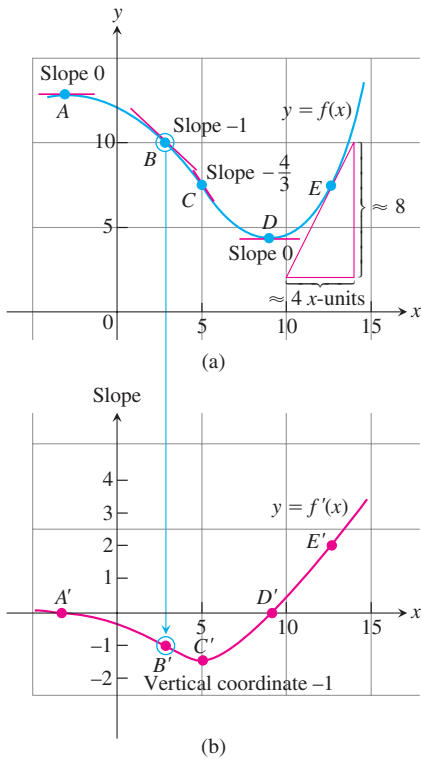


FIGURE 3.6 We made the graph of $y = f'(x)$ in (b) by plotting slopes from the graph of $y = f(x)$ in (a). The vertical coordinate of B' is the slope at B and so on. The slope at E is approximately $8/4 = 2$. In (b) we see that the rate of change of f is negative for x between A' and D' ; the rate of change is positive for x to the right of D' .

EXAMPLE 3 Graph the derivative of the function $y = f(x)$ in Figure 3.6a.

Solution We sketch the tangents to the graph of f at frequent intervals and use their slopes to estimate the values of $f'(x)$ at these points. We plot the corresponding $(x, f'(x))$ pairs and connect them with a smooth curve as sketched in Figure 3.6b. ■

What can we learn from the graph of $y = f'(x)$? At a glance we can see

1. where the rate of change of f is positive, negative, or zero;
2. the rough size of the growth rate at any x and its size in relation to the size of $f(x)$;
3. where the rate of change itself is increasing or decreasing.

Differentiable on an Interval; One-Sided Derivatives

A function $y = f(x)$ is **differentiable on an open interval** (finite or infinite) if it has a derivative at each point of the interval. It is **differentiable on a closed interval** $[a, b]$ if it is differentiable on the interior (a, b) and if the limits

$$\lim_{h \rightarrow 0^+} \frac{f(a+h) - f(a)}{h} \quad \text{Right-hand derivative at } a$$

$$\lim_{h \rightarrow 0^-} \frac{f(b+h) - f(b)}{h} \quad \text{Left-hand derivative at } b$$

exist at the endpoints (Figure 3.7).

Right-hand and left-hand derivatives may be defined at any point of a function's domain. Because of Theorem 6, Section 2.4, a function has a derivative at a point if and only if it has left-hand and right-hand derivatives there, and these one-sided derivatives are equal.

EXAMPLE 4 Show that the function $y = |x|$ is differentiable on $(-\infty, 0)$ and $(0, \infty)$ but has no derivative at $x = 0$.

Solution From Section 3.1, the derivative of $y = mx + b$ is the slope m . Thus, to the right of the origin,

$$\frac{d}{dx}(|x|) = \frac{d}{dx}(x) = \frac{d}{dx}(1 \cdot x) = 1. \quad \frac{d}{dx}(mx + b) = m, |x| = x$$

To the left,

$$\frac{d}{dx}(|x|) = \frac{d}{dx}(-x) = \frac{d}{dx}(-1 \cdot x) = -1 \quad |x| = -x$$

(Figure 3.8). There is no derivative at the origin because the one-sided derivatives differ there:

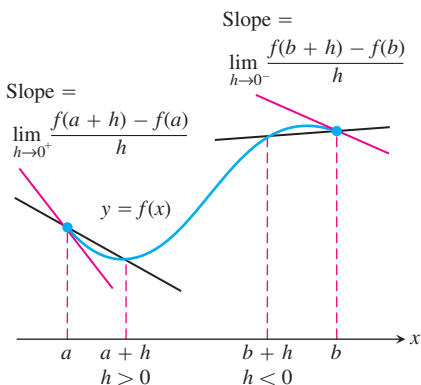


FIGURE 3.7 Derivatives at endpoints are one-sided limits.

$$\begin{aligned} \text{Right-hand derivative of } |x| \text{ at zero} &= \lim_{h \rightarrow 0^+} \frac{|0+h| - |0|}{h} = \lim_{h \rightarrow 0^+} \frac{|h|}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{h}{h} \quad |h| = h \text{ when } h > 0 \\ &= \lim_{h \rightarrow 0^+} 1 = 1 \end{aligned}$$

$$\begin{aligned} \text{Left-hand derivative of } |x| \text{ at zero} &= \lim_{h \rightarrow 0^-} \frac{|0+h| - |0|}{h} = \lim_{h \rightarrow 0^-} \frac{|h|}{h} \\ &= \lim_{h \rightarrow 0^-} \frac{-h}{h} \quad |h| = -h \text{ when } h < 0 \\ &= \lim_{h \rightarrow 0^-} -1 = -1. \end{aligned}$$

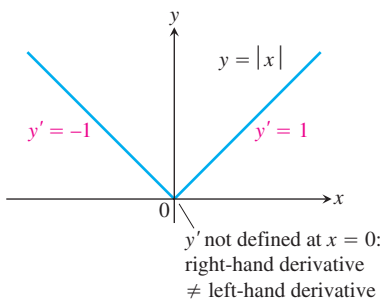


FIGURE 3.8 The function $y = |x|$ is not differentiable at the origin where the graph has a “corner” (Example 4).

EXAMPLE 5 In Example 2 we found that for $x > 0$,

$$\frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}}.$$

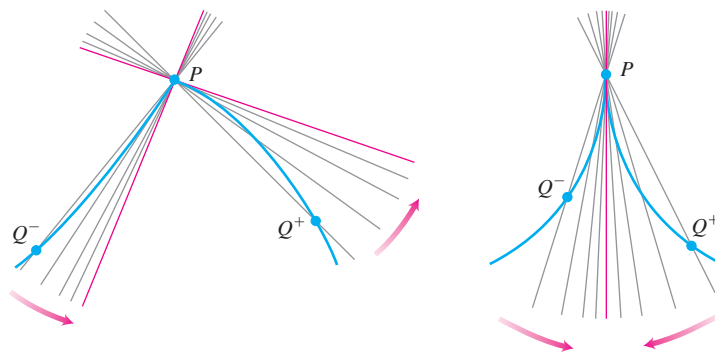
We apply the definition to examine if the derivative exists at $x = 0$:

$$\lim_{h \rightarrow 0^+} \frac{\sqrt{0+h} - \sqrt{0}}{h} = \lim_{h \rightarrow 0^+} \frac{1}{\sqrt{h}} = \infty.$$

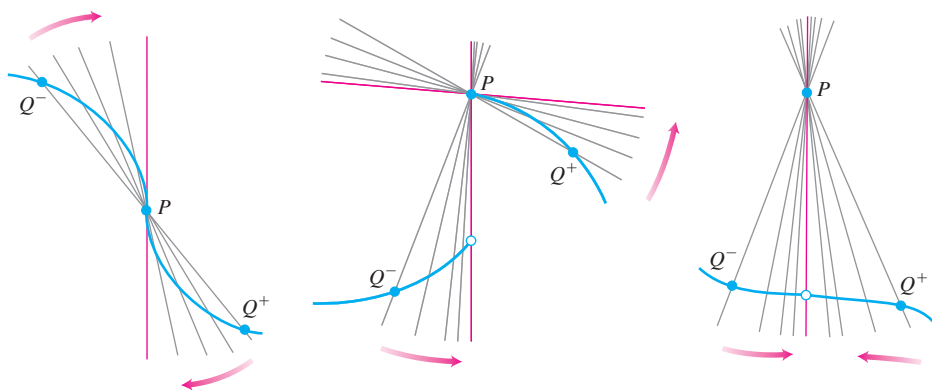
Since the (right-hand) limit is not finite, there is no derivative at $x = 0$. Since the slopes of the secant lines joining the origin to the points (h, \sqrt{h}) on a graph of $y = \sqrt{x}$ approach ∞ , the graph has a *vertical tangent* at the origin. (See Figure 1.17 on page 9). ■

When Does a Function Not Have a Derivative at a Point?

A function has a derivative at a point x_0 if the slopes of the secant lines through $P(x_0, f(x_0))$ and a nearby point Q on the graph approach a finite limit as Q approaches P . Whenever the secants fail to take up a limiting position or become vertical as Q approaches P , the derivative does not exist. Thus differentiability is a “smoothness” condition on the graph of f . A function can fail to have a derivative at a point for many reasons, including the existence of points where the graph has



1. a *corner*, where the one-sided derivatives differ.
2. a *cusp*, where the slope of PQ approaches ∞ from one side and $-\infty$ from the other.



3. a *vertical tangent*, where the slope of PQ approaches ∞ from both sides or approaches $-\infty$ from both sides (here, $-\infty$).
4. a *discontinuity* (two examples shown).

Another case in which the derivative may fail to exist occurs when the function's slope is oscillating rapidly near P , as with $f(x) = \sin(1/x)$ near the origin, where it is discontinuous (see Figure 2.31).

Differentiable Functions Are Continuous

A function is continuous at every point where it has a derivative.

THEOREM 1—Differentiability Implies Continuity If f has a derivative at $x = c$, then f is continuous at $x = c$.

Proof Given that $f'(c)$ exists, we must show that $\lim_{x \rightarrow c} f(x) = f(c)$, or equivalently, that $\lim_{h \rightarrow 0} f(c + h) = f(c)$. If $h \neq 0$, then

$$\begin{aligned} f(c + h) &= f(c) + (f(c + h) - f(c)) \\ &= f(c) + \frac{f(c + h) - f(c)}{h} \cdot h. \end{aligned}$$

Now take limits as $h \rightarrow 0$. By Theorem 1 of Section 2.2,

$$\begin{aligned} \lim_{h \rightarrow 0} f(c + h) &= \lim_{h \rightarrow 0} f(c) + \lim_{h \rightarrow 0} \frac{f(c + h) - f(c)}{h} \cdot \lim_{h \rightarrow 0} h \\ &= f(c) + f'(c) \cdot 0 \\ &= f(c) + 0 \\ &= f(c). \end{aligned} \quad \blacksquare$$

Similar arguments with one-sided limits show that if f has a derivative from one side (right or left) at $x = c$ then f is continuous from that side at $x = c$.

Theorem 1 says that if a function has a discontinuity at a point (for instance, a jump discontinuity), then it cannot be differentiable there. The greatest integer function $y = \lfloor x \rfloor$ fails to be differentiable at every integer $x = n$ (Example 4, Section 2.5).

Caution The converse of Theorem 1 is false. A function need not have a derivative at a point where it is continuous, as we saw in Example 4.

Exercises 3.2

Finding Derivative Functions and Values

Using the definition, calculate the derivatives of the functions in Exercises 1–6. Then find the values of the derivatives as specified.

- $f(x) = 4 - x^2$; $f'(-3), f'(0), f'(1)$
- $F(x) = (x - 1)^2 + 1$; $F'(-1), F'(0), F'(2)$
- $g(t) = \frac{1}{t^2}$; $g'(-1), g'(2), g'(\sqrt{3})$
- $k(z) = \frac{1-z}{2z}$; $k'(-1), k'(1), k'(\sqrt{2})$
- $p(\theta) = \sqrt{3\theta}$; $p'(1), p'(3), p'(2/3)$

$$6. r(s) = \sqrt{2s + 1}; \quad r'(0), r'(1), r'(1/2)$$

In Exercises 7–12, find the indicated derivatives.

- $\frac{dy}{dx}$ if $y = 2x^3$
- $\frac{dr}{ds}$ if $r = s^3 - 2s^2 + 3$
- $\frac{ds}{dt}$ if $s = \frac{t}{2t + 1}$
- $\frac{dv}{dt}$ if $v = t - \frac{1}{t}$
- $\frac{dp}{dq}$ if $p = \frac{1}{\sqrt{q + 1}}$
- $\frac{dz}{dw}$ if $z = \frac{1}{\sqrt{3w - 2}}$

Slopes and Tangent Lines

In Exercises 13–16, differentiate the functions and find the slope of the tangent line at the given value of the independent variable.

13. $f(x) = x + \frac{9}{x}$, $x = -3$ 14. $k(x) = \frac{1}{2+x}$, $x = 2$

15. $s = t^3 - t^2$, $t = -1$ 16. $y = \frac{x+3}{1-x}$, $x = -2$

In Exercises 17–18, differentiate the functions. Then find an equation of the tangent line at the indicated point on the graph of the function.

17. $y = f(x) = \frac{8}{\sqrt{x-2}}$, $(x, y) = (6, 4)$

18. $w = g(z) = 1 + \sqrt{4-z}$, $(z, w) = (3, 2)$

In Exercises 19–22, find the values of the derivatives.

19. $\left. \frac{ds}{dt} \right|_{t=-1}$ if $s = 1 - 3t^2$

20. $\left. \frac{dy}{dx} \right|_{x=\sqrt{3}}$ if $y = 1 - \frac{1}{x}$

21. $\left. \frac{dr}{d\theta} \right|_{\theta=0}$ if $r = \frac{2}{\sqrt{4-\theta}}$

22. $\left. \frac{dw}{dz} \right|_{z=4}$ if $w = z + \sqrt{z}$

Using the Alternative Formula for Derivatives

Use the formula

$$f'(x) = \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x}$$

to find the derivative of the functions in Exercises 23–26.

23. $f(x) = \frac{1}{x+2}$

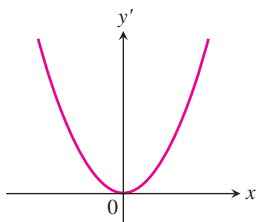
24. $f(x) = x^2 - 3x + 4$

25. $g(x) = \frac{x}{x-1}$

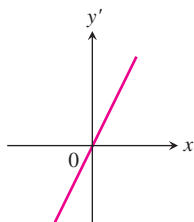
26. $g(x) = 1 + \sqrt{x}$

Graphs

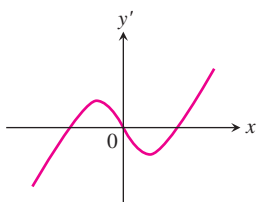
Match the functions graphed in Exercises 27–30 with the derivatives graphed in the accompanying figures (a)–(d).



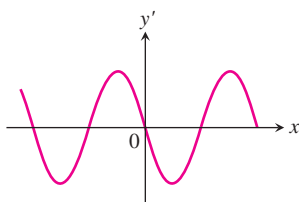
(a)



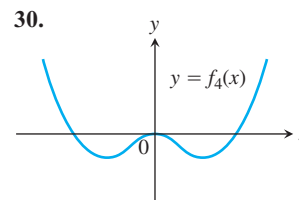
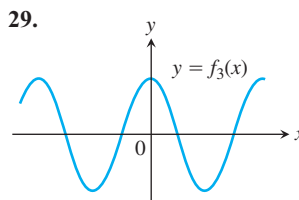
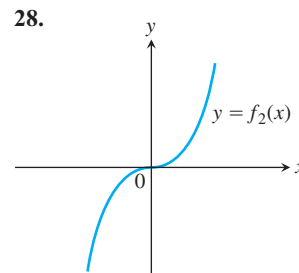
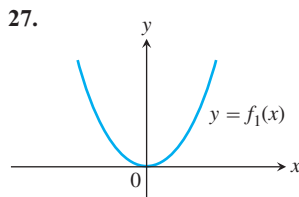
(b)



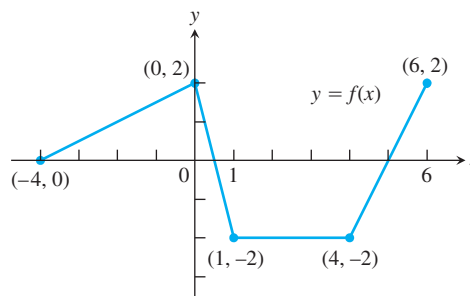
(c)



(d)



31. a. The graph in the accompanying figure is made of line segments joined end to end. At which points of the interval $[-4, 6]$ is f' not defined? Give reasons for your answer.

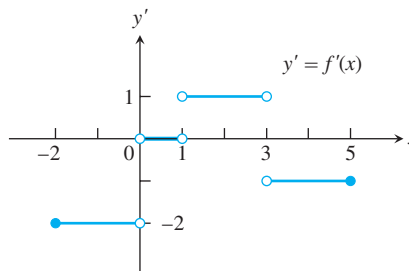


b. Graph the derivative of f .
The graph should show a step function.

32. Recovering a function from its derivative

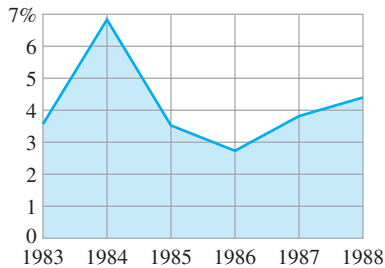
a. Use the following information to graph the function f over the closed interval $[-2, 5]$.

- i) The graph of f is made of closed line segments joined end to end.
- ii) The graph starts at the point $(-2, 3)$.
- iii) The derivative of f is the step function in the figure shown here.



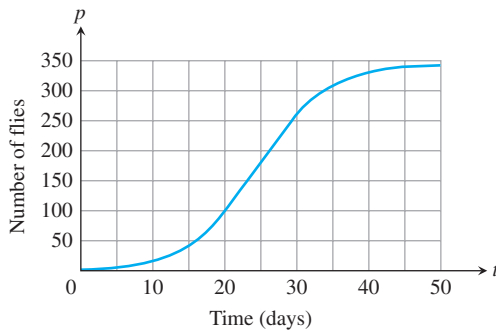
b. Repeat part (a) assuming that the graph starts at $(-2, 0)$ instead of $(-2, 3)$.

- 33. Growth in the economy** The graph in the accompanying figure shows the average annual percentage change $y = f(t)$ in the U.S. gross national product (GNP) for the years 1983–1988. Graph dy/dt (where defined).



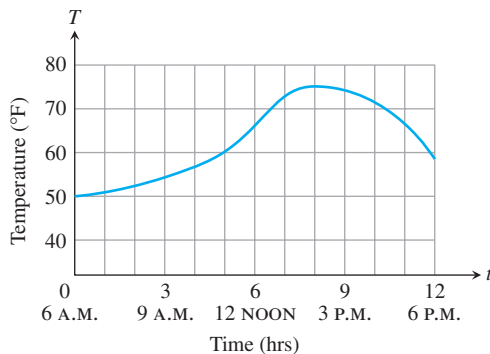
- 34. Fruit flies** (Continuation of Example 4, Section 2.1.) Populations starting out in closed environments grow slowly at first, when there are relatively few members, then more rapidly as the number of reproducing individuals increases and resources are still abundant, then slowly again as the population reaches the carrying capacity of the environment.

- a. Use the graphical technique of Example 3 to graph the derivative of the fruit fly population. The graph of the population is reproduced here.



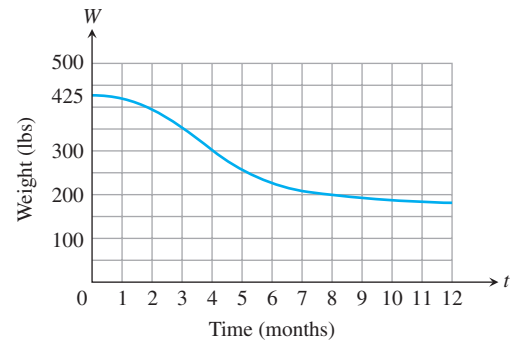
- b. During what days does the population seem to be increasing fastest? Slowest?

- 35. Temperature** The given graph shows the temperature T in $^{\circ}\text{F}$ at Davis, CA, on April 18, 2008, between 6 A.M. and 6 P.M.



- a. Estimate the rate of temperature change at the times
i) 7 A.M. ii) 9 A.M. iii) 2 P.M. iv) 4 P.M.
- b. At what time does the temperature increase most rapidly? Decrease most rapidly? What is the rate for each of those times?
- c. Use the graphical technique of Example 3 to graph the derivative of temperature T versus time t .

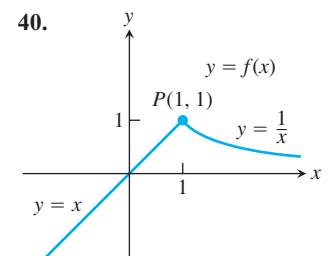
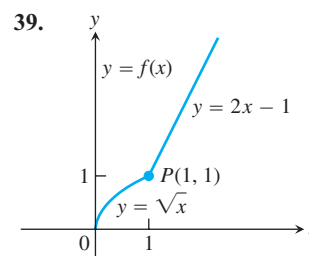
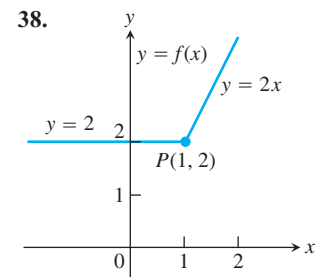
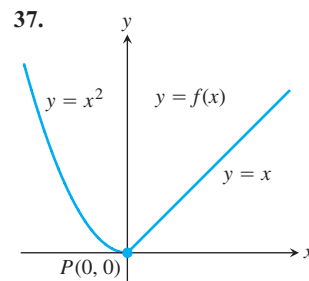
- 36. Weight loss** Jared Fogle, also known as the “Subway Sandwich Guy,” weighed 425 lb in 1997 before losing more than 240 lb in 12 months (http://en.wikipedia.org/wiki/Jared_Fogle). A chart showing his possible dramatic weight loss is given in the accompanying figure.



- a. Estimate Jared's rate of weight loss when
i) $t = 1$ ii) $t = 4$ iii) $t = 11$
- b. When does Jared lose weight most rapidly and what is this rate of weight loss?
- c. Use the graphical technique of Example 3 to graph the derivative of weight W .

One-Sided Derivatives

Compute the right-hand and left-hand derivatives as limits to show that the functions in Exercises 37–40 are not differentiable at the point P .



In Exercises 41 and 42, determine if the piecewise defined function is differentiable at the origin.

- 41.** $f(x) = \begin{cases} 2x - 1, & x \geq 0 \\ x^2 + 2x + 7, & x < 0 \end{cases}$
- 42.** $g(x) = \begin{cases} x^{2/3}, & x \geq 0 \\ x^{1/3}, & x < 0 \end{cases}$

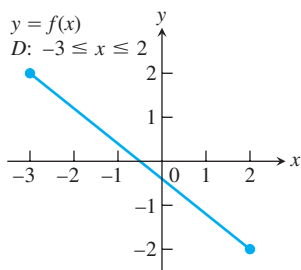
Differentiability and Continuity on an Interval

Each figure in Exercises 43–48 shows the graph of a function over a closed interval D . At what domain points does the function appear to be

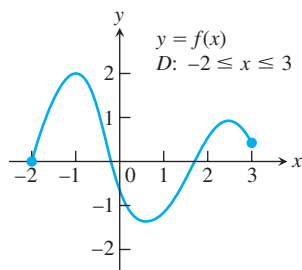
- differentiable?
- continuous but not differentiable?
- neither continuous nor differentiable?

Give reasons for your answers.

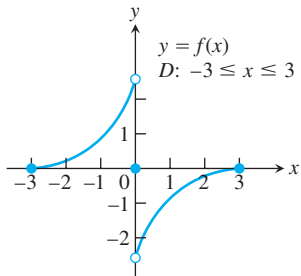
43.



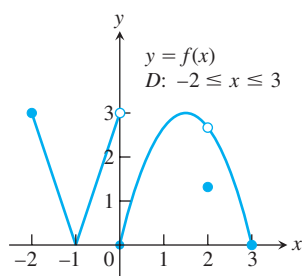
44.



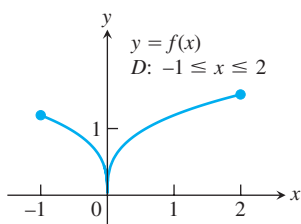
45.



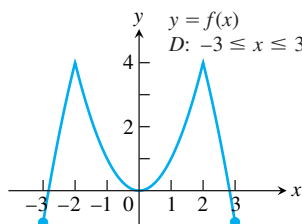
46.



47.



48.



Theory and Examples

In Exercises 49–52,

- Find the derivative $f'(x)$ of the given function $y = f(x)$.
- Graph $y = f(x)$ and $y = f'(x)$ side by side using separate sets of coordinate axes, and answer the following questions.
- For what values of x , if any, is f' positive? Zero? Negative?
- Over what intervals of x -values, if any, does the function $y = f(x)$ increase as x increases? Decrease as x increases? How is this related to what you found in part (c)? (We will say more about this relationship in Section 4.3.)

49. $y = -x^2$

50. $y = -1/x$

51. $y = x^3/3$

52. $y = x^4/4$

53. **Tangent to a parabola** Does the parabola $y = 2x^2 - 13x + 5$ have a tangent whose slope is -1 ? If so, find an equation for the line and the point of tangency. If not, why not?

54. **Tangent to $y = \sqrt{x}$** Does any tangent to the curve $y = \sqrt{x}$ cross the x -axis at $x = -1$? If so, find an equation for the line and the point of tangency. If not, why not?

55. **Derivative of $-f$** Does knowing that a function $f(x)$ is differentiable at $x = x_0$ tell you anything about the differentiability of the function $-f$ at $x = x_0$? Give reasons for your answer.

56. **Derivative of multiples** Does knowing that a function $g(t)$ is differentiable at $t = 7$ tell you anything about the differentiability of the function $3g$ at $t = 7$? Give reasons for your answer.

57. **Limit of a quotient** Suppose that functions $g(t)$ and $h(t)$ are defined for all values of t and $g(0) = h(0) = 0$. Can $\lim_{t \rightarrow 0} (g(t))/(h(t))$ exist? If it does exist, must it equal zero? Give reasons for your answers.

58. a. Let $f(x)$ be a function satisfying $|f(x)| \leq x^2$ for $-1 \leq x \leq 1$. Show that f is differentiable at $x = 0$ and find $f'(0)$.

b. Show that

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is differentiable at $x = 0$ and find $f'(0)$.

- T 59. Graph $y = 1/(2\sqrt{x})$ in a window that has $0 \leq x \leq 2$. Then, on the same screen, graph

$$y = \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

for $h = 1, 0.5, 0.1$. Then try $h = -1, -0.5, -0.1$. Explain what is going on.

- T 60. Graph $y = 3x^2$ in a window that has $-2 \leq x \leq 2, 0 \leq y \leq 3$. Then, on the same screen, graph

$$y = \frac{(x+h)^3 - x^3}{h}$$

for $h = 2, 1, 0.2$. Then try $h = -2, -1, -0.2$. Explain what is going on.

61. **Derivative of $y = |x|$** Graph the derivative of $f(x) = |x|$. Then graph $y = (|x| - 0)/(x - 0) = |x|/x$. What can you conclude?

- T 62. **Weierstrass's nowhere differentiable continuous function** The sum of the first eight terms of the Weierstrass function $f(x) = \sum_{n=0}^{\infty} (2/3)^n \cos(9^n \pi x)$ is

$$g(x) = \cos(\pi x) + (2/3)^1 \cos(9\pi x) + (2/3)^2 \cos(9^2 \pi x) + (2/3)^3 \cos(9^3 \pi x) + \cdots + (2/3)^7 \cos(9^7 \pi x).$$

Graph this sum. Zoom in several times. How wiggly and bumpy is this graph? Specify a viewing window in which the displayed portion of the graph is smooth.

COMPUTER EXPLORATIONS

Use a CAS to perform the following steps for the functions in Exercises 63–68.

- Plot $y = f(x)$ to see that function's global behavior.
- Define the difference quotient q at a general point x , with general step size h .
- Take the limit as $h \rightarrow 0$. What formula does this give?
- Substitute the value $x = x_0$ and plot the function $y = f(x)$ together with its tangent line at that point.

- e. Substitute various values for x larger and smaller than x_0 into the formula obtained in part (c). Do the numbers make sense with your picture?
- f. Graph the formula obtained in part (c). What does it mean when its values are negative? Zero? Positive? Does this make sense with your plot from part (a)? Give reasons for your answer.
63. $f(x) = x^3 + x^2 - x$, $x_0 = 1$
64. $f(x) = x^{1/3} + x^{2/3}$, $x_0 = 1$
65. $f(x) = \frac{4x}{x^2 + 1}$, $x_0 = 2$
66. $f(x) = \frac{x - 1}{3x^2 + 1}$, $x_0 = -1$
67. $f(x) = \sin 2x$, $x_0 = \pi/2$
68. $f(x) = x^2 \cos x$, $x_0 = \pi/4$

3.3 Differentiation Rules

This section introduces several rules that allow us to differentiate constant functions, power functions, polynomials, exponential functions, rational functions, and certain combinations of them, simply and directly, without having to take limits each time.

Powers, Multiples, Sums, and Differences

A simple rule of differentiation is that the derivative of every constant function is zero.

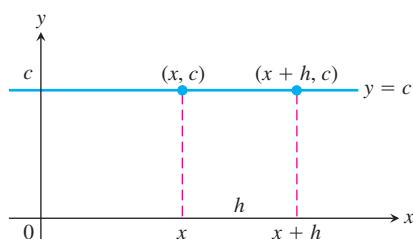


FIGURE 3.9 The rule $(d/dx)(c) = 0$ is another way to say that the values of constant functions never change and that the slope of a horizontal line is zero at every point.

Derivative of a Constant Function

If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

Proof We apply the definition of the derivative to $f(x) = c$, the function whose outputs have the constant value c (Figure 3.9). At every value of x , we find that

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{c - c}{h} = \lim_{h \rightarrow 0} 0 = 0. \quad \blacksquare$$

From Section 3.1, we know that

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}, \quad \text{or} \quad \frac{d}{dx}(x^{-1}) = -x^{-2}.$$

From Example 2 of the last section we also know that

$$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}, \quad \text{or} \quad \frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{-1/2}.$$

These two examples illustrate a general rule for differentiating a power x^n . We first prove the rule when n is a positive integer.

Power Rule for Positive Integers:

If n is a positive integer, then

$$\frac{d}{dx}x^n = nx^{n-1}.$$

HISTORICAL BIOGRAPHY

Richard Courant
(1888–1972)

Proof of the Positive Integer Power Rule The formula

$$z^n - x^n = (z - x)(z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-1})$$

can be verified by multiplying out the right-hand side. Then from the alternative formula for the definition of the derivative,

$$\begin{aligned} f'(x) &= \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x} = \lim_{z \rightarrow x} \frac{z^n - x^n}{z - x} \\ &= \lim_{z \rightarrow x} (z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-1}) \quad n \text{ terms} \\ &= nx^{n-1}. \end{aligned}$$

The Power Rule is actually valid for all real numbers n . We have seen examples for a negative integer and fractional power, but n could be an irrational number as well. To apply the Power Rule, we subtract 1 from the original exponent n and multiply the result by n . Here we state the general version of the rule, but postpone its proof until Section 3.8.

Power Rule (General Version)

If n is any real number, then

$$\frac{d}{dx} x^n = nx^{n-1},$$

for all x where the powers x^n and x^{n-1} are defined.

EXAMPLE 1 Differentiate the following powers of x .

(a) x^3 (b) $x^{2/3}$ (c) $x^{\sqrt{2}}$ (d) $\frac{1}{x^4}$ (e) $x^{-4/3}$ (f) $\sqrt{x^{2+\pi}}$

Solution

(a) $\frac{d}{dx}(x^3) = 3x^{3-1} = 3x^2$ (b) $\frac{d}{dx}(x^{2/3}) = \frac{2}{3}x^{(2/3)-1} = \frac{2}{3}x^{-1/3}$

(c) $\frac{d}{dx}(x^{\sqrt{2}}) = \sqrt{2}x^{\sqrt{2}-1}$ (d) $\frac{d}{dx}\left(\frac{1}{x^4}\right) = \frac{d}{dx}(x^{-4}) = -4x^{-4-1} = -4x^{-5} = -\frac{4}{x^5}$

(e) $\frac{d}{dx}(x^{-4/3}) = -\frac{4}{3}x^{-(4/3)-1} = -\frac{4}{3}x^{-7/3}$

(f) $\frac{d}{dx}(\sqrt{x^{2+\pi}}) = \frac{d}{dx}(x^{1+(\pi/2)}) = \left(1 + \frac{\pi}{2}\right)x^{1+(\pi/2)-1} = \frac{1}{2}(2 + \pi)\sqrt{x^\pi}$ ■

The next rule says that when a differentiable function is multiplied by a constant, its derivative is multiplied by the same constant.

Derivative Constant Multiple Rule

If u is a differentiable function of x , and c is a constant, then

$$\frac{d}{dx}(cu) = c \frac{du}{dx}.$$

In particular, if n is any real number, then

$$\frac{d}{dx}(cx^n) = cnx^{n-1}.$$

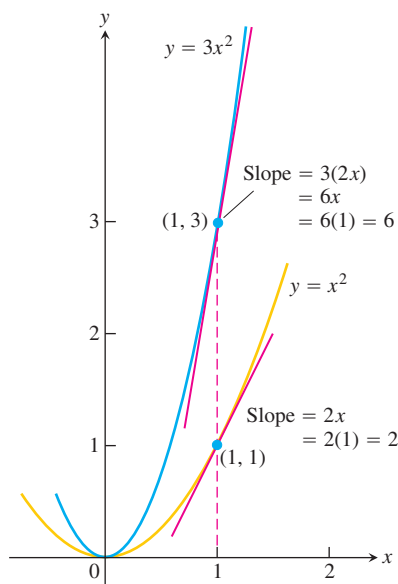


FIGURE 3.10 The graphs of $y = x^2$ and $y = 3x^2$. Tripling the y -coordinate triples the slope (Example 2).

Denoting Functions by u and v

The functions we are working with when we need a differentiation formula are likely to be denoted by letters like f and g . We do not want to use these same letters when stating general differentiation rules, so we use letters like u and v instead that are not likely to be already in use.

Proof

$$\begin{aligned} \frac{d}{dx} cu &= \lim_{h \rightarrow 0} \frac{cu(x+h) - cu(x)}{h} && \text{Derivative definition} \\ &&& \text{with } f(x) = cu(x) \\ &= c \lim_{h \rightarrow 0} \frac{u(x+h) - u(x)}{h} && \text{Constant Multiple Limit Property} \\ &= c \frac{du}{dx} && u \text{ is differentiable.} \quad \blacksquare \end{aligned}$$

EXAMPLE 2

(a) The derivative formula

$$\frac{d}{dx}(3x^2) = 3 \cdot 2x = 6x$$

says that if we rescale the graph of $y = x^2$ by multiplying each y -coordinate by 3, then we multiply the slope at each point by 3 (Figure 3.10).

(b) **Negative of a function**

The derivative of the negative of a differentiable function u is the negative of the function's derivative. The Constant Multiple Rule with $c = -1$ gives

$$\frac{d}{dx}(-u) = \frac{d}{dx}(-1 \cdot u) = -1 \cdot \frac{d}{dx}(u) = -\frac{du}{dx}. \quad \blacksquare$$

The next rule says that the derivative of the sum of two differentiable functions is the sum of their derivatives.

Derivative Sum Rule

If u and v are differentiable functions of x , then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

For example, if $y = x^4 + 12x$, then y is the sum of $u(x) = x^4$ and $v(x) = 12x$. We then have

$$\frac{dy}{dx} = \frac{d}{dx}(x^4) + \frac{d}{dx}(12x) = 4x^3 + 12.$$

Proof We apply the definition of the derivative to $f(x) = u(x) + v(x)$:

$$\begin{aligned} \frac{d}{dx}[u(x) + v(x)] &= \lim_{h \rightarrow 0} \frac{[u(x+h) + v(x+h)] - [u(x) + v(x)]}{h} \\ &= \lim_{h \rightarrow 0} \left[\frac{u(x+h) - u(x)}{h} + \frac{v(x+h) - v(x)}{h} \right] \\ &= \lim_{h \rightarrow 0} \frac{u(x+h) - u(x)}{h} + \lim_{h \rightarrow 0} \frac{v(x+h) - v(x)}{h} = \frac{du}{dx} + \frac{dv}{dx}. \quad \blacksquare \end{aligned}$$

Combining the Sum Rule with the Constant Multiple Rule gives the **Difference Rule**, which says that the derivative of a *difference* of differentiable functions is the difference of their derivatives:

$$\frac{d}{dx}(u - v) = \frac{d}{dx}[u + (-1)v] = \frac{du}{dx} + (-1)\frac{dv}{dx} = \frac{du}{dx} - \frac{dv}{dx}.$$

The Sum Rule also extends to finite sums of more than two functions. If u_1, u_2, \dots, u_n are differentiable at x , then so is $u_1 + u_2 + \dots + u_n$, and

$$\frac{d}{dx}(u_1 + u_2 + \dots + u_n) = \frac{du_1}{dx} + \frac{du_2}{dx} + \dots + \frac{du_n}{dx}.$$

For instance, to see that the rule holds for three functions we compute

$$\frac{d}{dx}(u_1 + u_2 + u_3) = \frac{d}{dx}((u_1 + u_2) + u_3) = \frac{d}{dx}(u_1 + u_2) + \frac{du_3}{dx} = \frac{du_1}{dx} + \frac{du_2}{dx} + \frac{du_3}{dx}.$$

A proof by mathematical induction for any finite number of terms is given in Appendix 2.

EXAMPLE 3 Find the derivative of the polynomial $y = x^3 + \frac{4}{3}x^2 - 5x + 1$.

Solution $\frac{dy}{dx} = \frac{d}{dx}x^3 + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1)$ Sum and Difference Rules

$$= 3x^2 + \frac{4}{3} \cdot 2x - 5 + 0 = 3x^2 + \frac{8}{3}x - 5$$

We can differentiate any polynomial term by term, the way we differentiated the polynomial in Example 3. All polynomials are differentiable at all values of x .

EXAMPLE 4 Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so, where?

Solution The horizontal tangents, if any, occur where the slope dy/dx is zero. We have

$$\frac{dy}{dx} = \frac{d}{dx}(x^4 - 2x^2 + 2) = 4x^3 - 4x.$$

Now solve the equation $\frac{dy}{dx} = 0$ for x :

$$\begin{aligned} 4x^3 - 4x &= 0 \\ 4x(x^2 - 1) &= 0 \\ x &= 0, 1, -1. \end{aligned}$$

The curve $y = x^4 - 2x^2 + 2$ has horizontal tangents at $x = 0, 1$, and -1 . The corresponding points on the curve are $(0, 2)$, $(1, 1)$ and $(-1, 1)$. See Figure 3.11. We will see in Chapter 4 that finding the values of x where the derivative of a function is equal to zero is an important and useful procedure.

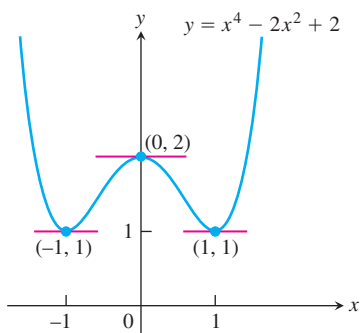


FIGURE 3.11 The curve in Example 4 and its horizontal tangents.

Derivatives of Exponential Functions

We briefly reviewed exponential functions in Section 1.5. When we apply the definition of the derivative to $f(x) = a^x$, we get

$$\begin{aligned} \frac{d}{dx}(a^x) &= \lim_{h \rightarrow 0} \frac{a^{x+h} - a^x}{h} && \text{Derivative definition} \\ &= \lim_{h \rightarrow 0} \frac{a^x \cdot a^h - a^x}{h} && a^{x+h} = a^x \cdot a^h \\ &= \lim_{h \rightarrow 0} a^x \cdot \frac{a^h - 1}{h} && \text{Factoring out } a^x \\ &= a^x \cdot \lim_{h \rightarrow 0} \frac{a^h - 1}{h} && a^x \text{ is constant as } h \rightarrow 0. \\ &= \underbrace{\left(\lim_{h \rightarrow 0} \frac{a^h - 1}{h} \right)}_{\text{a fixed number } L} \cdot a^x. && (1) \end{aligned}$$

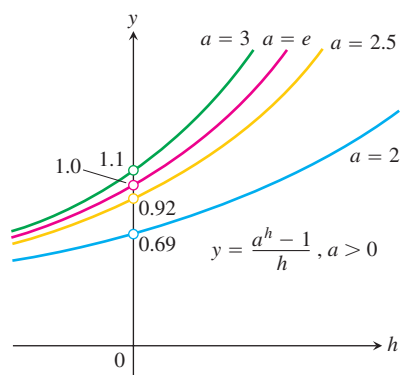


FIGURE 3.12 The position of the curve $y = (a^h - 1)/h, a > 0$, varies continuously with a .

Thus we see that the derivative of a^x is a constant multiple L of a^x . The constant L is a limit unlike any we have encountered before. Note, however, that it equals the derivative of $f(x) = a^x$ at $x = 0$:

$$f'(0) = \lim_{h \rightarrow 0} \frac{a^h - a^0}{h} = \lim_{h \rightarrow 0} \frac{a^h - 1}{h} = L.$$

The limit L is therefore the slope of the graph of $f(x) = a^x$ where it crosses the y -axis. In Chapter 7, where we carefully develop the logarithmic and exponential functions, we prove that the limit L exists and has the value $\ln a$. For now we investigate values of L by graphing the function $y = (a^h - 1)/h$ and studying its behavior as h approaches 0.

Figure 3.12 shows the graphs of $y = (a^h - 1)/h$ for four different values of a . The limit L is approximately 0.69 if $a = 2$, about 0.92 if $a = 2.5$, and about 1.1 if $a = 3$. It appears that the value of L is 1 at some number a chosen between 2.5 and 3. That number is given by $a = e \approx 2.718281828$. With this choice of base we obtain the natural exponential function $f(x) = e^x$ as in Section 1.5, and see that it satisfies the property

$$f'(0) = \lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1. \tag{2}$$

That the limit is 1 implies an important relationship between the natural exponential function e^x and its derivative:

$$\begin{aligned} \frac{d}{dx}(e^x) &= \lim_{h \rightarrow 0} \left(\frac{e^h - 1}{h} \right) \cdot e^x && \text{Eq. (1) with } a = e \\ &= 1 \cdot e^x = e^x. && \text{Eq. (2)} \end{aligned}$$

Therefore the natural exponential function is its own derivative.

Derivative of the Natural Exponential Function

$$\frac{d}{dx}(e^x) = e^x$$

EXAMPLE 5 Find an equation for a line that is tangent to the graph of $y = e^x$ and goes through the origin.

Solution Since the line passes through the origin, its equation is of the form $y = mx$, where m is the slope. If it is tangent to the graph at the point (a, e^a) , the slope is $m = (e^a - 0)/(a - 0)$. The slope of the natural exponential at $x = a$ is e^a . Because these slopes are the same, we then have that $e^a = e^a/a$. It follows that $a = 1$ and $m = e$, so the equation of the tangent line is $y = ex$. See Figure 3.13. ■

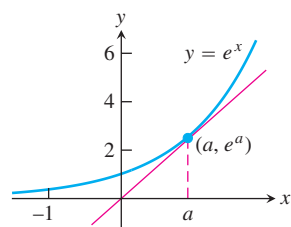


FIGURE 3.13 The line through the origin is tangent to the graph of $y = e^x$ when $a = 1$ (Example 5).

We might ask if there are functions *other* than the natural exponential function that are their own derivatives. The answer is that the only functions that satisfy the property that $f'(x) = f(x)$ are functions that are constant multiples of the natural exponential function, $f(x) = c \cdot e^x$, c any constant. We prove this fact in Section 7.2. Note from the Constant Multiple Rule that indeed

$$\frac{d}{dx}(c \cdot e^x) = c \cdot \frac{d}{dx}(e^x) = c \cdot e^x.$$

Products and Quotients

While the derivative of the sum of two functions is the sum of their derivatives, the derivative of the product of two functions is *not* the product of their derivatives. For instance,

$$\frac{d}{dx}(x \cdot x) = \frac{d}{dx}(x^2) = 2x, \quad \text{while} \quad \frac{d}{dx}(x) \cdot \frac{d}{dx}(x) = 1 \cdot 1 = 1.$$

The derivative of a product of two functions is the sum of *two* products, as we now explain.

Derivative Product Rule

If u and v are differentiable at x , then so is their product uv , and

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}.$$

The derivative of the product uv is u times the derivative of v plus v times the derivative of u . In *prime notation*, $(uv)' = uv' + vu'$. In function notation,

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x).$$

EXAMPLE 6 Find the derivative of (a) $y = \frac{1}{x}(x^2 + e^x)$, (b) $y = e^{2x}$.

Solution

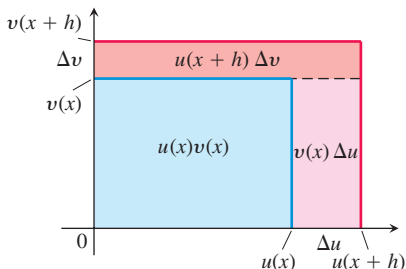
(a) We apply the Product Rule with $u = 1/x$ and $v = x^2 + e^x$:

$$\begin{aligned} \frac{d}{dx} \left[\frac{1}{x}(x^2 + e^x) \right] &= \frac{1}{x}(2x + e^x) + (x^2 + e^x) \left(-\frac{1}{x^2} \right) && \frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}, \text{ and} \\ &= 2 + \frac{e^x}{x} - 1 - \frac{e^x}{x^2} && \frac{d}{dx} \left(\frac{1}{x} \right) = -\frac{1}{x^2} \\ &= 1 + (x - 1) \frac{e^x}{x^2}. \end{aligned}$$

(b) $\frac{d}{dx}(e^{2x}) = \frac{d}{dx}(e^x \cdot e^x) = e^x \cdot \frac{d}{dx}(e^x) + e^x \cdot \frac{d}{dx}(e^x) = 2e^x \cdot e^x = 2e^{2x}$ ■

Picturing the Product Rule

Suppose $u(x)$ and $v(x)$ are positive and increase when x increases, and $h > 0$.



Then the change in the product uv is the difference in areas of the larger and smaller “squares,” which is the sum of the upper and right-hand reddish-shaded rectangles. That is,

$$\begin{aligned} \Delta(uv) &= u(x+h)v(x+h) - u(x)v(x) \\ &= u(x+h)\Delta v + v(x)\Delta u. \end{aligned}$$

Division by h gives

$$\frac{\Delta(uv)}{h} = u(x+h) \frac{\Delta v}{h} + v(x) \frac{\Delta u}{h}.$$

The limit as $h \rightarrow 0^+$ gives the Product Rule.

Proof of the Derivative Product Rule

$$\frac{d}{dx}(uv) = \lim_{h \rightarrow 0} \frac{u(x+h)v(x+h) - u(x)v(x)}{h}$$

To change this fraction into an equivalent one that contains difference quotients for the derivatives of u and v , we subtract and add $u(x+h)v(x)$ in the numerator:

$$\begin{aligned} \frac{d}{dx}(uv) &= \lim_{h \rightarrow 0} \frac{u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x)}{h} \\ &= \lim_{h \rightarrow 0} \left[u(x+h) \frac{v(x+h) - v(x)}{h} + v(x) \frac{u(x+h) - u(x)}{h} \right] \\ &= \lim_{h \rightarrow 0} u(x+h) \cdot \lim_{h \rightarrow 0} \frac{v(x+h) - v(x)}{h} + v(x) \cdot \lim_{h \rightarrow 0} \frac{u(x+h) - u(x)}{h}. \end{aligned}$$

As h approaches zero, $u(x+h)$ approaches $u(x)$ because u , being differentiable at x , is continuous at x . The two fractions approach the values of dv/dx at x and du/dx at x . In short,

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}. \quad \blacksquare$$

EXAMPLE 7 Find the derivative of $y = (x^2 + 1)(x^3 + 3)$.

Solution

(a) From the Product Rule with $u = x^2 + 1$ and $v = x^3 + 3$, we find

$$\begin{aligned}\frac{d}{dx}[(x^2 + 1)(x^3 + 3)] &= (x^2 + 1)(3x^2) + (x^3 + 3)(2x) & \frac{d}{dx}(uv) &= u \frac{dv}{dx} + v \frac{du}{dx} \\ &= 3x^4 + 3x^2 + 2x^4 + 6x \\ &= 5x^4 + 3x^2 + 6x.\end{aligned}$$

(b) This particular product can be differentiated as well (perhaps better) by multiplying out the original expression for y and differentiating the resulting polynomial:

$$\begin{aligned}y &= (x^2 + 1)(x^3 + 3) = x^5 + x^3 + 3x^2 + 3 \\ \frac{dy}{dx} &= 5x^4 + 3x^2 + 6x.\end{aligned}$$

This is in agreement with our first calculation. ■

The derivative of the quotient of two functions is given by the Quotient Rule.

Derivative Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x , and

$$\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}.$$

In function notation,

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$

EXAMPLE 8 Find the derivative of (a) $y = \frac{t^2 - 1}{t^3 + 1}$, (b) $y = e^{-x}$.

Solution

(a) We apply the Quotient Rule with $u = t^2 - 1$ and $v = t^3 + 1$:

$$\begin{aligned}\frac{dy}{dt} &= \frac{(t^3 + 1) \cdot 2t - (t^2 - 1) \cdot 3t^2}{(t^3 + 1)^2} & \frac{d}{dt} \left(\frac{u}{v} \right) &= \frac{v(du/dt) - u(dv/dt)}{v^2} \\ &= \frac{2t^4 + 2t - 3t^4 + 3t^2}{(t^3 + 1)^2} \\ &= \frac{-t^4 + 3t^2 + 2t}{(t^3 + 1)^2}.\end{aligned}$$

(b) $\frac{d}{dx}(e^{-x}) = \frac{d}{dx} \left(\frac{1}{e^x} \right) = \frac{e^x \cdot 0 - 1 \cdot e^x}{(e^x)^2} = \frac{-1}{e^x} = -e^{-x}$ ■

Proof of the Derivative Quotient Rule

$$\begin{aligned}\frac{d}{dx}\left(\frac{u}{v}\right) &= \lim_{h \rightarrow 0} \frac{\frac{u(x+h)}{v(x+h)} - \frac{u(x)}{v(x)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{v(x)u(x+h) - u(x)v(x+h)}{hv(x+h)v(x)}\end{aligned}$$

To change the last fraction into an equivalent one that contains the difference quotients for the derivatives of u and v , we subtract and add $v(x)u(x)$ in the numerator. We then get

$$\begin{aligned}\frac{d}{dx}\left(\frac{u}{v}\right) &= \lim_{h \rightarrow 0} \frac{v(x)u(x+h) - v(x)u(x) + v(x)u(x) - u(x)v(x+h)}{hv(x+h)v(x)} \\ &= \lim_{h \rightarrow 0} \frac{v(x)\frac{u(x+h) - u(x)}{h} - u(x)\frac{v(x+h) - v(x)}{h}}{v(x+h)v(x)}.\end{aligned}$$

Taking the limits in the numerator and denominator now gives the Quotient Rule. ■

The choice of which rules to use in solving a differentiation problem can make a difference in how much work you have to do. Here is an example.

EXAMPLE 9 Rather than using the Quotient Rule to find the derivative of

$$y = \frac{(x-1)(x^2-2x)}{x^4},$$

expand the numerator and divide by x^4 :

$$y = \frac{(x-1)(x^2-2x)}{x^4} = \frac{x^3 - 3x^2 + 2x}{x^4} = x^{-1} - 3x^{-2} + 2x^{-3}.$$

Then use the Sum and Power Rules:

$$\begin{aligned}\frac{dy}{dx} &= -x^{-2} - 3(-2)x^{-3} + 2(-3)x^{-4} \\ &= -\frac{1}{x^2} + \frac{6}{x^3} - \frac{6}{x^4}.\end{aligned}$$

Second- and Higher-Order Derivatives

If $y = f(x)$ is a differentiable function, then its derivative $f'(x)$ is also a function. If f' is also differentiable, then we can differentiate f' to get a new function of x denoted by f'' . So $f'' = (f')'$. The function f'' is called the **second derivative** of f because it is the derivative of the first derivative. It is written in several ways:

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = y'' = D^2(f)(x) = D_x^2 f(x).$$

The symbol D^2 means the operation of differentiation is performed twice.

If $y = x^6$, then $y' = 6x^5$ and we have

$$y'' = \frac{dy'}{dx} = \frac{d}{dx}(6x^5) = 30x^4.$$

Thus $D^2(x^6) = 30x^4$.

How to Read the Symbols for Derivatives

y'	“y prime”
y''	“y double prime”
$\frac{d^2y}{dx^2}$	“d squared y dx squared”
y'''	“y triple prime”
$y^{(n)}$	“y super n”
$\frac{d^ny}{dx^n}$	“d to the n of y by dx to the n”
D^n	“D to the n”

If y'' is differentiable, its derivative, $y''' = dy''/dx = d^3y/dx^3$, is the **third derivative** of y with respect to x . The names continue as you imagine, with

$$y^{(n)} = \frac{d}{dx}y^{(n-1)} = \frac{d^ny}{dx^n} = D^ny$$

denoting the **n th derivative** of y with respect to x for any positive integer n .

We can interpret the second derivative as the rate of change of the slope of the tangent to the graph of $y = f(x)$ at each point. You will see in the next chapter that the second derivative reveals whether the graph bends upward or downward from the tangent line as we move off the point of tangency. In the next section, we interpret both the second and third derivatives in terms of motion along a straight line.

EXAMPLE 10 The first four derivatives of $y = x^3 - 3x^2 + 2$ are

$$\text{First derivative: } y' = 3x^2 - 6x$$

$$\text{Second derivative: } y'' = 6x - 6$$

$$\text{Third derivative: } y''' = 6$$

$$\text{Fourth derivative: } y^{(4)} = 0.$$

The function has derivatives of all orders, the fifth and later derivatives all being zero. ■

Exercises 3.3

Derivative Calculations

In Exercises 1–12, find the first and second derivatives.

- | | |
|---|---|
| 1. $y = -x^2 + 3$ | 2. $y = x^2 + x + 8$ |
| 3. $s = 5t^3 - 3t^5$ | 4. $w = 3z^7 - 7z^3 + 21z^2$ |
| 5. $y = \frac{4x^3}{3} - x + 2e^x$ | 6. $y = \frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{4}$ |
| 7. $w = 3z^{-2} - \frac{1}{z}$ | 8. $s = -2t^{-1} + \frac{4}{t^2}$ |
| 9. $y = 6x^2 - 10x - 5x^{-2}$ | 10. $y = 4 - 2x - x^{-3}$ |
| 11. $r = \frac{1}{3s^2} - \frac{5}{2s}$ | 12. $r = \frac{12}{\theta} - \frac{4}{\theta^3} + \frac{1}{\theta^4}$ |

In Exercises 13–16, find y' (a) by applying the Product Rule and (b) by multiplying the factors to produce a sum of simpler terms to differentiate.

- | | |
|---|---------------------------------------|
| 13. $y = (3 - x^2)(x^3 - x + 1)$ | 14. $y = (2x + 3)(5x^2 - 4x)$ |
| 15. $y = (x^2 + 1)\left(x + 5 + \frac{1}{x}\right)$ | 16. $y = (1 + x^2)(x^{3/4} - x^{-3})$ |

Find the derivatives of the functions in Exercises 17–40.

- | | |
|--|---|
| 17. $y = \frac{2x + 5}{3x - 2}$ | 18. $z = \frac{4 - 3x}{3x^2 + x}$ |
| 19. $g(x) = \frac{x^2 - 4}{x + 0.5}$ | 20. $f(t) = \frac{t^2 - 1}{t^2 + t - 2}$ |
| 21. $v = (1 - t)(1 + t^2)^{-1}$ | 22. $w = (2x - 7)^{-1}(x + 5)$ |
| 23. $f(s) = \frac{\sqrt{s} - 1}{\sqrt{s} + 1}$ | 24. $u = \frac{5x + 1}{2\sqrt{x}}$ |
| 25. $v = \frac{1 + x - 4\sqrt{x}}{x}$ | 26. $r = 2\left(\frac{1}{\sqrt{\theta}} + \sqrt{\theta}\right)$ |

- | | |
|--|---|
| 27. $y = \frac{1}{(x^2 - 1)(x^2 + x + 1)}$ | 28. $y = \frac{(x + 1)(x + 2)}{(x - 1)(x - 2)}$ |
| 29. $y = 2e^{-x} + e^{3x}$ | 30. $y = \frac{x^2 + 3e^x}{2e^x - x}$ |
| 31. $y = x^3e^x$ | 32. $w = re^{-r}$ |
| 33. $y = x^{9/4} + e^{-2x}$ | 34. $y = x^{-3/5} + \pi^{3/2}$ |
| 35. $s = 2t^{3/2} + 3e^2$ | 36. $w = \frac{1}{z^{1.4}} + \frac{\pi}{\sqrt{z}}$ |
| 37. $y = \sqrt[3]{x^2} - x^e$ | 38. $y = \sqrt[3]{x^{9.6}} + 2e^{1.3}$ |
| 39. $r = \frac{e^s}{s}$ | 40. $r = e^\theta\left(\frac{1}{\theta^2} + \theta^{-\pi/2}\right)$ |

Find the derivatives of all orders of the functions in Exercises 41–44.

- | | |
|--|------------------------------|
| 41. $y = \frac{x^4}{2} - \frac{3}{2}x^2 - x$ | 42. $y = \frac{x^5}{120}$ |
| 43. $y = (x - 1)(x^2 + 3x - 5)$ | 44. $y = (4x^3 + 3x)(2 - x)$ |

Find the first and second derivatives of the functions in Exercises 45–52.

- | | |
|--|---|
| 45. $y = \frac{x^3 + 7}{x}$ | 46. $s = \frac{t^2 + 5t - 1}{t^2}$ |
| 47. $r = \frac{(\theta - 1)(\theta^2 + \theta + 1)}{\theta^3}$ | 48. $u = \frac{(x^2 + x)(x^2 - x + 1)}{x^4}$ |
| 49. $w = \left(\frac{1 + 3z}{3z}\right)(3 - z)$ | 50. $p = \frac{q^2 + 3}{(q - 1)^3 + (q + 1)^3}$ |
| 51. $w = 3z^2e^{2z}$ | 52. $w = e^z(z - 1)(z^2 + 1)$ |

53. Suppose u and v are functions of x that are differentiable at $x = 0$ and that

$$u(0) = 5, \quad u'(0) = -3, \quad v(0) = -1, \quad v'(0) = 2.$$

Find the values of the following derivatives at $x = 0$.

a. $\frac{d}{dx}(uv)$ b. $\frac{d}{dx}\left(\frac{u}{v}\right)$ c. $\frac{d}{dx}\left(\frac{v}{u}\right)$ d. $\frac{d}{dx}(7v - 2u)$

54. Suppose u and v are differentiable functions of x and that

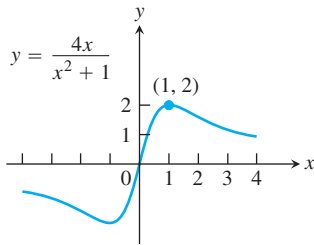
$$u(1) = 2, \quad u'(1) = 0, \quad v(1) = 5, \quad v'(1) = -1.$$

Find the values of the following derivatives at $x = 1$.

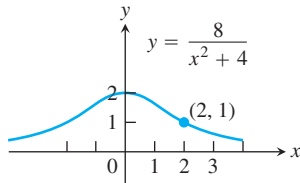
a. $\frac{d}{dx}(uv)$ b. $\frac{d}{dx}\left(\frac{u}{v}\right)$ c. $\frac{d}{dx}\left(\frac{v}{u}\right)$ d. $\frac{d}{dx}(7v - 2u)$

Slopes and Tangents

55. a. **Normal to a curve** Find an equation for the line perpendicular to the tangent to the curve $y = x^3 - 4x + 1$ at the point $(2, 1)$.
 b. **Smallest slope** What is the smallest slope on the curve? At what point on the curve does the curve have this slope?
 c. **Tangents having specified slope** Find equations for the tangents to the curve at the points where the slope of the curve is 8.
56. a. **Horizontal tangents** Find equations for the horizontal tangents to the curve $y = x^3 - 3x - 2$. Also find equations for the lines that are perpendicular to these tangents at the points of tangency.
 b. **Smallest slope** What is the smallest slope on the curve? At what point on the curve does the curve have this slope? Find an equation for the line that is perpendicular to the curve's tangent at this point.
57. Find the tangents to *Newton's serpentine* (graphed here) at the origin and the point $(1, 2)$.

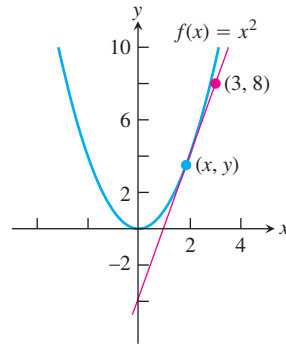


58. Find the tangent to the *Witch of Agnesi* (graphed here) at the point $(2, 1)$.



59. **Quadratic tangent to identity function** The curve $y = ax^2 + bx + c$ passes through the point $(1, 2)$ and is tangent to the line $y = x$ at the origin. Find a , b , and c .
60. **Quadratics having a common tangent** The curves $y = x^2 + ax + b$ and $y = cx - x^2$ have a common tangent line at the point $(1, 0)$. Find a , b , and c .
61. Find all points (x, y) on the graph of $f(x) = 3x^2 - 4x$ with tangent lines parallel to the line $y = 8x + 5$.

62. Find all points (x, y) on the graph of $g(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 1$ with tangent lines parallel to the line $8x - 2y = 1$.
63. Find all points (x, y) on the graph of $y = x/(x - 2)$ with tangent lines perpendicular to the line $y = 2x + 3$.
64. Find all points (x, y) on the graph of $f(x) = x^2$ with tangent lines passing through the point $(3, 8)$.



65. a. Find an equation for the line that is tangent to the curve $y = x^3 - x$ at the point $(-1, 0)$.
 T b. Graph the curve and tangent line together. The tangent intersects the curve at another point. Use Zoom and Trace to estimate the point's coordinates.
 T c. Confirm your estimates of the coordinates of the second intersection point by solving the equations for the curve and tangent simultaneously (Solver key).
66. a. Find an equation for the line that is tangent to the curve $y = x^3 - 6x^2 + 5x$ at the origin.
 T b. Graph the curve and tangent together. The tangent intersects the curve at another point. Use Zoom and Trace to estimate the point's coordinates.
 T c. Confirm your estimates of the coordinates of the second intersection point by solving the equations for the curve and tangent simultaneously (Solver key).

Theory and Examples

For Exercises 67 and 68 evaluate each limit by first converting each to a derivative at a particular x -value.

67. $\lim_{x \rightarrow 1} \frac{x^{50} - 1}{x - 1}$ 68. $\lim_{x \rightarrow -1} \frac{x^{2/9} - 1}{x + 1}$

69. Find the value of a that makes the following function differentiable for all x -values.

$$g(x) = \begin{cases} ax, & \text{if } x < 0 \\ x^2 - 3x, & \text{if } x \geq 0 \end{cases}$$

70. Find the values of a and b that make the following function differentiable for all x -values.

$$f(x) = \begin{cases} ax + b, & x > -1 \\ bx^2 - 3, & x \leq -1 \end{cases}$$

71. The general polynomial of degree n has the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$$

where $a_n \neq 0$. Find $P'(x)$.

- 72. The body's reaction to medicine** The reaction of the body to a dose of medicine can sometimes be represented by an equation of the form

$$R = M^2 \left(\frac{C}{2} - \frac{M}{3} \right),$$

where C is a positive constant and M is the amount of medicine absorbed in the blood. If the reaction is a change in blood pressure, R is measured in millimeters of mercury. If the reaction is a change in temperature, R is measured in degrees, and so on.

Find dR/dM . This derivative, as a function of M , is called the sensitivity of the body to the medicine. In Section 4.5, we will see how to find the amount of medicine to which the body is most sensitive.

- 73.** Suppose that the function v in the Derivative Product Rule has a constant value c . What does the Derivative Product Rule then say? What does this say about the Derivative Constant Multiple Rule?

74. The Reciprocal Rule

- a.** The *Reciprocal Rule* says that at any point where the function $v(x)$ is differentiable and different from zero,

$$\frac{d}{dx} \left(\frac{1}{v} \right) = -\frac{1}{v^2} \frac{dv}{dx}.$$

Show that the Reciprocal Rule is a special case of the Derivative Quotient Rule.

- b.** Show that the Reciprocal Rule and the Derivative Product Rule together imply the Derivative Quotient Rule.

- 75. Generalizing the Product Rule** The Derivative Product Rule gives the formula

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$$

for the derivative of the product uv of two differentiable functions of x .

- a.** What is the analogous formula for the derivative of the product uvw of *three* differentiable functions of x ?
- b.** What is the formula for the derivative of the product $u_1 u_2 u_3 u_4$ of *four* differentiable functions of x ?

- c.** What is the formula for the derivative of a product $u_1 u_2 u_3 \cdots u_n$ of a finite number n of differentiable functions of x ?

- 76. Power Rule for negative integers** Use the Derivative Quotient Rule to prove the Power Rule for negative integers, that is,

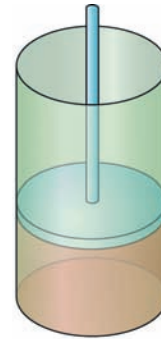
$$\frac{d}{dx}(x^{-m}) = -mx^{-m-1}$$

where m is a positive integer.

- 77. Cylinder pressure** If gas in a cylinder is maintained at a constant temperature T , the pressure P is related to the volume V by a formula of the form

$$P = \frac{nRT}{V - nb} - \frac{an^2}{V^2},$$

in which a , b , n , and R are constants. Find dP/dV . (See accompanying figure.)



- 78. The best quantity to order** One of the formulas for inventory management says that the average weekly cost of ordering, paying for, and holding merchandise is

$$A(q) = \frac{km}{q} + cm + \frac{hq}{2},$$

where q is the quantity you order when things run low (shoes, radios, brooms, or whatever the item might be); k is the cost of placing an order (the same, no matter how often you order); c is the cost of one item (a constant); m is the number of items sold each week (a constant); and h is the weekly holding cost per item (a constant that takes into account things such as space, utilities, insurance, and security). Find dA/dq and d^2A/dq^2 .

3.4

The Derivative as a Rate of Change

In Section 2.1 we introduced average and instantaneous rates of change. In this section we study further applications in which derivatives model the rates at which things change. It is natural to think of a quantity changing with respect to time, but other variables can be treated in the same way. For example, an economist may want to study how the cost of producing steel varies with the number of tons produced, or an engineer may want to know how the power output of a generator varies with its temperature.

Instantaneous Rates of Change

If we interpret the difference quotient $(f(x+h) - f(x))/h$ as the average rate of change in f over the interval from x to $x+h$, we can interpret its limit as $h \rightarrow 0$ as the rate at which f is changing at the point x .

DEFINITION The **instantaneous rate of change** of f with respect to x at x_0 is the derivative

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided the limit exists.

Thus, instantaneous rates are limits of average rates.

It is conventional to use the word *instantaneous* even when x does not represent time. The word is, however, frequently omitted. When we say *rate of change*, we mean *instantaneous rate of change*.

EXAMPLE 1 The area A of a circle is related to its diameter by the equation

$$A = \frac{\pi}{4} D^2.$$

How fast does the area change with respect to the diameter when the diameter is 10 m?

Solution The rate of change of the area with respect to the diameter is

$$\frac{dA}{dD} = \frac{\pi}{4} \cdot 2D = \frac{\pi D}{2}.$$

When $D = 10$ m, the area is changing with respect to the diameter at the rate of $(\pi/2)10 = 5\pi \text{ m}^2/\text{m} \approx 15.71 \text{ m}^2/\text{m}$. ■

Motion Along a Line: Displacement, Velocity, Speed, Acceleration, and Jerk

Suppose that an object is moving along a coordinate line (an s -axis), usually horizontal or vertical, so that we know its position s on that line as a function of time t :

$$s = f(t).$$

The **displacement** of the object over the time interval from t to $t + \Delta t$ (Figure 3.14) is

$$\Delta s = f(t + \Delta t) - f(t),$$

and the **average velocity** of the object over that time interval is

$$v_{av} = \frac{\text{displacement}}{\text{travel time}} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

To find the body's velocity at the exact instant t , we take the limit of the average velocity over the interval from t to $t + \Delta t$ as Δt shrinks to zero. This limit is the derivative of f with respect to t .

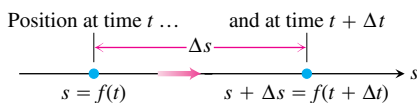


FIGURE 3.14 The positions of a body moving along a coordinate line at time t and shortly later at time $t + \Delta t$. Here the coordinate line is horizontal.

DEFINITION **Velocity (instantaneous velocity)** is the derivative of position with respect to time. If a body's position at time t is $s = f(t)$, then the body's velocity at time t is

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \rightarrow 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

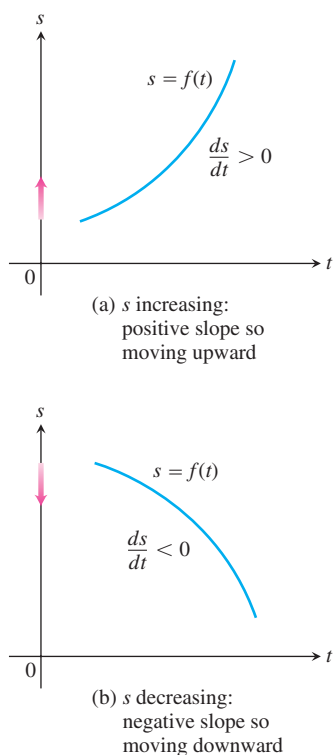


FIGURE 3.15 For motion $s = f(t)$ along a straight line (the vertical axis), $v = ds/dt$ is (a) positive when s increases and (b) negative when s decreases.

HISTORICAL BIOGRAPHY

Bernard Bolzano
(1781–1848)

Besides telling how fast an object is moving along the horizontal line in Figure 3.14, its velocity tells the direction of motion. When the object is moving forward (s increasing), the velocity is positive; when the object is moving backward (s decreasing), the velocity is negative. If the coordinate line is vertical, the object moves upward for positive velocity and downward for negative velocity. The blue curves in Figure 3.15 represent position along the line over time; they do not portray the path of motion, which lies along the s -axis.

If we drive to a friend's house and back at 30 mph, say, the speedometer will show 30 on the way over but it will not show -30 on the way back, even though our distance from home is decreasing. The speedometer always shows *speed*, which is the absolute value of velocity. Speed measures the rate of progress regardless of direction.

DEFINITION **Speed** is the absolute value of velocity.

$$\text{Speed} = |v(t)| = \left| \frac{ds}{dt} \right|$$

EXAMPLE 2 Figure 3.16 shows the graph of the velocity $v = f'(t)$ of a particle moving along a horizontal line (as opposed to showing a position function $s = f(t)$ such as in Figure 3.15). In the graph of the velocity function, it's not the slope of the curve that tells us if the particle is moving forward or backward along the line (which is not shown in the figure), but rather the sign of the velocity. Looking at Figure 3.16, we see that the particle moves forward for the first 3 sec (when the velocity is positive), moves backward for the next 2 sec (the velocity is negative), stands motionless for a full second, and then moves forward again. The particle is speeding up when its positive velocity increases during the first second, moves at a steady speed during the next second, and then slows down as the velocity decreases to zero during the third second. It stops for an instant at $t = 3$ sec (when the velocity is zero) and reverses direction as the velocity starts to become negative. The particle is now moving backward and gaining in speed until $t = 4$ sec, at which time it achieves its greatest speed during its backward motion. Continuing its backward motion at time $t = 4$, the particle starts to slow down again until it finally stops at time $t = 5$ (when the velocity is once again zero). The particle now remains motionless for one full second, and then moves forward again at $t = 6$ sec, speeding up during the final second of the forward motion indicated in the velocity graph. ■

The rate at which a body's velocity changes is the body's *acceleration*. The acceleration measures how quickly the body picks up or loses speed.

A sudden change in acceleration is called a *jerk*. When a ride in a car or a bus is jerky, it is not that the accelerations involved are necessarily large but that the changes in acceleration are abrupt.

DEFINITIONS **Acceleration** is the derivative of velocity with respect to time. If a body's position at time t is $s = f(t)$, then the body's acceleration at time t is

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

Jerk is the derivative of acceleration with respect to time:

$$j(t) = \frac{da}{dt} = \frac{d^3s}{dt^3}.$$

Near the surface of the Earth all bodies fall with the same constant acceleration. Galileo's experiments with free fall (see Section 2.1) lead to the equation

$$s = \frac{1}{2}gt^2,$$

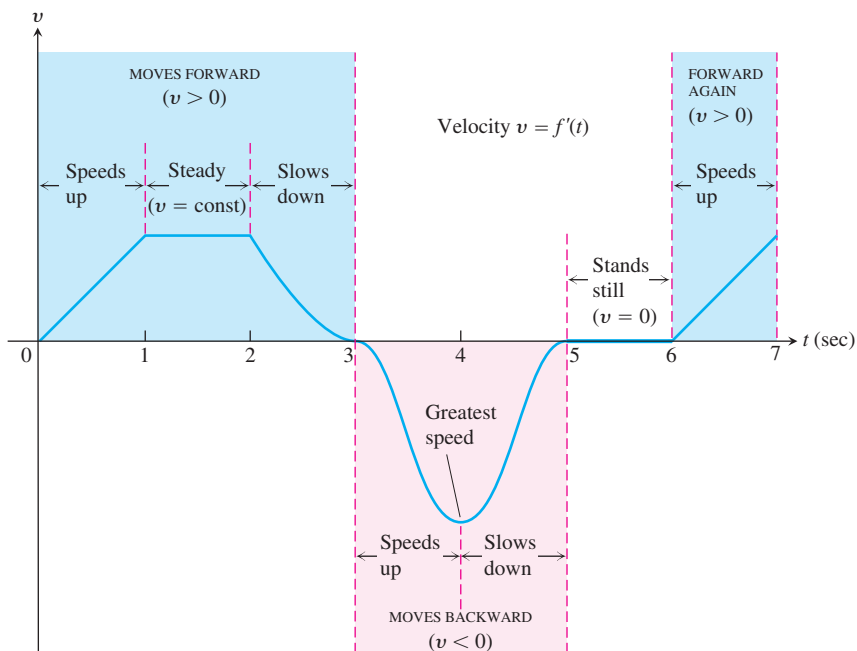


FIGURE 3.16 The velocity graph of a particle moving along a horizontal line, discussed in Example 2.

where s is the distance fallen and g is the acceleration due to Earth’s gravity. This equation holds in a vacuum, where there is no air resistance, and closely models the fall of dense, heavy objects, such as rocks or steel tools, for the first few seconds of their fall, before the effects of air resistance are significant.

The value of g in the equation $s = (1/2)gt^2$ depends on the units used to measure t and s . With t in seconds (the usual unit), the value of g determined by measurement at sea level is approximately 32 ft/sec^2 (feet per second squared) in English units, and $g = 9.8 \text{ m/sec}^2$ (meters per second squared) in metric units. (These gravitational constants depend on the distance from Earth’s center of mass, and are slightly lower on top of Mt. Everest, for example.)

The jerk associated with the constant acceleration of gravity ($g = 32 \text{ ft/sec}^2$) is zero:

$$j = \frac{d}{dt}(g) = 0.$$

An object does not exhibit jerkiness during free fall.

EXAMPLE 3 Figure 3.17 shows the free fall of a heavy ball bearing released from rest at time $t = 0 \text{ sec}$.

- (a) How many meters does the ball fall in the first 2 sec?
- (b) What is its velocity, speed, and acceleration when $t = 2$?

Solution

- (a) The metric free-fall equation is $s = 4.9t^2$. During the first 2 sec, the ball falls

$$s(2) = 4.9(2)^2 = 19.6 \text{ m}.$$

- (b) At any time t , *velocity* is the derivative of position:

$$v(t) = s'(t) = \frac{d}{dt}(4.9t^2) = 9.8t.$$

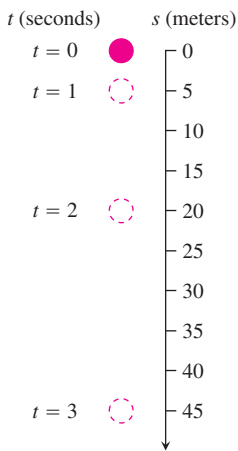


FIGURE 3.17 A ball bearing falling from rest (Example 3).

At $t = 2$, the velocity is

$$v(2) = 19.6 \text{ m/sec}$$

in the downward (increasing s) direction. The *speed* at $t = 2$ is

$$\text{speed} = |v(2)| = 19.6 \text{ m/sec}.$$

The *acceleration* at any time t is

$$a(t) = v'(t) = s''(t) = 9.8 \text{ m/sec}^2.$$

At $t = 2$, the acceleration is 9.8 m/sec^2 . ■

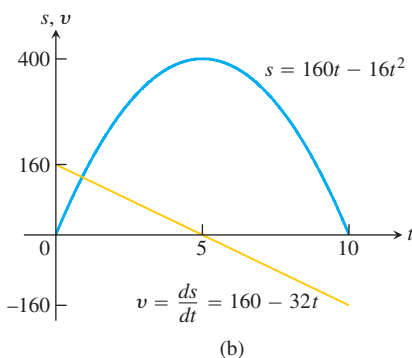
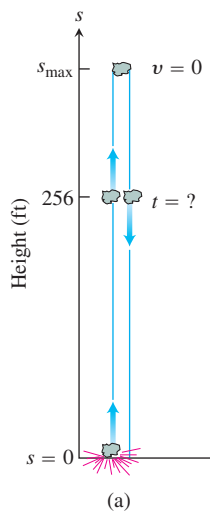


FIGURE 3.18 (a) The rock in Example 4. (b) The graphs of s and v as functions of time; s is largest when $v = ds/dt = 0$. The graph of s is *not* the path of the rock: It is a plot of height versus time. The slope of the plot is the rock's velocity, graphed here as a straight line.

EXAMPLE 4 A dynamite blast blows a heavy rock straight up with a launch velocity of 160 ft/sec (about 109 mph) (Figure 3.18a). It reaches a height of $s = 160t - 16t^2$ ft after t sec.

- How high does the rock go?
- What are the velocity and speed of the rock when it is 256 ft above the ground on the way up? On the way down?
- What is the acceleration of the rock at any time t during its flight (after the blast)?
- When does the rock hit the ground again?

Solution

- In the coordinate system we have chosen, s measures height from the ground up, so the velocity is positive on the way up and negative on the way down. The instant the rock is at its highest point is the one instant during the flight when the velocity is 0. To find the maximum height, all we need to do is to find when $v = 0$ and evaluate s at this time.

At any time t during the rock's motion, its velocity is

$$v = \frac{ds}{dt} = \frac{d}{dt}(160t - 16t^2) = 160 - 32t \text{ ft/sec}.$$

The velocity is zero when

$$160 - 32t = 0 \quad \text{or} \quad t = 5 \text{ sec}.$$

The rock's height at $t = 5$ sec is

$$s_{\max} = s(5) = 160(5) - 16(5)^2 = 800 - 400 = 400 \text{ ft}.$$

See Figure 3.18b.

- To find the rock's velocity at 256 ft on the way up and again on the way down, we first find the two values of t for which

$$s(t) = 160t - 16t^2 = 256.$$

To solve this equation, we write

$$16t^2 - 160t + 256 = 0$$

$$16(t^2 - 10t + 16) = 0$$

$$(t - 2)(t - 8) = 0$$

$$t = 2 \text{ sec}, t = 8 \text{ sec}.$$

The rock is 256 ft above the ground 2 sec after the explosion and again 8 sec after the explosion. The rock's velocities at these times are

$$v(2) = 160 - 32(2) = 160 - 64 = 96 \text{ ft/sec}.$$

$$v(8) = 160 - 32(8) = 160 - 256 = -96 \text{ ft/sec}.$$

At both instants, the rock's speed is 96 ft/sec. Since $v(2) > 0$, the rock is moving upward (s is increasing) at $t = 2$ sec; it is moving downward (s is decreasing) at $t = 8$ because $v(8) < 0$.

- (c) At any time during its flight following the explosion, the rock's acceleration is a constant

$$a = \frac{dv}{dt} = \frac{d}{dt}(160 - 32t) = -32 \text{ ft/sec}^2.$$

The acceleration is always downward. As the rock rises, it slows down; as it falls, it speeds up.

- (d) The rock hits the ground at the positive time t for which $s = 0$. The equation $160t - 16t^2 = 0$ factors to give $16t(10 - t) = 0$, so it has solutions $t = 0$ and $t = 10$. At $t = 0$, the blast occurred and the rock was thrown upward. It returned to the ground 10 sec later. ■

Derivatives in Economics

Engineers use the terms *velocity* and *acceleration* to refer to the derivatives of functions describing motion. Economists, too, have a specialized vocabulary for rates of change and derivatives. They call them *marginals*.

In a manufacturing operation, the *cost of production* $c(x)$ is a function of x , the number of units produced. The **marginal cost of production** is the rate of change of cost with respect to level of production, so it is dc/dx .

Suppose that $c(x)$ represents the dollars needed to produce x tons of steel in one week. It costs more to produce $x + h$ tons per week, and the cost difference, divided by h , is the average cost of producing each additional ton:

$$\frac{c(x + h) - c(x)}{h} = \text{average cost of each of the additional } h \text{ tons of steel produced.}$$

The limit of this ratio as $h \rightarrow 0$ is the *marginal cost* of producing more steel per week when the current weekly production is x tons (Figure 3.19):

$$\frac{dc}{dx} = \lim_{h \rightarrow 0} \frac{c(x + h) - c(x)}{h} = \text{marginal cost of production.}$$

Sometimes the marginal cost of production is loosely defined to be the extra cost of producing one additional unit:

$$\frac{\Delta c}{\Delta x} = \frac{c(x + 1) - c(x)}{1},$$

which is approximated by the value of dc/dx at x . This approximation is acceptable if the slope of the graph of c does not change quickly near x . Then the difference quotient will be close to its limit dc/dx , which is the rise in the tangent line if $\Delta x = 1$ (Figure 3.20). The approximation works best for large values of x .

Economists often represent a total cost function by a cubic polynomial

$$c(x) = \alpha x^3 + \beta x^2 + \gamma x + \delta$$

where δ represents *fixed costs* such as rent, heat, equipment capitalization, and management costs. The other terms represent *variable costs* such as the costs of raw materials, taxes, and labor. Fixed costs are independent of the number of units produced, whereas variable costs depend on the quantity produced. A cubic polynomial is usually adequate to capture the cost behavior on a realistic quantity interval.

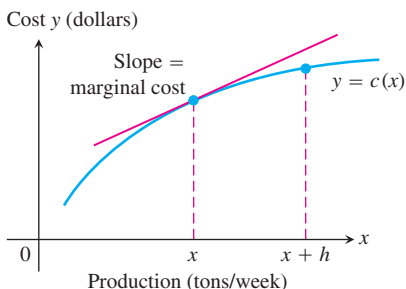


FIGURE 3.19 Weekly steel production: $c(x)$ is the cost of producing x tons per week. The cost of producing an additional h tons is $c(x + h) - c(x)$.

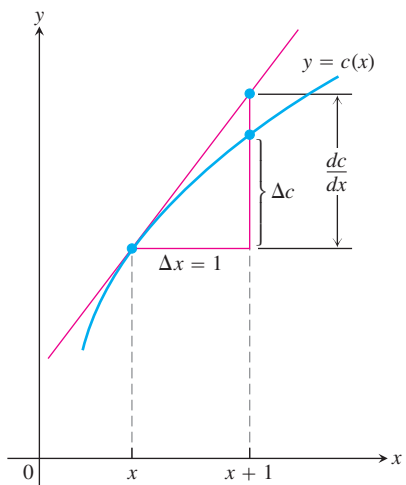


FIGURE 3.20 The marginal cost dc/dx is approximately the extra cost Δc of producing $\Delta x = 1$ more unit.

EXAMPLE 5 Suppose that it costs

$$c(x) = x^3 - 6x^2 + 15x$$

dollars to produce x radiators when 8 to 30 radiators are produced and that

$$r(x) = x^3 - 3x^2 + 12x$$

gives the dollar revenue from selling x radiators. Your shop currently produces 10 radiators a day. About how much extra will it cost to produce one more radiator a day, and what is your estimated increase in revenue for selling 11 radiators a day?

Solution The cost of producing one more radiator a day when 10 are produced is about $c'(10)$:

$$c'(x) = \frac{d}{dx}(x^3 - 6x^2 + 15x) = 3x^2 - 12x + 15$$

$$c'(10) = 3(100) - 12(10) + 15 = 195.$$

The additional cost will be about \$195. The marginal revenue is

$$r'(x) = \frac{d}{dx}(x^3 - 3x^2 + 12x) = 3x^2 - 6x + 12.$$

The marginal revenue function estimates the increase in revenue that will result from selling one additional unit. If you currently sell 10 radiators a day, you can expect your revenue to increase by about

$$r'(10) = 3(100) - 6(10) + 12 = \$252$$

if you increase sales to 11 radiators a day. ■

EXAMPLE 6 To get some feel for the language of marginal rates, consider marginal tax rates. If your marginal income tax rate is 28% and your income increases by \$1000, you can expect to pay an extra \$280 in taxes. This does not mean that you pay 28% of your entire income in taxes. It just means that at your current income level I , the rate of increase of taxes T with respect to income is $dT/dI = 0.28$. You will pay \$0.28 in taxes out of every extra dollar you earn. Of course, if you earn a lot more, you may land in a higher tax bracket and your marginal rate will increase. ■

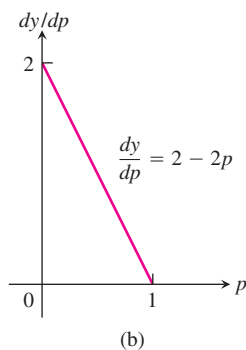
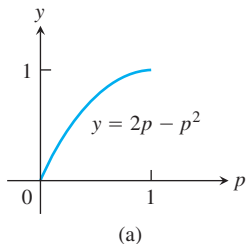


FIGURE 3.21 (a) The graph of $y = 2p - p^2$, describing the proportion of smooth-skinned peas in the next generation. (b) The graph of dy/dp (Example 7).

Sensitivity to Change

When a small change in x produces a large change in the value of a function $f(x)$, we say that the function is relatively **sensitive** to changes in x . The derivative $f'(x)$ is a measure of this sensitivity.

EXAMPLE 7 Genetic Data and Sensitivity to Change

The Austrian monk Gregor Johann Mendel (1822–1884), working with garden peas and other plants, provided the first scientific explanation of hybridization.

His careful records showed that if p (a number between 0 and 1) is the frequency of the gene for smooth skin in peas (dominant) and $(1 - p)$ is the frequency of the gene for wrinkled skin in peas, then the proportion of smooth-skinned peas in the next generation will be

$$y = 2p(1 - p) + p^2 = 2p - p^2.$$

The graph of y versus p in Figure 3.21a suggests that the value of y is more sensitive to a change in p when p is small than when p is large. Indeed, this fact is borne out by the derivative graph in Figure 3.21b, which shows that dy/dp is close to 2 when p is near 0 and close to 0 when p is near 1.

The implication for genetics is that introducing a few more smooth skin genes into a population where the frequency of wrinkled skin peas is large will have a more dramatic effect on later generations than will a similar increase when the population has a large proportion of smooth skin peas. ■

Exercises 3.4

Motion Along a Coordinate Line

Exercises 1–6 give the positions $s = f(t)$ of a body moving on a coordinate line, with s in meters and t in seconds.

- Find the body's displacement and average velocity for the given time interval.
 - Find the body's speed and acceleration at the endpoints of the interval.
 - When, if ever, during the interval does the body change direction?
- $s = t^2 - 3t + 2$, $0 \leq t \leq 2$
 - $s = 6t - t^2$, $0 \leq t \leq 6$
 - $s = -t^3 + 3t^2 - 3t$, $0 \leq t \leq 3$
 - $s = (t^4/4) - t^3 + t^2$, $0 \leq t \leq 3$
 - $s = \frac{25}{t^2} - \frac{5}{t}$, $1 \leq t \leq 5$
 - $s = \frac{25}{t+5}$, $-4 \leq t \leq 0$
- Particle motion** At time t , the position of a body moving along the s -axis is $s = t^3 - 6t^2 + 9t$ m.
 - Find the body's acceleration each time the velocity is zero.
 - Find the body's speed each time the acceleration is zero.
 - Find the total distance traveled by the body from $t = 0$ to $t = 2$.
 - Particle motion** At time $t \geq 0$, the velocity of a body moving along the horizontal s -axis is $v = t^2 - 4t + 3$.
 - Find the body's acceleration each time the velocity is zero.
 - When is the body moving forward? Backward?
 - When is the body's velocity increasing? Decreasing?

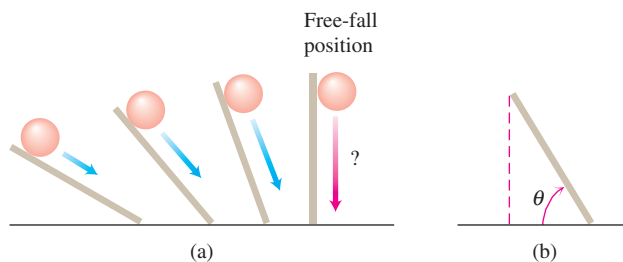
Free-Fall Applications

- Free fall on Mars and Jupiter** The equations for free fall at the surfaces of Mars and Jupiter (s in meters, t in seconds) are $s = 1.86t^2$ on Mars and $s = 11.44t^2$ on Jupiter. How long does it take a rock falling from rest to reach a velocity of 27.8 m/sec (about 100 km/h) on each planet?
- Lunar projectile motion** A rock thrown vertically upward from the surface of the moon at a velocity of 24 m/sec (about 86 km/h) reaches a height of $s = 24t - 0.8t^2$ m in t sec.
 - Find the rock's velocity and acceleration at time t . (The acceleration in this case is the acceleration of gravity on the moon.)
 - How long does it take the rock to reach its highest point?
 - How high does the rock go?
 - How long does it take the rock to reach half its maximum height?
 - How long is the rock aloft?
- Finding g on a small airless planet** Explorers on a small airless planet used a spring gun to launch a ball bearing vertically upward from the surface at a launch velocity of 15 m/sec. Because the acceleration of gravity at the planet's surface was g_s m/sec², the explorers expected the ball bearing to reach a height of $s = 15t - (1/2)g_s t^2$ m t sec later. The ball bearing reached its maximum height 20 sec after being launched. What was the value of g_s ?

- Speeding bullet** A 45-caliber bullet shot straight up from the surface of the moon would reach a height of $s = 832t - 2.6t^2$ ft after t sec. On Earth, in the absence of air, its height would be $s = 832t - 16t^2$ ft after t sec. How long will the bullet be aloft in each case? How high will the bullet go?
- Free fall from the Tower of Pisa** Had Galileo dropped a cannonball from the Tower of Pisa, 179 ft above the ground, the ball's height above the ground t sec into the fall would have been $s = 179 - 16t^2$.
 - What would have been the ball's velocity, speed, and acceleration at time t ?
 - About how long would it have taken the ball to hit the ground?
 - What would have been the ball's velocity at the moment of impact?
- Galileo's free-fall formula** Galileo developed a formula for a body's velocity during free fall by rolling balls from rest down increasingly steep inclined planks and looking for a limiting formula that would predict a ball's behavior when the plank was vertical and the ball fell freely; see part (a) of the accompanying figure. He found that, for any given angle of the plank, the ball's velocity t sec into motion was a constant multiple of t . That is, the velocity was given by a formula of the form $v = kt$. The value of the constant k depended on the inclination of the plank.

In modern notation—part (b) of the figure—with distance in meters and time in seconds, what Galileo determined by experiment was that, for any given angle θ , the ball's velocity t sec into the roll was

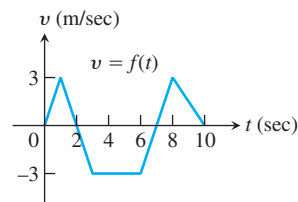
$$v = 9.8(\sin \theta)t \text{ m/sec.}$$



- What is the equation for the ball's velocity during free fall?
- Building on your work in part (a), what constant acceleration does a freely falling body experience near the surface of Earth?

Understanding Motion from Graphs

- The accompanying figure shows the velocity $v = ds/dt = f(t)$ (m/sec) of a body moving along a coordinate line.

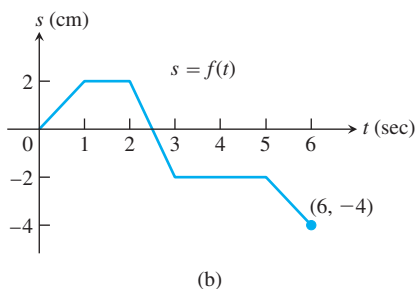
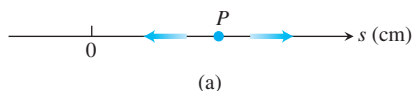


- When does the body reverse direction?
- When (approximately) is the body moving at a constant speed?

c. Graph the body's speed for $0 \leq t \leq 10$.

d. Graph the acceleration, where defined.

16. A particle P moves on the number line shown in part (a) of the accompanying figure. Part (b) shows the position of P as a function of time t .



a. When is P moving to the left? Moving to the right? Standing still?

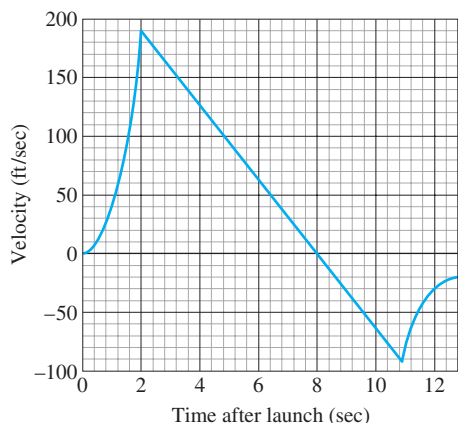
b. Graph the particle's velocity and speed (where defined).

17. **Launching a rocket** When a model rocket is launched, the propellant burns for a few seconds, accelerating the rocket upward. After burnout, the rocket coasts upward for a while and then begins to fall. A small explosive charge pops out a parachute shortly after the rocket starts down. The parachute slows the rocket to keep it from breaking when it lands.

The figure here shows velocity data from the flight of the model rocket. Use the data to answer the following.

a. How fast was the rocket climbing when the engine stopped?

b. For how many seconds did the engine burn?



c. When did the rocket reach its highest point? What was its velocity then?

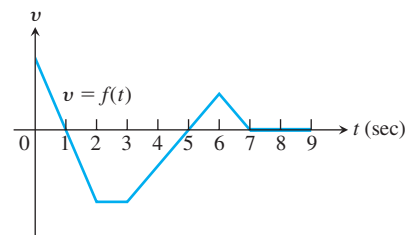
d. When did the parachute pop out? How fast was the rocket falling then?

e. How long did the rocket fall before the parachute opened?

f. When was the rocket's acceleration greatest?

g. When was the acceleration constant? What was its value then (to the nearest integer)?

18. The accompanying figure shows the velocity $v = f(t)$ of a particle moving on a horizontal coordinate line.



a. When does the particle move forward? Move backward?

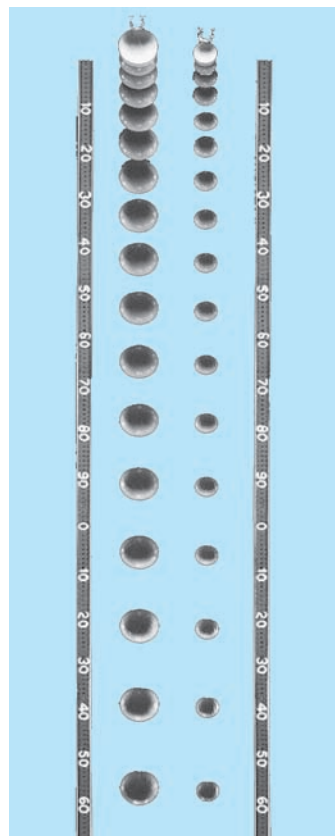
Speed up? Slow down?

b. When is the particle's acceleration positive? Negative? Zero?

c. When does the particle move at its greatest speed?

d. When does the particle stand still for more than an instant?

19. **Two falling balls** The multiflash photograph in the accompanying figure shows two balls falling from rest. The vertical rulers are marked in centimeters. Use the equation $s = 490t^2$ (the free-fall equation for s in centimeters and t in seconds) to answer the following questions.



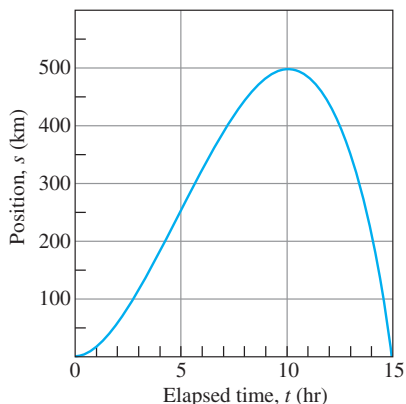
a. How long did it take the balls to fall the first 160 cm? What was their average velocity for the period?

b. How fast were the balls falling when they reached the 160-cm mark? What was their acceleration then?

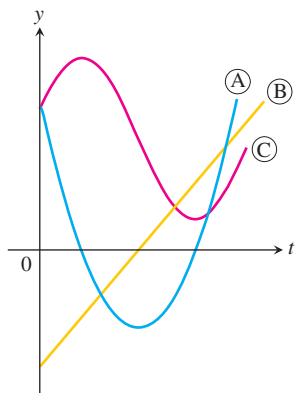
c. About how fast was the light flashing (flashes per second)?

20. A traveling truck The accompanying graph shows the position s of a truck traveling on a highway. The truck starts at $t = 0$ and returns 15 h later at $t = 15$.

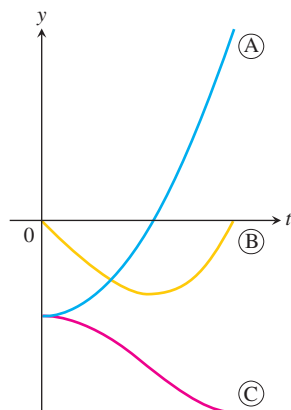
- Use the technique described in Section 3.2, Example 3, to graph the truck's velocity $v = ds/dt$ for $0 \leq t \leq 15$. Then repeat the process, with the velocity curve, to graph the truck's acceleration dv/dt .
- Suppose that $s = 15t^2 - t^3$. Graph ds/dt and d^2s/dt^2 and compare your graphs with those in part (a).



21. The graphs in the accompanying figure show the position s , velocity $v = ds/dt$, and acceleration $a = d^2s/dt^2$ of a body moving along a coordinate line as functions of time t . Which graph is which? Give reasons for your answers.



22. The graphs in the accompanying figure show the position s , the velocity $v = ds/dt$, and the acceleration $a = d^2s/dt^2$ of a body moving along the coordinate line as functions of time t . Which graph is which? Give reasons for your answers.



Economics

23. Marginal cost Suppose that the dollar cost of producing x washing machines is $c(x) = 2000 + 100x - 0.1x^2$.

- Find the average cost per machine of producing the first 100 washing machines.
- Find the marginal cost when 100 washing machines are produced.
- Show that the marginal cost when 100 washing machines are produced is approximately the cost of producing one more washing machine after the first 100 have been made, by calculating the latter cost directly.

24. Marginal revenue Suppose that the revenue from selling x washing machines is

$$r(x) = 20,000 \left(1 - \frac{1}{x} \right)$$

dollars.

- Find the marginal revenue when 100 machines are produced.
- Use the function $r'(x)$ to estimate the increase in revenue that will result from increasing production from 100 machines a week to 101 machines a week.
- Find the limit of $r'(x)$ as $x \rightarrow \infty$. How would you interpret this number?

Additional Applications

25. Bacterium population When a bactericide was added to a nutrient broth in which bacteria were growing, the bacterium population continued to grow for a while, but then stopped growing and began to decline. The size of the population at time t (hours) was $b = 10^6 + 10^4t - 10^3t^2$. Find the growth rates at

- $t = 0$ hours.
- $t = 5$ hours.
- $t = 10$ hours.

26. Draining a tank The number of gallons of water in a tank t minutes after the tank has started to drain is $Q(t) = 200(30 - t)^2$. How fast is the water running out at the end of 10 min? What is the average rate at which the water flows out during the first 10 min?

T 27. Draining a tank It takes 12 hours to drain a storage tank by opening the valve at the bottom. The depth y of fluid in the tank t hours after the valve is opened is given by the formula

$$y = 6 \left(1 - \frac{t}{12} \right)^2 \text{ m.}$$

- Find the rate dy/dt (m/h) at which the tank is draining at time t .
- When is the fluid level in the tank falling fastest? Slowest? What are the values of dy/dt at these times?
- Graph y and dy/dt together and discuss the behavior of y in relation to the signs and values of dy/dt .

28. Inflating a balloon The volume $V = (4/3)\pi r^3$ of a spherical balloon changes with the radius.

- At what rate (ft^3/ft) does the volume change with respect to the radius when $r = 2$ ft?
- By approximately how much does the volume increase when the radius changes from 2 to 2.2 ft?

- 29. Airplane takeoff** Suppose that the distance an aircraft travels along a runway before takeoff is given by $D = (10/9)t^2$, where D is measured in meters from the starting point and t is measured in seconds from the time the brakes are released. The aircraft will become airborne when its speed reaches 200 km/h. How long will it take to become airborne, and what distance will it travel in that time?
- 30. Volcanic lava fountains** Although the November 1959 Kilauea Iki eruption on the island of Hawaii began with a line of fountains along the wall of the crater, activity was later confined to a single vent in the crater's floor, which at one point shot lava 1900 ft straight into the air (a Hawaiian record). What was the lava's exit velocity in feet per second? In miles per hour? (*Hint:* If v_0 is the exit velocity of a particle of lava, its height t sec later will be $s = v_0t - 16t^2$ ft. Begin by finding the time at which $ds/dt = 0$. Neglect air resistance.)

Analyzing Motion Using Graphs

- T** Exercises 31–34 give the position function $s = f(t)$ of an object moving along the s -axis as a function of time t . Graph f together with the

velocity function $v(t) = ds/dt = f'(t)$ and the acceleration function $a(t) = d^2s/dt^2 = f''(t)$. Comment on the object's behavior in relation to the signs and values of v and a . Include in your commentary such topics as the following:

- When is the object momentarily at rest?
 - When does it move to the left (down) or to the right (up)?
 - When does it change direction?
 - When does it speed up and slow down?
 - When is it moving fastest (highest speed)? Slowest?
 - When is it farthest from the axis origin?
- 31.** $s = 200t - 16t^2$, $0 \leq t \leq 12.5$ (a heavy object fired straight up from Earth's surface at 200 ft/sec)
- 32.** $s = t^2 - 3t + 2$, $0 \leq t \leq 5$
- 33.** $s = t^3 - 6t^2 + 7t$, $0 \leq t \leq 4$
- 34.** $s = 4 - 7t + 6t^2 - t^3$, $0 \leq t \leq 4$

3.5

Derivatives of Trigonometric Functions

Many phenomena of nature are approximately periodic (electromagnetic fields, heart rhythms, tides, weather). The derivatives of sines and cosines play a key role in describing periodic changes. This section shows how to differentiate the six basic trigonometric functions.

Derivative of the Sine Function

To calculate the derivative of $f(x) = \sin x$, for x measured in radians, we combine the limits in Example 5a and Theorem 7 in Section 2.4 with the angle sum identity for the sine function:

$$\sin(x + h) = \sin x \cos h + \cos x \sin h.$$

If $f(x) = \sin x$, then

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} && \text{Derivative definition} \\ &= \lim_{h \rightarrow 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h} = \lim_{h \rightarrow 0} \frac{\sin x (\cos h - 1) + \cos x \sin h}{h} \\ &= \lim_{h \rightarrow 0} \left(\sin x \cdot \frac{\cos h - 1}{h} \right) + \lim_{h \rightarrow 0} \left(\cos x \cdot \frac{\sin h}{h} \right) \\ &= \sin x \cdot \underbrace{\lim_{h \rightarrow 0} \frac{\cos h - 1}{h}}_{\text{limit 0}} + \cos x \cdot \underbrace{\lim_{h \rightarrow 0} \frac{\sin h}{h}}_{\text{limit 1}} = \sin x \cdot 0 + \cos x \cdot 1 = \cos x. \end{aligned}$$

Example 5a and
Theorem 7, Section 2.4

The derivative of the sine function is the cosine function:

$$\frac{d}{dx}(\sin x) = \cos x.$$

EXAMPLE 1 We find derivatives of the sine function involving differences, products, and quotients.

(a) $y = x^2 - \sin x$: $\frac{dy}{dx} = 2x - \frac{d}{dx}(\sin x)$ Difference Rule
 $= 2x - \cos x$

(b) $y = e^x \sin x$: $\frac{dy}{dx} = e^x \frac{d}{dx}(\sin x) + \frac{d}{dx}(e^x) \sin x$ Product Rule
 $= e^x \cos x + e^x \sin x$
 $= e^x (\cos x + \sin x)$

(c) $y = \frac{\sin x}{x}$: $\frac{dy}{dx} = \frac{x \cdot \frac{d}{dx}(\sin x) - \sin x \cdot 1}{x^2}$ Quotient Rule
 $= \frac{x \cos x - \sin x}{x^2}$ ■

Derivative of the Cosine Function

With the help of the angle sum formula for the cosine function,

$$\cos(x + h) = \cos x \cos h - \sin x \sin h,$$

we can compute the limit of the difference quotient:

$$\begin{aligned} \frac{d}{dx}(\cos x) &= \lim_{h \rightarrow 0} \frac{\cos(x + h) - \cos x}{h} && \text{Derivative definition} \\ &= \lim_{h \rightarrow 0} \frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h} && \text{Cosine angle sum identity} \\ &= \lim_{h \rightarrow 0} \frac{\cos x(\cos h - 1) - \sin x \sin h}{h} \\ &= \lim_{h \rightarrow 0} \cos x \cdot \frac{\cos h - 1}{h} - \lim_{h \rightarrow 0} \sin x \cdot \frac{\sin h}{h} \\ &= \cos x \cdot \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} - \sin x \cdot \lim_{h \rightarrow 0} \frac{\sin h}{h} \\ &= \cos x \cdot 0 - \sin x \cdot 1 && \text{Example 5a and Theorem 7, Section 2.4} \\ &= -\sin x. \end{aligned}$$

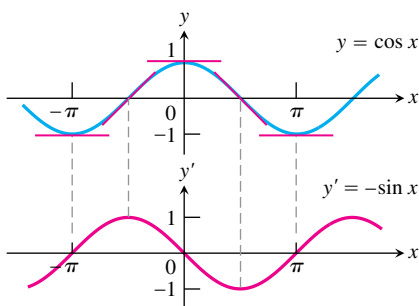


FIGURE 3.22 The curve $y' = -\sin x$ as the graph of the slopes of the tangents to the curve $y = \cos x$.

The derivative of the cosine function is the negative of the sine function:

$$\frac{d}{dx}(\cos x) = -\sin x.$$

Figure 3.22 shows a way to visualize this result in the same way we did for graphing derivatives in Section 3.2, Figure 3.6.

EXAMPLE 2 We find derivatives of the cosine function in combinations with other functions.

(a) $y = 5e^x + \cos x$:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(5e^x) + \frac{d}{dx}(\cos x) && \text{Sum Rule} \\ &= 5e^x - \sin x\end{aligned}$$

(b) $y = \sin x \cos x$:

$$\begin{aligned}\frac{dy}{dx} &= \sin x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(\sin x) && \text{Product Rule} \\ &= \sin x(-\sin x) + \cos x(\cos x) \\ &= \cos^2 x - \sin^2 x\end{aligned}$$

(c) $y = \frac{\cos x}{1 - \sin x}$:

$$\begin{aligned}\frac{dy}{dx} &= \frac{(1 - \sin x) \frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(1 - \sin x)}{(1 - \sin x)^2} && \text{Quotient Rule} \\ &= \frac{(1 - \sin x)(-\sin x) - \cos x(0 - \cos x)}{(1 - \sin x)^2} \\ &= \frac{1 - \sin x}{(1 - \sin x)^2} && \sin^2 x + \cos^2 x = 1 \\ &= \frac{1}{1 - \sin x}\end{aligned}$$

Simple Harmonic Motion

The motion of an object or weight bobbing freely up and down with no resistance on the end of a spring is an example of *simple harmonic motion*. The motion is periodic and repeats indefinitely, so we represent it using trigonometric functions. The next example describes a case in which there are no opposing forces such as friction or buoyancy to slow the motion.

EXAMPLE 3 A weight hanging from a spring (Figure 3.23) is stretched down 5 units beyond its rest position and released at time $t = 0$ to bob up and down. Its position at any later time t is

$$s = 5 \cos t.$$

What are its velocity and acceleration at time t ?

Solution We have

$$\text{Position: } s = 5 \cos t$$

$$\text{Velocity: } v = \frac{ds}{dt} = \frac{d}{dt}(5 \cos t) = -5 \sin t$$

$$\text{Acceleration: } a = \frac{dv}{dt} = \frac{d}{dt}(-5 \sin t) = -5 \cos t.$$

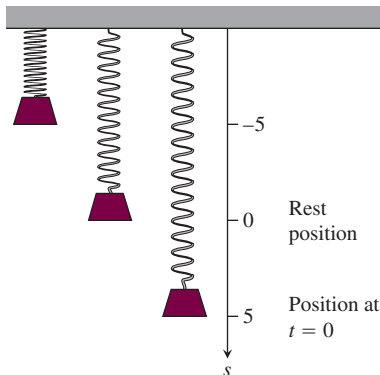


FIGURE 3.23 A weight hanging from a vertical spring and then displaced oscillates above and below its rest position (Example 3).

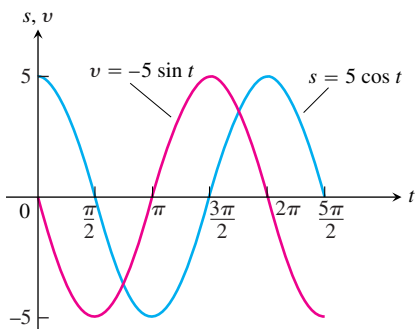


FIGURE 3.24 The graphs of the position and velocity of the weight in Example 3.

Notice how much we can learn from these equations:

1. As time passes, the weight moves down and up between $s = -5$ and $s = 5$ on the s -axis. The amplitude of the motion is 5. The period of the motion is 2π , the period of the cosine function.
2. The velocity $v = -5 \sin t$ attains its greatest magnitude, 5, when $\cos t = 0$, as the graphs show in Figure 3.24. Hence, the speed of the weight, $|v| = 5|\sin t|$, is greatest when $\cos t = 0$, that is, when $s = 0$ (the rest position). The speed of the weight is zero when $\sin t = 0$. This occurs when $s = 5 \cos t = \pm 5$, at the endpoints of the interval of motion.
3. The acceleration value is always the exact opposite of the position value. When the weight is above the rest position, gravity is pulling it back down; when the weight is below the rest position, the spring is pulling it back up.
4. The acceleration, $a = -5 \cos t$, is zero only at the rest position, where $\cos t = 0$ and the force of gravity and the force from the spring balance each other. When the weight is anywhere else, the two forces are unequal and acceleration is nonzero. The acceleration is greatest in magnitude at the points farthest from the rest position, where $\cos t = \pm 1$. ■

EXAMPLE 4 The jerk associated with the simple harmonic motion in Example 3 is

$$j = \frac{da}{dt} = \frac{d}{dt}(-5 \cos t) = 5 \sin t.$$

It has its greatest magnitude when $\sin t = \pm 1$, not at the extremes of the displacement but at the rest position, where the acceleration changes direction and sign. ■

Derivatives of the Other Basic Trigonometric Functions

Because $\sin x$ and $\cos x$ are differentiable functions of x , the related functions

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \text{and} \quad \csc x = \frac{1}{\sin x}$$

are differentiable at every value of x at which they are defined. Their derivatives, calculated from the Quotient Rule, are given by the following formulas. Notice the negative signs in the derivative formulas for the cofunctions.

The derivatives of the other trigonometric functions:

$$\begin{aligned} \frac{d}{dx}(\tan x) &= \sec^2 x & \frac{d}{dx}(\cot x) &= -\csc^2 x \\ \frac{d}{dx}(\sec x) &= \sec x \tan x & \frac{d}{dx}(\csc x) &= -\csc x \cot x \end{aligned}$$

To show a typical calculation, we find the derivative of the tangent function. The other derivations are left to Exercise 60.

EXAMPLE 5 Find $d(\tan x)/dx$.

Solution We use the Derivative Quotient Rule to calculate the derivative:

$$\begin{aligned} \frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x} && \text{Quotient Rule} \\ &= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} = \sec^2 x. \end{aligned}$$

EXAMPLE 6 Find y'' if $y = \sec x$.

Solution Finding the second derivative involves a combination of trigonometric derivatives.

$$\begin{aligned} y &= \sec x \\ y' &= \sec x \tan x && \text{Derivative rule for secant function} \\ y'' &= \frac{d}{dx}(\sec x \tan x) \\ &= \sec x \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(\sec x) && \text{Derivative Product Rule} \\ &= \sec x(\sec^2 x) + \tan x(\sec x \tan x) && \text{Derivative rules} \\ &= \sec^3 x + \sec x \tan^2 x \end{aligned}$$

The differentiability of the trigonometric functions throughout their domains gives another proof of their continuity at every point in their domains (Theorem 1, Section 3.2). So we can calculate limits of algebraic combinations and composites of trigonometric functions by direct substitution.

EXAMPLE 7 We can use direct substitution in computing limits provided there is no division by zero, which is algebraically undefined.

$$\lim_{x \rightarrow 0} \frac{\sqrt{2 + \sec x}}{\cos(\pi - \tan x)} = \frac{\sqrt{2 + \sec 0}}{\cos(\pi - \tan 0)} = \frac{\sqrt{2 + 1}}{\cos(\pi - 0)} = \frac{\sqrt{3}}{-1} = -\sqrt{3}$$

Exercises 3.5

Derivatives

In Exercises 1–18, find dy/dx .

- $y = -10x + 3 \cos x$
- $y = \frac{3}{x} + 5 \sin x$
- $y = x^2 \cos x$
- $y = \sqrt{x} \sec x + 3$
- $y = \csc x - 4\sqrt{x} + 7$
- $y = x^2 \cot x - \frac{1}{x^2}$
- $f(x) = \sin x \tan x$
- $g(x) = \csc x \cot x$
- $y = (\sec x + \tan x)(\sec x - \tan x)$
- $y = (\sin x + \cos x) \sec x$

11. $y = \frac{\cot x}{1 + \cot x}$ 12. $y = \frac{\cos x}{1 + \sin x}$
13. $y = \frac{4}{\cos x} + \frac{1}{\tan x}$ 14. $y = \frac{\cos x}{x} + \frac{x}{\cos x}$
15. $y = x^2 \sin x + 2x \cos x - 2 \sin x$
16. $y = x^2 \cos x - 2x \sin x - 2 \cos x$
17. $f(x) = x^3 \sin x \cos x$ 18. $g(x) = (2 - x) \tan^2 x$

In Exercises 19–22, find ds/dt .

19. $s = \tan t - e^{-t}$ 20. $s = t^2 - \sec t + 5e^t$
21. $s = \frac{1 + \csc t}{1 - \csc t}$ 22. $s = \frac{\sin t}{1 - \cos t}$

In Exercises 23–26, find $dr/d\theta$.

23. $r = 4 - \theta^2 \sin \theta$ 24. $r = \theta \sin \theta + \cos \theta$
25. $r = \sec \theta \csc \theta$ 26. $r = (1 + \sec \theta) \sin \theta$

In Exercises 27–32, find dp/dq .

27. $p = 5 + \frac{1}{\cot q}$ 28. $p = (1 + \csc q) \cos q$
29. $p = \frac{\sin q + \cos q}{\cos q}$ 30. $p = \frac{\tan q}{1 + \tan q}$
31. $p = \frac{q \sin q}{q^2 - 1}$ 32. $p = \frac{3q + \tan q}{q \sec q}$

33. Find y'' if

- a. $y = \csc x$. b. $y = \sec x$.

34. Find $y^{(4)} = d^4 y/dx^4$ if

- a. $y = -2 \sin x$. b. $y = 9 \cos x$.

Tangent Lines

In Exercises 35–38, graph the curves over the given intervals, together with their tangents at the given values of x . Label each curve and tangent with its equation.

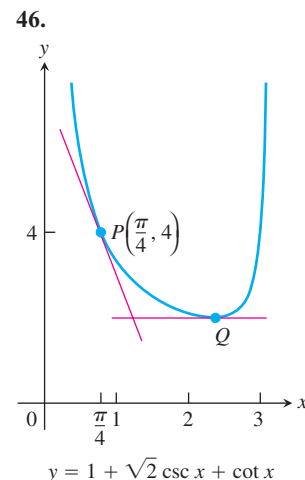
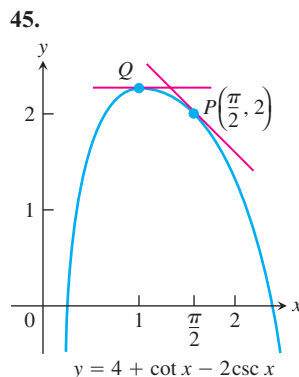
35. $y = \sin x$, $-3\pi/2 \leq x \leq 2\pi$
 $x = -\pi, 0, 3\pi/2$
36. $y = \tan x$, $-\pi/2 < x < \pi/2$
 $x = -\pi/3, 0, \pi/3$
37. $y = \sec x$, $-\pi/2 < x < \pi/2$
 $x = -\pi/3, \pi/4$
38. $y = 1 + \cos x$, $-3\pi/2 \leq x \leq 2\pi$
 $x = -\pi/3, 3\pi/2$

T Do the graphs of the functions in Exercises 39–42 have any horizontal tangents in the interval $0 \leq x \leq 2\pi$? If so, where? If not, why not? Visualize your findings by graphing the functions with a grapher.

39. $y = x + \sin x$
40. $y = 2x + \sin x$
41. $y = x - \cot x$
42. $y = x + 2 \cos x$
43. Find all points on the curve $y = \tan x$, $-\pi/2 < x < \pi/2$, where the tangent line is parallel to the line $y = 2x$. Sketch the curve and tangent(s) together, labeling each with its equation.

44. Find all points on the curve $y = \cot x$, $0 < x < \pi$, where the tangent line is parallel to the line $y = -x$. Sketch the curve and tangent(s) together, labeling each with its equation.

In Exercises 45 and 46, find an equation for (a) the tangent to the curve at P and (b) the horizontal tangent to the curve at Q .



Trigonometric Limits

Find the limits in Exercises 47–54.

47. $\lim_{x \rightarrow 2} \sin\left(\frac{1}{x} - \frac{1}{2}\right)$
48. $\lim_{x \rightarrow -\pi/6} \sqrt{1 + \cos(\pi \csc x)}$
49. $\lim_{\theta \rightarrow \pi/6} \frac{\sin \theta - \frac{1}{2}}{\theta - \frac{\pi}{6}}$ 50. $\lim_{\theta \rightarrow \pi/4} \frac{\tan \theta - 1}{\theta - \frac{\pi}{4}}$
51. $\lim_{x \rightarrow 0} \sec\left[e^x + \pi \tan\left(\frac{\pi}{4 \sec x}\right) - 1\right]$
52. $\lim_{x \rightarrow 0} \sin\left(\frac{\pi + \tan x}{\tan x - 2 \sec x}\right)$
53. $\lim_{t \rightarrow 0} \tan\left(1 - \frac{\sin t}{t}\right)$ 54. $\lim_{\theta \rightarrow 0} \cos\left(\frac{\pi \theta}{\sin \theta}\right)$

Theory and Examples

The equations in Exercises 55 and 56 give the position $s = f(t)$ of a body moving on a coordinate line (s in meters, t in seconds). Find the body's velocity, speed, acceleration, and jerk at time $t = \pi/4$ sec.

55. $s = 2 - 2 \sin t$ 56. $s = \sin t + \cos t$
57. Is there a value of c that will make

$$f(x) = \begin{cases} \frac{\sin^2 3x}{x^2}, & x \neq 0 \\ c, & x = 0 \end{cases}$$

continuous at $x = 0$? Give reasons for your answer.

58. Is there a value of b that will make

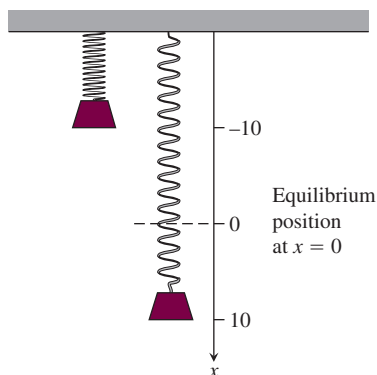
$$g(x) = \begin{cases} x + b, & x < 0 \\ \cos x, & x \geq 0 \end{cases}$$

continuous at $x = 0$? Differentiable at $x = 0$? Give reasons for your answers.

59. Find $d^{999}/dx^{999}(\cos x)$.
60. Derive the formula for the derivative with respect to x of
- $\sec x$.
 - $\csc x$.
 - $\cot x$.
61. A weight is attached to a spring and reaches its equilibrium position ($x = 0$). It is then set in motion resulting in a displacement of

$$x = 10 \cos t,$$

where x is measured in centimeters and t is measured in seconds. See the accompanying figure.



- Find the spring's displacement when $t = 0$, $t = \pi/3$, and $t = 3\pi/4$.
 - Find the spring's velocity when $t = 0$, $t = \pi/3$, and $t = 3\pi/4$.
62. Assume that a particle's position on the x -axis is given by

$$x = 3 \cos t + 4 \sin t,$$

where x is measured in feet and t is measured in seconds.

- Find the particle's position when $t = 0$, $t = \pi/2$, and $t = \pi$.
- Find the particle's velocity when $t = 0$, $t = \pi/2$, and $t = \pi$.

T 63. Graph $y = \cos x$ for $-\pi \leq x \leq 2\pi$. On the same screen, graph

$$y = \frac{\sin(x+h) - \sin x}{h}$$

for $h = 1, 0.5, 0.3$, and 0.1 . Then, in a new window, try $h = -1, -0.5$, and -0.3 . What happens as $h \rightarrow 0^+$? As $h \rightarrow 0^-$? What phenomenon is being illustrated here?

T 64. Graph $y = -\sin x$ for $-\pi \leq x \leq 2\pi$. On the same screen, graph

$$y = \frac{\cos(x+h) - \cos x}{h}$$

for $h = 1, 0.5, 0.3$, and 0.1 . Then, in a new window, try $h = -1, -0.5$, and -0.3 . What happens as $h \rightarrow 0^+$? As $h \rightarrow 0^-$? What phenomenon is being illustrated here?

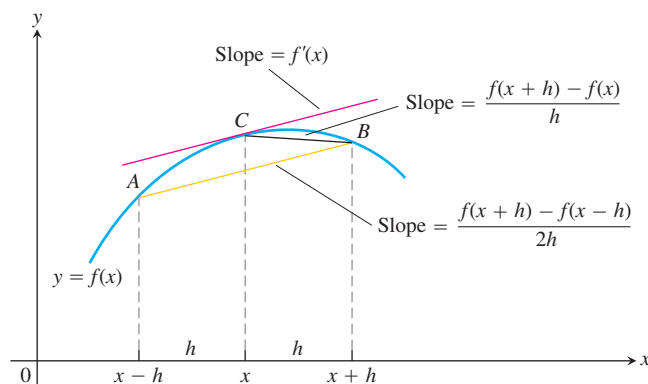
T 65. **Centered difference quotients** The *centered difference quotient*

$$\frac{f(x+h) - f(x-h)}{2h}$$

is used to approximate $f'(x)$ in numerical work because (1) its limit as $h \rightarrow 0$ equals $f'(x)$ when $f'(x)$ exists, and (2) it usually gives a better approximation of $f'(x)$ for a given value of h than the difference quotient

$$\frac{f(x+h) - f(x)}{h}$$

See the accompanying figure.



- To see how rapidly the centered difference quotient for $f(x) = \sin x$ converges to $f'(x) = \cos x$, graph $y = \cos x$ together with

$$y = \frac{\sin(x+h) - \sin(x-h)}{2h}$$

over the interval $[-\pi, 2\pi]$ for $h = 1, 0.5$, and 0.3 . Compare the results with those obtained in Exercise 63 for the same values of h .

- To see how rapidly the centered difference quotient for $f(x) = \cos x$ converges to $f'(x) = -\sin x$, graph $y = -\sin x$ together with

$$y = \frac{\cos(x+h) - \cos(x-h)}{2h}$$

over the interval $[-\pi, 2\pi]$ for $h = 1, 0.5$, and 0.3 . Compare the results with those obtained in Exercise 64 for the same values of h .

66. **A caution about centered difference quotients** (Continuation of Exercise 65.) The quotient

$$\frac{f(x+h) - f(x-h)}{2h}$$

may have a limit as $h \rightarrow 0$ when f has no derivative at x . As a case in point, take $f(x) = |x|$ and calculate

$$\lim_{h \rightarrow 0} \frac{|0+h| - |0-h|}{2h}$$

As you will see, the limit exists even though $f(x) = |x|$ has no derivative at $x = 0$. *Moral:* Before using a centered difference quotient, be sure the derivative exists.

- T** 67. **Slopes on the graph of the tangent function** Graph $y = \tan x$ and its derivative together on $(-\pi/2, \pi/2)$. Does the graph of the tangent function appear to have a smallest slope? A largest slope? Is the slope ever negative? Give reasons for your answers.

T 68. Slopes on the graph of the cotangent function Graph $y = \cot x$ and its derivative together for $0 < x < \pi$. Does the graph of the cotangent function appear to have a smallest slope? A largest slope? Is the slope ever positive? Give reasons for your answers.

T 69. Exploring $(\sin kx)/x$ Graph $y = (\sin x)/x$, $y = (\sin 2x)/x$, and $y = (\sin 4x)/x$ together over the interval $-2 \leq x \leq 2$. Where does each graph appear to cross the y -axis? Do the graphs really intersect the axis? What would you expect the graphs of $y = (\sin 5x)/x$ and $y = (\sin(-3x))/x$ to do as $x \rightarrow 0$? Why? What about the graph of $y = (\sin kx)/x$ for other values of k ? Give reasons for your answers.

T 70. Radians versus degrees: degree mode derivatives What happens to the derivatives of $\sin x$ and $\cos x$ if x is measured in degrees instead of radians? To find out, take the following steps.

- a. With your graphing calculator or computer grapher in *degree mode*, graph

$$f(h) = \frac{\sin h}{h}$$

and estimate $\lim_{h \rightarrow 0} f(h)$. Compare your estimate with $\pi/180$. Is there any reason to believe the limit *should* be $\pi/180$?

- b. With your grapher still in degree mode, estimate

$$\lim_{h \rightarrow 0} \frac{\cos h - 1}{h}$$

- c. Now go back to the derivation of the formula for the derivative of $\sin x$ in the text and carry out the steps of the derivation using degree-mode limits. What formula do you obtain for the derivative?
- d. Work through the derivation of the formula for the derivative of $\cos x$ using degree-mode limits. What formula do you obtain for the derivative?
- e. The disadvantages of the degree-mode formulas become apparent as you start taking derivatives of higher order. Try it. What are the second and third degree-mode derivatives of $\sin x$ and $\cos x$?

3.6 The Chain Rule

How do we differentiate $F(x) = \sin(x^2 - 4)$? This function is the composite $f \circ g$ of two functions $y = f(u) = \sin u$ and $u = g(x) = x^2 - 4$ that we know how to differentiate. The answer, given by the *Chain Rule*, says that the derivative is the product of the derivatives of f and g . We develop the rule in this section.

Derivative of a Composite Function

The function $y = \frac{3}{2}x = \frac{1}{2}(3x)$ is the composite of the functions $y = \frac{1}{2}u$ and $u = 3x$.

We have

$$\frac{dy}{dx} = \frac{3}{2}, \quad \frac{dy}{du} = \frac{1}{2}, \quad \text{and} \quad \frac{du}{dx} = 3.$$

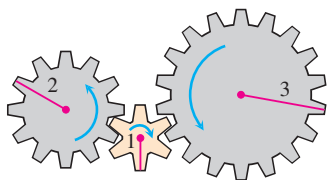
Since $\frac{3}{2} = \frac{1}{2} \cdot 3$, we see in this case that

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

If we think of the derivative as a rate of change, our intuition allows us to see that this relationship is reasonable. If $y = f(u)$ changes half as fast as u and $u = g(x)$ changes three times as fast as x , then we expect y to change $3/2$ times as fast as x . This effect is much like that of a multiple gear train (Figure 3.25). Let's look at another example.

EXAMPLE 1 The function

$$y = (3x^2 + 1)^2$$



C: y turns B: u turns A: x turns

FIGURE 3.25 When gear A makes x turns, gear B makes u turns and gear C makes y turns. By comparing circumferences or counting teeth, we see that $y = u/2$ (C turns one-half turn for each B turn) and $u = 3x$ (B turns three times for A's one), so $y = 3x/2$. Thus, $dy/dx = 3/2 = (1/2)(3) = (dy/du)(du/dx)$.

is the composite of $y = f(u) = u^2$ and $u = g(x) = 3x^2 + 1$. Calculating derivatives, we see that

$$\begin{aligned}\frac{dy}{du} \cdot \frac{du}{dx} &= 2u \cdot 6x \\ &= 2(3x^2 + 1) \cdot 6x \\ &= 36x^3 + 12x.\end{aligned}$$

Calculating the derivative from the expanded formula $(3x^2 + 1)^2 = 9x^4 + 6x^2 + 1$ gives the same result:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(9x^4 + 6x^2 + 1) \\ &= 36x^3 + 12x.\end{aligned}$$

The derivative of the composite function $f(g(x))$ at x is the derivative of f at $g(x)$ times the derivative of g at x . This is known as the Chain Rule (Figure 3.26).

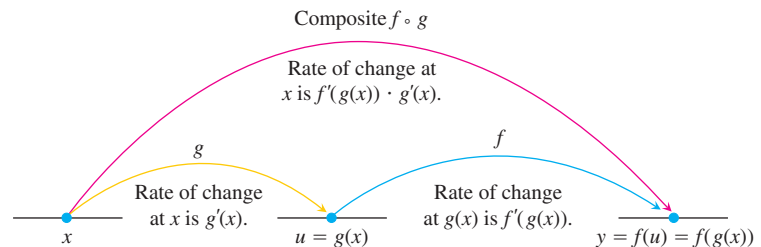


FIGURE 3.26 Rates of change multiply: The derivative of $f \circ g$ at x is the derivative of f at $g(x)$ times the derivative of g at x .

THEOREM 2—The Chain Rule If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x , then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x , and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

Intuitive "Proof" of the Chain Rule:

Let Δu be the change in u when x changes by Δx , so that

$$\Delta u = g(x + \Delta x) - g(x).$$

Then the corresponding change in y is

$$\Delta y = f(u + \Delta u) - f(u).$$

If $\Delta u \neq 0$, we can write the fraction $\Delta y/\Delta x$ as the product

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} \tag{1}$$

and take the limit as $\Delta x \rightarrow 0$:

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} \\ &= \lim_{\Delta u \rightarrow 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} && \text{(Note that } \Delta u \rightarrow 0 \text{ as } \Delta x \rightarrow 0 \\ &= \frac{dy}{du} \cdot \frac{du}{dx}. && \text{since } g \text{ is continuous.)} \end{aligned}$$

The problem with this argument is that it could be true that $\Delta u = 0$ even when $\Delta x \neq 0$, so the cancellation of Δu in Equation (1) would be invalid. A proof requires a different approach that avoids this flaw, and we give one such proof in Section 3.11. ■

EXAMPLE 2 An object moves along the x -axis so that its position at any time $t \geq 0$ is given by $x(t) = \cos(t^2 + 1)$. Find the velocity of the object as a function of t .

Solution We know that the velocity is dx/dt . In this instance, x is a composite function: $x = \cos(u)$ and $u = t^2 + 1$. We have

$$\begin{aligned} \frac{dx}{du} &= -\sin(u) && x = \cos(u) \\ \frac{du}{dt} &= 2t. && u = t^2 + 1 \end{aligned}$$

By the Chain Rule,

$$\begin{aligned} \frac{dx}{dt} &= \frac{dx}{du} \cdot \frac{du}{dt} \\ &= -\sin(u) \cdot 2t && \frac{dx}{du} \text{ evaluated at } u \\ &= -\sin(t^2 + 1) \cdot 2t \\ &= -2t \sin(t^2 + 1). \end{aligned}$$

“Outside-Inside” Rule

A difficulty with the Leibniz notation is that it doesn’t state specifically where the derivatives in the Chain Rule are supposed to be evaluated. So it sometimes helps to think about the Chain Rule using functional notation. If $y = f(g(x))$, then

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x).$$

In words, differentiate the “outside” function f and evaluate it at the “inside” function $g(x)$ left alone; then multiply by the derivative of the “inside function.”

EXAMPLE 3 Differentiate $\sin(x^2 + e^x)$ with respect to x .

Solution We apply the Chain Rule directly and find

$$\frac{d}{dx} \sin(\underbrace{x^2 + e^x}_{\text{inside}}) = \cos(\underbrace{x^2 + e^x}_{\text{inside left alone}}) \cdot \underbrace{(2x + e^x)}_{\text{derivative of the inside}}.$$

EXAMPLE 4 Differentiate $y = e^{\cos x}$.

Solution Here the inside function is $u = g(x) = \cos x$ and the outside function is the exponential function $f(x) = e^x$. Applying the Chain Rule, we get

$$\frac{dy}{dx} = \frac{d}{dx}(e^{\cos x}) = e^{\cos x} \frac{d}{dx}(\cos x) = e^{\cos x}(-\sin x) = -e^{\cos x} \sin x. \quad \blacksquare$$

Generalizing Example 4, we see that the Chain Rule gives the formula

$$\frac{d}{dx} e^u = e^u \frac{du}{dx}.$$

Thus, for example,

$$\frac{d}{dx}(e^{kx}) = e^{kx} \cdot \frac{d}{dx}(kx) = ke^{kx}, \quad \text{for any constant } k$$

and

$$\frac{d}{dx}(e^{x^2}) = e^{x^2} \cdot \frac{d}{dx}(x^2) = 2xe^{x^2}.$$

Repeated Use of the Chain Rule

We sometimes have to use the Chain Rule two or more times to find a derivative.

EXAMPLE 5 Find the derivative of $g(t) = \tan(5 - \sin 2t)$.

Solution Notice here that the tangent is a function of $5 - \sin 2t$, whereas the sine is a function of $2t$, which is itself a function of t . Therefore, by the Chain Rule,

$$\begin{aligned} g'(t) &= \frac{d}{dt}(\tan(5 - \sin 2t)) && \text{Derivative of } \tan u \text{ with } \\ &= \sec^2(5 - \sin 2t) \cdot \frac{d}{dt}(5 - \sin 2t) && u = 5 - \sin 2t \\ &= \sec^2(5 - \sin 2t) \cdot \left(0 - \cos 2t \cdot \frac{d}{dt}(2t)\right) && \text{Derivative of } 5 - \sin u \\ &= \sec^2(5 - \sin 2t) \cdot (-\cos 2t) \cdot 2 && \text{with } u = 2t \\ &= -2(\cos 2t) \sec^2(5 - \sin 2t). \quad \blacksquare \end{aligned}$$

The Chain Rule with Powers of a Function

If f is a differentiable function of u and if u is a differentiable function of x , then substituting $y = f(u)$ into the Chain Rule formula

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

leads to the formula

$$\frac{d}{dx} f(u) = f'(u) \frac{du}{dx}.$$

If n is any real number and f is a power function, $f(u) = u^n$, the Power Rule tells us that $f'(u) = nu^{n-1}$. If u is a differentiable function of x , then we can use the Chain Rule to extend this to the **Power Chain Rule**:

$$\frac{d}{dx}(u^n) = nu^{n-1} \frac{du}{dx}. \quad \frac{d}{du}(u^n) = nu^{n-1}$$

HISTORICAL BIOGRAPHY

Johann Bernoulli
(1667–1748)

EXAMPLE 6 The Power Chain Rule simplifies computing the derivative of a power of an expression.

$$\begin{aligned} \text{(a)} \quad \frac{d}{dx}(5x^3 - x^4)^7 &= 7(5x^3 - x^4)^6 \frac{d}{dx}(5x^3 - x^4) && \text{Power Chain Rule with} \\ & && u = 5x^3 - x^4, n = 7 \\ &= 7(5x^3 - x^4)^6(5 \cdot 3x^2 - 4x^3) \\ &= 7(5x^3 - x^4)^6(15x^2 - 4x^3) \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \frac{d}{dx}\left(\frac{1}{3x-2}\right) &= \frac{d}{dx}(3x-2)^{-1} \\ &= -1(3x-2)^{-2} \frac{d}{dx}(3x-2) && \text{Power Chain Rule with} \\ & && u = 3x-2, n = -1 \\ &= -1(3x-2)^{-2}(3) \\ &= -\frac{3}{(3x-2)^2} \end{aligned}$$

In part (b) we could also find the derivative with the Derivative Quotient Rule.

$$\begin{aligned} \text{(c)} \quad \frac{d}{dx}(\sin^5 x) &= 5 \sin^4 x \cdot \frac{d}{dx} \sin x && \text{Power Chain Rule with } u = \sin x, n = 5, \\ & && \text{because } \sin^n x \text{ means } (\sin x)^n, n \neq -1. \\ &= 5 \sin^4 x \cos x \end{aligned}$$

$$\begin{aligned} \text{(d)} \quad \frac{d}{dx}(e^{\sqrt{3x+1}}) &= e^{\sqrt{3x+1}} \cdot \frac{d}{dx}(\sqrt{3x+1}) \\ &= e^{\sqrt{3x+1}} \cdot \frac{1}{2}(3x+1)^{-1/2} \cdot 3 && \text{Power Chain Rule with } u = 3x+1, n = 1/2 \\ &= \frac{3}{2\sqrt{3x+1}} e^{\sqrt{3x+1}} \quad \blacksquare \end{aligned}$$

EXAMPLE 7 In Section 3.2, we saw that the absolute value function $y = |x|$ is not differentiable at $x = 0$. However, the function *is* differentiable at all other real numbers as we now show. Since $|x| = \sqrt{x^2}$, we can derive the following formula:

$$\begin{aligned} \frac{d}{dx}(|x|) &= \frac{d}{dx} \sqrt{x^2} \\ &= \frac{1}{2\sqrt{x^2}} \cdot \frac{d}{dx}(x^2) && \text{Power Chain Rule with} \\ & && u = x^2, n = 1/2, x \neq 0 \\ &= \frac{1}{2|x|} \cdot 2x && \sqrt{x^2} = |x| \\ &= \frac{x}{|x|}, \quad x \neq 0. \quad \blacksquare \end{aligned}$$

Derivative of the Absolute Value Function

$$\frac{d}{dx}(|x|) = \frac{x}{|x|}, \quad x \neq 0$$

EXAMPLE 8 Show that the slope of every line tangent to the curve $y = 1/(1 - 2x)^3$ is positive.

Solution We find the derivative:

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(1 - 2x)^{-3} \\ &= -3(1 - 2x)^{-4} \cdot \frac{d}{dx}(1 - 2x) && \text{Power Chain Rule with } u = (1 - 2x), n = -3 \\ &= -3(1 - 2x)^{-4} \cdot (-2) \\ &= \frac{6}{(1 - 2x)^4}. \end{aligned}$$

At any point (x, y) on the curve, $x \neq 1/2$ and the slope of the tangent line is

$$\frac{dy}{dx} = \frac{6}{(1 - 2x)^4},$$

the quotient of two positive numbers. ■

EXAMPLE 9 The formulas for the derivatives of both $\sin x$ and $\cos x$ were obtained under the assumption that x is measured in radians, *not* degrees. The Chain Rule gives us new insight into the difference between the two. Since $180^\circ = \pi$ radians, $x^\circ = \pi x/180$ radians where x° is the size of the angle measured in degrees.

By the Chain Rule,

$$\frac{d}{dx} \sin(x^\circ) = \frac{d}{dx} \sin\left(\frac{\pi x}{180}\right) = \frac{\pi}{180} \cos\left(\frac{\pi x}{180}\right) = \frac{\pi}{180} \cos(x^\circ).$$

See Figure 3.27. Similarly, the derivative of $\cos(x^\circ)$ is $-(\pi/180) \sin(x^\circ)$.

The factor $\pi/180$ would compound with repeated differentiation. We see here the advantage for the use of radian measure in computations. ■

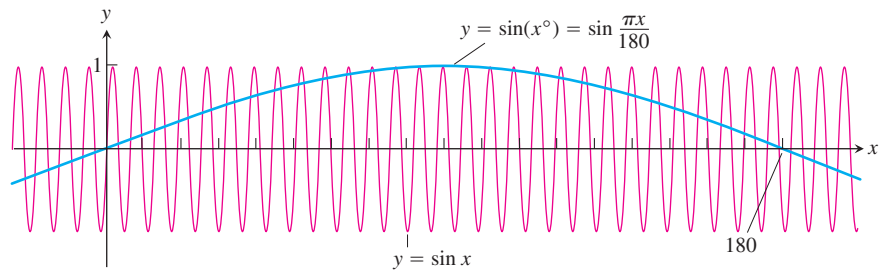


FIGURE 3.27 $\sin(x^\circ)$ oscillates only $\pi/180$ times as often as $\sin x$ oscillates. Its maximum slope is $\pi/180$ at $x = 0$ (Example 9).

Exercises 3.6

Derivative Calculations

In Exercises 1–8, given $y = f(u)$ and $u = g(x)$, find $dy/dx = f'(g(x))g'(x)$.

1. $y = 6u - 9$, $u = (1/2)x^4$
2. $y = 2u^3$, $u = 8x - 1$
3. $y = \sin u$, $u = 3x + 1$
4. $y = \cos u$, $u = -x/3$
5. $y = \cos u$, $u = \sin x$
6. $y = \sin u$, $u = x - \cos x$
7. $y = \tan u$, $u = 10x - 5$
8. $y = -\sec u$, $u = x^2 + 7x$

In Exercises 9–22, write the function in the form $y = f(u)$ and $u = g(x)$. Then find dy/dx as a function of x .

9. $y = (2x + 1)^5$
10. $y = (4 - 3x)^9$
11. $y = \left(1 - \frac{x}{7}\right)^{-7}$
12. $y = \left(\frac{x}{2} - 1\right)^{-10}$
13. $y = \left(\frac{x^2}{8} + x - \frac{1}{x}\right)^4$
14. $y = \sqrt{3x^2 - 4x + 6}$
15. $y = \sec(\tan x)$
16. $y = \cot\left(\pi - \frac{1}{x}\right)$
17. $y = \sin^3 x$
18. $y = 5 \cos^{-4} x$

19. $y = e^{-5x}$

20. $y = e^{2x/3}$

21. $y = e^{5-7x}$

22. $y = e^{(4\sqrt{x+x^2})}$

Find the derivatives of the functions in Exercises 23–50.

23. $p = \sqrt{3 - t}$

24. $q = \sqrt[3]{2r - r^2}$

25. $s = \frac{4}{3\pi} \sin 3t + \frac{4}{5\pi} \cos 5t$

26. $s = \sin\left(\frac{3\pi t}{2}\right) + \cos\left(\frac{3\pi t}{2}\right)$

27. $r = (\csc \theta + \cot \theta)^{-1}$

28. $r = 6(\sec \theta - \tan \theta)^{3/2}$

29. $y = x^2 \sin^4 x + x \cos^{-2} x$

30. $y = \frac{1}{x} \sin^{-5} x - \frac{x}{3} \cos^3 x$

31. $y = \frac{1}{21} (3x - 2)^7 + \left(4 - \frac{1}{2x^2}\right)^{-1}$

32. $y = (5 - 2x)^{-3} + \frac{1}{8} \left(\frac{2}{x} + 1\right)^4$

33. $y = (4x + 3)^4 (x + 1)^{-3}$

34. $y = (2x - 5)^{-1} (x^2 - 5x)^6$

35. $y = xe^{-x} + e^{3x}$

36. $y = (1 + 2x)e^{-2x}$

37. $y = (x^2 - 2x + 2)e^{5x/2}$

38. $y = (9x^2 - 6x + 2)e^{x^3}$

39. $h(x) = x \tan(2\sqrt{x}) + 7$

40. $k(x) = x^2 \sec\left(\frac{1}{x}\right)$

$$41. f(x) = \sqrt{7 + x \sec x} \quad 42. g(x) = \frac{\tan 3x}{(x+7)^4}$$

$$43. f(\theta) = \left(\frac{\sin \theta}{1 + \cos \theta} \right)^2 \quad 44. g(t) = \left(\frac{1 + \sin 3t}{3 - 2t} \right)^{-1}$$

$$45. r = \sin(\theta^2) \cos(2\theta) \quad 46. r = \sec \sqrt{\theta} \tan \left(\frac{1}{\theta} \right)$$

$$47. q = \sin \left(\frac{t}{\sqrt{t+1}} \right) \quad 48. q = \cot \left(\frac{\sin t}{t} \right)$$

$$49. y = \cos(e^{-\theta^2}) \quad 50. y = \theta^3 e^{-2\theta} \cos 5\theta$$

In Exercises 51–70, find dy/dt .

$$51. y = \sin^2(\pi t - 2) \quad 52. y = \sec^2 \pi t$$

$$53. y = (1 + \cos 2t)^{-4} \quad 54. y = (1 + \cot(t/2))^{-2}$$

$$55. y = (t \tan t)^{10} \quad 56. y = (t^{-3/4} \sin t)^{4/3}$$

$$57. y = e^{\cos^2(\pi t - 1)} \quad 58. y = (e^{\sin(t/2)})^3$$

$$59. y = \left(\frac{t^2}{t^3 - 4t} \right)^3 \quad 60. y = \left(\frac{3t - 4}{5t + 2} \right)^{-5}$$

$$61. y = \sin(\cos(2t - 5)) \quad 62. y = \cos \left(5 \sin \left(\frac{t}{3} \right) \right)$$

$$63. y = \left(1 + \tan^4 \left(\frac{t}{12} \right) \right)^3 \quad 64. y = \frac{1}{6} (1 + \cos^2(7t))^3$$

$$65. y = \sqrt{1 + \cos(t^2)} \quad 66. y = 4 \sin(\sqrt{1 + \sqrt{t}})$$

$$67. y = \tan^2(\sin^3 t) \quad 68. y = \cos^4(\sec^2 3t)$$

$$69. y = 3t(2t^2 - 5)^4 \quad 70. y = \sqrt{3t + \sqrt{2 + \sqrt{1 - t}}}$$

Second Derivatives

Find y'' in Exercises 71–78.

$$71. y = \left(1 + \frac{1}{x} \right)^3 \quad 72. y = (1 - \sqrt{x})^{-1}$$

$$73. y = \frac{1}{9} \cot(3x - 1) \quad 74. y = 9 \tan \left(\frac{x}{3} \right)$$

$$75. y = x(2x + 1)^4 \quad 76. y = x^2(x^3 - 1)^5$$

$$77. y = e^{x^2} + 5x \quad 78. y = \sin(x^2 e^x)$$

Finding Derivative Values

In Exercises 79–84, find the value of $(f \circ g)'$ at the given value of x .

$$79. f(u) = u^5 + 1, \quad u = g(x) = \sqrt{x}, \quad x = 1$$

$$80. f(u) = 1 - \frac{1}{u}, \quad u = g(x) = \frac{1}{1-x}, \quad x = -1$$

$$81. f(u) = \cot \frac{\pi u}{10}, \quad u = g(x) = 5\sqrt{x}, \quad x = 1$$

$$82. f(u) = u + \frac{1}{\cos^2 u}, \quad u = g(x) = \pi x, \quad x = 1/4$$

$$83. f(u) = \frac{2u}{u^2 + 1}, \quad u = g(x) = 10x^2 + x + 1, \quad x = 0$$

$$84. f(u) = \left(\frac{u-1}{u+1} \right)^2, \quad u = g(x) = \frac{1}{x^2} - 1, \quad x = -1$$

85. Assume that $f'(3) = -1$, $g'(2) = 5$, $g(2) = 3$, and $y = f(g(x))$. What is y' at $x = 2$?

86. If $r = \sin(f(t))$, $f(0) = \pi/3$, and $f'(0) = 4$, then what is dr/dt at $t = 0$?

87. Suppose that functions f and g and their derivatives with respect to x have the following values at $x = 2$ and $x = 3$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	8	2	1/3	-3
3	3	-4	2π	5

Find the derivatives with respect to x of the following combinations at the given value of x .

- a. $2f(x)$, $x = 2$ b. $f(x) + g(x)$, $x = 3$
c. $f(x) \cdot g(x)$, $x = 3$ d. $f(x)/g(x)$, $x = 2$
e. $f(g(x))$, $x = 2$ f. $\sqrt{f(x)}$, $x = 2$
g. $1/g^2(x)$, $x = 3$ h. $\sqrt{f^2(x) + g^2(x)}$, $x = 2$
88. Suppose that the functions f and g and their derivatives with respect to x have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	1	1	5	1/3
1	3	-4	-1/3	-8/3

Find the derivatives with respect to x of the following combinations at the given value of x .

- a. $5f(x) - g(x)$, $x = 1$ b. $f(x)g^3(x)$, $x = 0$
c. $\frac{f(x)}{g(x) + 1}$, $x = 1$ d. $f(g(x))$, $x = 0$
e. $g(f(x))$, $x = 0$ f. $(x^{11} + f(x))^{-2}$, $x = 1$
g. $f(x + g(x))$, $x = 0$

89. Find ds/dt when $\theta = 3\pi/2$ if $s = \cos \theta$ and $d\theta/dt = 5$.

90. Find dy/dt when $x = 1$ if $y = x^2 + 7x - 5$ and $dx/dt = 1/3$.

Theory and Examples

What happens if you can write a function as a composite in different ways? Do you get the same derivative each time? The Chain Rule says you should. Try it with the functions in Exercises 91 and 92.

91. Find dy/dx if $y = x$ by using the Chain Rule with y as a composite of

- a. $y = (u/5) + 7$ and $u = 5x - 35$
b. $y = 1 + (1/u)$ and $u = 1/(x - 1)$.

92. Find dy/dx if $y = x^{3/2}$ by using the Chain Rule with y as a composite of

- a. $y = u^3$ and $u = \sqrt{x}$
b. $y = \sqrt{u}$ and $u = x^3$.

93. Find the tangent to $y = ((x - 1)/(x + 1))^2$ at $x = 0$.

94. Find the tangent to $y = \sqrt{x^2 - x + 7}$ at $x = 2$.

95. a. Find the tangent to the curve $y = 2 \tan(\pi x/4)$ at $x = 1$.

- b. **Slopes on a tangent curve** What is the smallest value the slope of the curve can ever have on the interval $-2 < x < 2$? Give reasons for your answer.

96. **Slopes on sine curves**

- a. Find equations for the tangents to the curves $y = \sin 2x$ and $y = -\sin(x/2)$ at the origin. Is there anything special about how the tangents are related? Give reasons for your answer.

- b. Can anything be said about the tangents to the curves $y = \sin mx$ and $y = -\sin(x/m)$ at the origin (m a constant $\neq 0$)? Give reasons for your answer.
- c. For a given m , what are the largest values the slopes of the curves $y = \sin mx$ and $y = -\sin(x/m)$ can ever have? Give reasons for your answer.
- d. The function $y = \sin x$ completes one period on the interval $[0, 2\pi]$, the function $y = \sin 2x$ completes two periods, the function $y = \sin(x/2)$ completes half a period, and so on. Is there any relation between the number of periods $y = \sin mx$ completes on $[0, 2\pi]$ and the slope of the curve $y = \sin mx$ at the origin? Give reasons for your answer.

- 97. Running machinery too fast** Suppose that a piston is moving straight up and down and that its position at time t sec is

$$s = A \cos(2\pi bt),$$

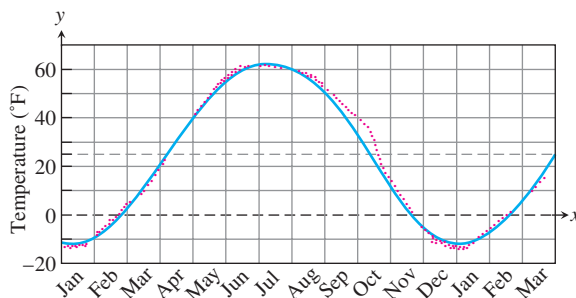
with A and b positive. The value of A is the amplitude of the motion, and b is the frequency (number of times the piston moves up and down each second). What effect does doubling the frequency have on the piston's velocity, acceleration, and jerk? (Once you find out, you will know why some machinery breaks when you run it too fast.)

- 98. Temperatures in Fairbanks, Alaska** The graph in the accompanying figure shows the average Fahrenheit temperature in Fairbanks, Alaska, during a typical 365-day year. The equation that approximates the temperature on day x is

$$y = 37 \sin \left[\frac{2\pi}{365} (x - 101) \right] + 25$$

and is graphed in the accompanying figure.

- a. On what day is the temperature increasing the fastest?
- b. About how many degrees per day is the temperature increasing when it is increasing at its fastest?



- 99. Particle motion** The position of a particle moving along a coordinate line is $s = \sqrt{1 + 4t}$, with s in meters and t in seconds. Find the particle's velocity and acceleration at $t = 6$ sec.
- 100. Constant acceleration** Suppose that the velocity of a falling body is $v = k\sqrt{s}$ m/sec (k a constant) at the instant the body has fallen s m from its starting point. Show that the body's acceleration is constant.
- 101. Falling meteorite** The velocity of a heavy meteorite entering Earth's atmosphere is inversely proportional to \sqrt{s} when it is s km from Earth's center. Show that the meteorite's acceleration is inversely proportional to s^2 .

- 102. Particle acceleration** A particle moves along the x -axis with velocity $dx/dt = f(x)$. Show that the particle's acceleration is $f(x)f'(x)$.

- 103. Temperature and the period of a pendulum** For oscillations of small amplitude (short swings), we may safely model the relationship between the period T and the length L of a simple pendulum with the equation

$$T = 2\pi\sqrt{\frac{L}{g}},$$

where g is the constant acceleration of gravity at the pendulum's location. If we measure g in centimeters per second squared, we measure L in centimeters and T in seconds. If the pendulum is made of metal, its length will vary with temperature, either increasing or decreasing at a rate that is roughly proportional to L . In symbols, with u being temperature and k the proportionality constant,

$$\frac{dL}{du} = kL.$$

Assuming this to be the case, show that the rate at which the period changes with respect to temperature is $kT/2$.

- 104. Chain Rule** Suppose that $f(x) = x^2$ and $g(x) = |x|$. Then the composites

$$(f \circ g)(x) = |x|^2 = x^2 \quad \text{and} \quad (g \circ f)(x) = |x^2| = x^2$$

are both differentiable at $x = 0$ even though g itself is not differentiable at $x = 0$. Does this contradict the Chain Rule? Explain.

- T 105. The derivative of $\sin 2x$** Graph the function $y = 2 \cos 2x$ for $-2 \leq x \leq 3.5$. Then, on the same screen, graph

$$y = \frac{\sin 2(x+h) - \sin 2x}{h}$$

for $h = 1.0, 0.5$, and 0.2 . Experiment with other values of h , including negative values. What do you see happening as $h \rightarrow 0$? Explain this behavior.

- 106. The derivative of $\cos(x^2)$** Graph $y = -2x \sin(x^2)$ for $-2 \leq x \leq 3$. Then, on the same screen, graph

$$y = \frac{\cos((x+h)^2) - \cos(x^2)}{h}$$

for $h = 1.0, 0.7$, and 0.3 . Experiment with other values of h . What do you see happening as $h \rightarrow 0$? Explain this behavior.

Using the Chain Rule, show that the Power Rule $(d/dx)x^n = nx^{n-1}$ holds for the functions x^n in Exercises 107 and 108.

- 107.** $x^{1/4} = \sqrt{\sqrt{x}}$ **108.** $x^{3/4} = \sqrt{x\sqrt{x}}$

COMPUTER EXPLORATIONS

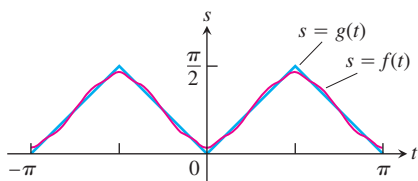
Trigonometric Polynomials

- 109.** As the accompanying figure shows, the trigonometric "polynomial"

$$s = f(t) = 0.78540 - 0.63662 \cos 2t - 0.07074 \cos 6t \\ - 0.02546 \cos 10t - 0.01299 \cos 14t$$

gives a good approximation of the sawtooth function $s = g(t)$ on the interval $[-\pi, \pi]$. How well does the derivative of f approximate the derivative of g at the points where dg/dt is defined? To find out, carry out the following steps.

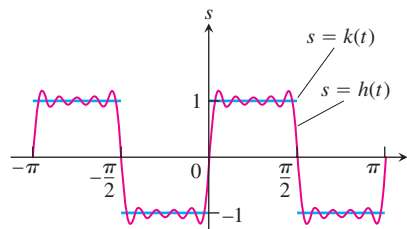
- a. Graph dg/dt (where defined) over $[-\pi, \pi]$.
- b. Find df/dt .
- c. Graph df/dt . Where does the approximation of dg/dt by df/dt seem to be best? Least good? Approximations by trigonometric polynomials are important in the theories of heat and oscillation, but we must not expect too much of them, as we see in the next exercise.



110. (Continuation of Exercise 109.) In Exercise 109, the trigonometric polynomial $f(t)$ that approximated the sawtooth function $g(t)$ on $[-\pi, \pi]$ had a derivative that approximated the derivative of the sawtooth function. It is possible, however, for a trigonometric polynomial to approximate a function in a reasonable way without its derivative approximating the function's derivative at all well. As a case in point, the "polynomial"

$$s = h(t) = 1.2732 \sin 2t + 0.4244 \sin 6t + 0.25465 \sin 10t + 0.18189 \sin 14t + 0.14147 \sin 18t$$

graphed in the accompanying figure approximates the step function $s = k(t)$ shown there. Yet the derivative of h is nothing like the derivative of k .



- a. Graph dk/dt (where defined) over $[-\pi, \pi]$.
- b. Find dh/dt .
- c. Graph dh/dt to see how badly the graph fits the graph of dk/dt . Comment on what you see.

3.7 Implicit Differentiation

Most of the functions we have dealt with so far have been described by an equation of the form $y = f(x)$ that expresses y explicitly in terms of the variable x . We have learned rules for differentiating functions defined in this way. Another situation occurs when we encounter equations like

$$x^3 + y^3 - 9xy = 0, \quad y^2 - x = 0, \quad \text{or} \quad x^2 + y^2 - 25 = 0.$$

(See Figures 3.28, 3.29, and 3.30.) These equations define an *implicit* relation between the variables x and y . In some cases we may be able to solve such an equation for y as an explicit function (or even several functions) of x . When we cannot put an equation $F(x, y) = 0$ in the form $y = f(x)$ to differentiate it in the usual way, we may still be able to find dy/dx by *implicit differentiation*. This section describes the technique.

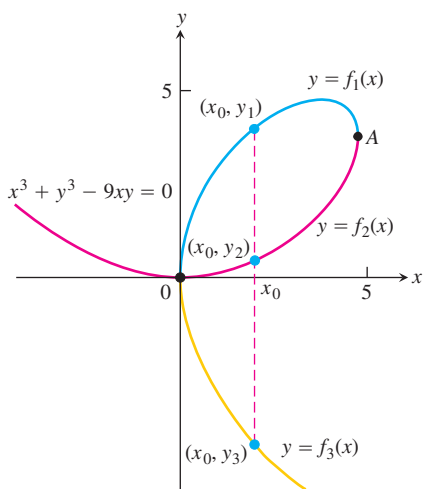


FIGURE 3.28 The curve $x^3 + y^3 - 9xy = 0$ is not the graph of any one function of x . The curve can, however, be divided into separate arcs that are the graphs of functions of x . This particular curve, called a *folium*, dates to Descartes in 1638.

Implicitly Defined Functions

We begin with examples involving familiar equations that we can solve for y as a function of x to calculate dy/dx in the usual way. Then we differentiate the equations implicitly, and find the derivative to compare the two methods. Following the examples, we summarize the steps involved in the new method. In the examples and exercises, it is always assumed that the given equation determines y implicitly as a differentiable function of x so that dy/dx exists.

EXAMPLE 1 Find dy/dx if $y^2 = x$.

Solution The equation $y^2 = x$ defines two differentiable functions of x that we can actually find, namely $y_1 = \sqrt{x}$ and $y_2 = -\sqrt{x}$ (Figure 3.29). We know how to calculate the derivative of each of these for $x > 0$:

$$\frac{dy_1}{dx} = \frac{1}{2\sqrt{x}} \quad \text{and} \quad \frac{dy_2}{dx} = -\frac{1}{2\sqrt{x}}.$$

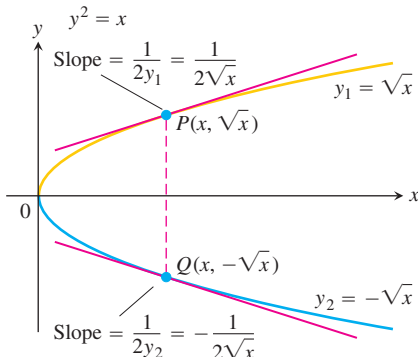


FIGURE 3.29 The equation $y^2 - x = 0$, or $y^2 = x$ as it is usually written, defines two differentiable functions of x on the interval $x > 0$. Example 1 shows how to find the derivatives of these functions without solving the equation $y^2 = x$ for y .

But suppose that we knew only that the equation $y^2 = x$ defined y as one or more differentiable functions of x for $x > 0$ without knowing exactly what these functions were. Could we still find dy/dx ?

The answer is yes. To find dy/dx , we simply differentiate both sides of the equation $y^2 = x$ with respect to x , treating $y = f(x)$ as a differentiable function of x :

$$\begin{aligned}
 y^2 &= x && \text{The Chain Rule gives } \frac{d}{dx}(y^2) = \\
 2y \frac{dy}{dx} &= 1 && \frac{d}{dx}[f(x)]^2 = 2f(x)f'(x) = 2y \frac{dy}{dx}. \\
 \frac{dy}{dx} &= \frac{1}{2y}.
 \end{aligned}$$

This one formula gives the derivatives we calculated for *both* explicit solutions $y_1 = \sqrt{x}$ and $y_2 = -\sqrt{x}$:

$$\frac{dy_1}{dx} = \frac{1}{2y_1} = \frac{1}{2\sqrt{x}} \quad \text{and} \quad \frac{dy_2}{dx} = \frac{1}{2y_2} = \frac{1}{2(-\sqrt{x})} = -\frac{1}{2\sqrt{x}}.$$

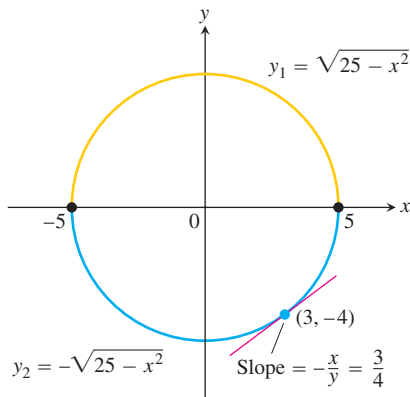


FIGURE 3.30 The circle combines the graphs of two functions. The graph of y_2 is the lower semicircle and passes through $(3, -4)$.

EXAMPLE 2 Find the slope of the circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

Solution The circle is not the graph of a single function of x . Rather it is the combined graphs of two differentiable functions, $y_1 = \sqrt{25 - x^2}$ and $y_2 = -\sqrt{25 - x^2}$ (Figure 3.30). The point $(3, -4)$ lies on the graph of y_2 , so we can find the slope by calculating the derivative directly, using the Power Chain Rule:

$$\frac{dy_2}{dx} \Big|_{x=3} = -\frac{-2x}{2\sqrt{25 - x^2}} \Big|_{x=3} = -\frac{-6}{2\sqrt{25 - 9}} = \frac{3}{4}. \quad \frac{d}{dx} - (25 - x^2)^{1/2} = -\frac{1}{2}(25 - x^2)^{-1/2}(-2x)$$

We can solve this problem more easily by differentiating the given equation of the circle implicitly with respect to x :

$$\begin{aligned}
 \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) &= \frac{d}{dx}(25) \\
 2x + 2y \frac{dy}{dx} &= 0 \\
 \frac{dy}{dx} &= -\frac{x}{y}.
 \end{aligned}$$

The slope at $(3, -4)$ is $-\frac{x}{y} \Big|_{(3, -4)} = -\frac{3}{-4} = \frac{3}{4}$.

Notice that unlike the slope formula for dy_2/dx , which applies only to points below the x -axis, the formula $dy/dx = -x/y$ applies everywhere the circle has a slope. Notice also that the derivative involves *both* variables x and y , not just the independent variable x .

To calculate the derivatives of other implicitly defined functions, we proceed as in Examples 1 and 2: We treat y as a differentiable implicit function of x and apply the usual rules to differentiate both sides of the defining equation.

Implicit Differentiation

1. Differentiate both sides of the equation with respect to x , treating y as a differentiable function of x .
2. Collect the terms with dy/dx on one side of the equation and solve for dy/dx .

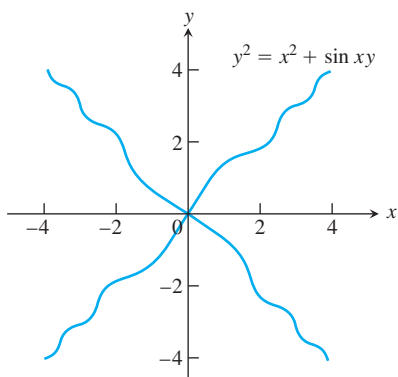


FIGURE 3.31 The graph of $y^2 = x^2 + \sin xy$ in Example 3.

EXAMPLE 3 Find dy/dx if $y^2 = x^2 + \sin xy$ (Figure 3.31).

Solution We differentiate the equation implicitly.

$$y^2 = x^2 + \sin xy$$

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x^2) + \frac{d}{dx}(\sin xy)$$

Differentiate both sides with respect to x ...

$$2y \frac{dy}{dx} = 2x + (\cos xy) \frac{d}{dx}(xy)$$

... treating y as a function of x and using the Chain Rule.

$$2y \frac{dy}{dx} = 2x + (\cos xy) \left(y + x \frac{dy}{dx} \right)$$

Treat xy as a product.

$$2y \frac{dy}{dx} - (\cos xy) \left(x \frac{dy}{dx} \right) = 2x + (\cos xy)y$$

Collect terms with dy/dx .

$$(2y - x \cos xy) \frac{dy}{dx} = 2x + y \cos xy$$

$$\frac{dy}{dx} = \frac{2x + y \cos xy}{2y - x \cos xy}$$

Solve for dy/dx .

Notice that the formula for dy/dx applies everywhere that the implicitly defined curve has a slope. Notice again that the derivative involves *both* variables x and y , not just the independent variable x . ■

Derivatives of Higher Order

Implicit differentiation can also be used to find higher derivatives.

EXAMPLE 4 Find d^2y/dx^2 if $2x^3 - 3y^2 = 8$.

Solution To start, we differentiate both sides of the equation with respect to x in order to find $y' = dy/dx$.

$$\frac{d}{dx}(2x^3 - 3y^2) = \frac{d}{dx}(8)$$

$$6x^2 - 6yy' = 0$$

Treat y as a function of x .

$$y' = \frac{x^2}{y}, \quad \text{when } y \neq 0$$

Solve for y' .

We now apply the Quotient Rule to find y'' .

$$y'' = \frac{d}{dx} \left(\frac{x^2}{y} \right) = \frac{2xy - x^2y'}{y^2} = \frac{2x}{y} - \frac{x^2}{y^2} \cdot y'$$

Finally, we substitute $y' = x^2/y$ to express y'' in terms of x and y .

$$y'' = \frac{2x}{y} - \frac{x^2}{y^2} \left(\frac{x^2}{y} \right) = \frac{2x}{y} - \frac{x^4}{y^3}, \quad \text{when } y \neq 0$$

■

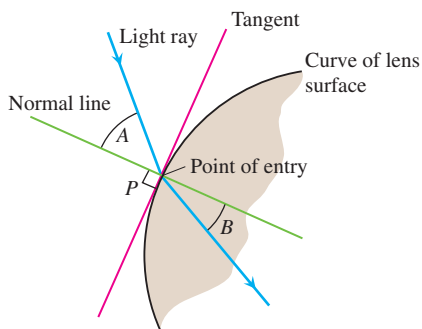


FIGURE 3.32 The profile of a lens, showing the bending (refraction) of a ray of light as it passes through the lens surface.

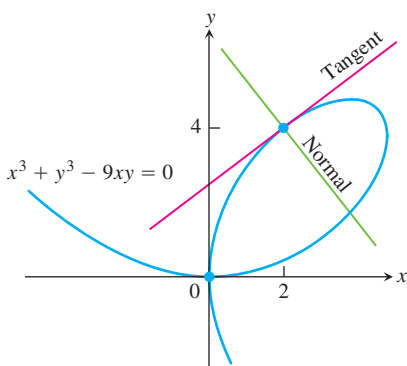


FIGURE 3.33 Example 5 shows how to find equations for the tangent and normal to the folium of Descartes at (2, 4).

Lenses, Tangents, and Normal Lines

In the law that describes how light changes direction as it enters a lens, the important angles are the angles the light makes with the line perpendicular to the surface of the lens at the point of entry (angles A and B in Figure 3.32). This line is called the *normal* to the surface at the point of entry. In a profile view of a lens like the one in Figure 3.32, the **normal** is the line perpendicular to the tangent of the profile curve at the point of entry.

EXAMPLE 5 Show that the point (2, 4) lies on the curve $x^3 + y^3 - 9xy = 0$. Then find the tangent and normal to the curve there (Figure 3.33).

Solution The point (2, 4) lies on the curve because its coordinates satisfy the equation given for the curve: $2^3 + 4^3 - 9(2)(4) = 8 + 64 - 72 = 0$.

To find the slope of the curve at (2, 4), we first use implicit differentiation to find a formula for dy/dx :

$$\begin{aligned}
 x^3 + y^3 - 9xy &= 0 \\
 \frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) - \frac{d}{dx}(9xy) &= \frac{d}{dx}(0) \\
 3x^2 + 3y^2 \frac{dy}{dx} - 9\left(x \frac{dy}{dx} + y \frac{dx}{dx}\right) &= 0 && \text{Differentiate both sides} \\
 (3y^2 - 9x) \frac{dy}{dx} + 3x^2 - 9y &= 0 && \text{with respect to } x. \\
 3(y^2 - 3x) \frac{dy}{dx} = 9y - 3x^2 &&& \text{Treat } xy \text{ as a product and } y \\
 \frac{dy}{dx} = \frac{3y - x^2}{y^2 - 3x} &&& \text{as a function of } x. \\
 &&& \text{Solve for } dy/dx.
 \end{aligned}$$

We then evaluate the derivative at $(x, y) = (2, 4)$:

$$\left. \frac{dy}{dx} \right|_{(2,4)} = \left. \frac{3y - x^2}{y^2 - 3x} \right|_{(2,4)} = \frac{3(4) - 2^2}{4^2 - 3(2)} = \frac{8}{10} = \frac{4}{5}.$$

The tangent at (2, 4) is the line through (2, 4) with slope $4/5$:

$$\begin{aligned}
 y &= 4 + \frac{4}{5}(x - 2) \\
 y &= \frac{4}{5}x + \frac{12}{5}.
 \end{aligned}$$

The normal to the curve at (2, 4) is the line perpendicular to the tangent there, the line through (2, 4) with slope $-5/4$:

$$\begin{aligned}
 y &= 4 - \frac{5}{4}(x - 2) \\
 y &= -\frac{5}{4}x + \frac{13}{2}.
 \end{aligned}$$

The quadratic formula enables us to solve a second-degree equation like $y^2 - 2xy + 3x^2 = 0$ for y in terms of x . There is a formula for the three roots of a cubic equation that is like the quadratic formula but much more complicated. If this formula is used to solve the equation $x^3 + y^3 = 9xy$ in Example 5 for y in terms of x , then three functions determined by the equation are

$$y = f(x) = \sqrt[3]{-\frac{x^3}{2} + \sqrt{\frac{x^6}{4} - 27x^3}} + \sqrt[3]{-\frac{x^3}{2} - \sqrt{\frac{x^6}{4} - 27x^3}}$$

and

$$y = \frac{1}{2} \left[-f(x) \pm \sqrt{-3} \left(\sqrt[3]{-\frac{x^3}{2} + \sqrt{\frac{x^6}{4} - 27x^3}} - \sqrt[3]{-\frac{x^3}{2} - \sqrt{\frac{x^6}{4} - 27x^3}} \right) \right].$$

Using implicit differentiation in Example 5 was much simpler than calculating dy/dx directly from any of the above formulas. Finding slopes on curves defined by higher-degree equations usually requires implicit differentiation.

Exercise 3.7

Differentiating Implicitly

Use implicit differentiation to find dy/dx in Exercises 1–16.

1. $x^2y + xy^2 = 6$
2. $x^3 + y^3 = 18xy$
3. $2xy + y^2 = x + y$
4. $x^3 - xy + y^3 = 1$
5. $x^2(x - y)^2 = x^2 - y^2$
6. $(3xy + 7)^2 = 6y$
7. $y^2 = \frac{x - 1}{x + 1}$
8. $x^3 = \frac{2x - y}{x + 3y}$
9. $x = \tan y$
10. $xy = \cot(xy)$
11. $x + \tan(xy) = 0$
12. $x^4 + \sin y = x^3y^2$
13. $y \sin\left(\frac{1}{y}\right) = 1 - xy$
14. $x \cos(2x + 3y) = y \sin x$
15. $e^{2x} = \sin(x + 3y)$
16. $e^{xy} = 2x + 2y$

Find $dr/d\theta$ in Exercises 17–20.

17. $\theta^{1/2} + r^{1/2} = 1$
18. $r - 2\sqrt{\theta} = \frac{3}{2}\theta^{2/3} + \frac{4}{3}\theta^{3/4}$
19. $\sin(r\theta) = \frac{1}{2}$
20. $\cos r + \cot \theta = e^{r\theta}$

Second Derivatives

In Exercises 21–26, use implicit differentiation to find dy/dx and then d^2y/dx^2 .

21. $x^2 + y^2 = 1$
22. $x^{2/3} + y^{2/3} = 1$
23. $y^2 = e^{x^2} + 2x$
24. $y^2 - 2x = 1 - 2y$
25. $2\sqrt{y} = x - y$
26. $xy + y^2 = 1$
27. If $x^3 + y^3 = 16$, find the value of d^2y/dx^2 at the point $(2, 2)$.
28. If $xy + y^2 = 1$, find the value of d^2y/dx^2 at the point $(0, -1)$.

In Exercises 29 and 30, find the slope of the curve at the given points.

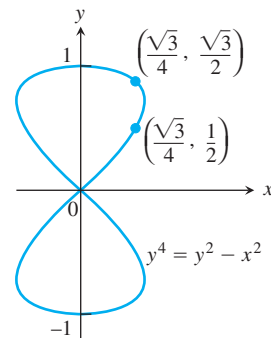
29. $y^2 + x^2 = y^4 - 2x$ at $(-2, 1)$ and $(-2, -1)$
30. $(x^2 + y^2)^2 = (x - y)^2$ at $(1, 0)$ and $(1, -1)$

Slopes, Tangents, and Normals

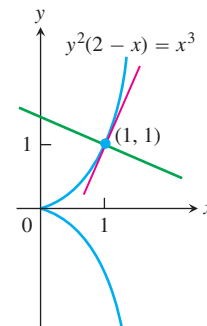
In Exercises 31–40, verify that the given point is on the curve and find the lines that are (a) tangent and (b) normal to the curve at the given point.

31. $x^2 + xy - y^2 = 1$, $(2, 3)$
32. $x^2 + y^2 = 25$, $(3, -4)$
33. $x^2y^2 = 9$, $(-1, 3)$
34. $y^2 - 2x - 4y - 1 = 0$, $(-2, 1)$
35. $6x^2 + 3xy + 2y^2 + 17y - 6 = 0$, $(-1, 0)$

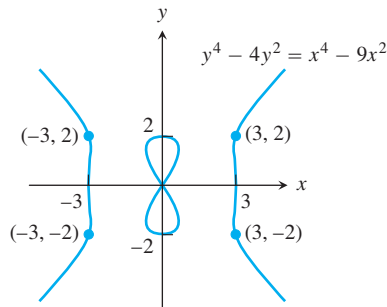
36. $x^2 - \sqrt{3}xy + 2y^2 = 5$, $(\sqrt{3}, 2)$
37. $2xy + \pi \sin y = 2\pi$, $(1, \pi/2)$
38. $x \sin 2y = y \cos 2x$, $(\pi/4, \pi/2)$
39. $y = 2 \sin(\pi x - y)$, $(1, 0)$
40. $x^2 \cos^2 y - \sin y = 0$, $(0, \pi)$
41. **Parallel tangents** Find the two points where the curve $x^2 + xy + y^2 = 7$ crosses the x -axis, and show that the tangents to the curve at these points are parallel. What is the common slope of these tangents?
42. **Normals parallel to a line** Find the normals to the curve $xy + 2x - y = 0$ that are parallel to the line $2x + y = 0$.
43. **The eight curve** Find the slopes of the curve $y^4 = y^2 - x^2$ at the two points shown here.



44. **The cissoid of Diocles (from about 200 B.C.)** Find equations for the tangent and normal to the cissoid of Diocles $y^2(2 - x) = x^3$ at $(1, 1)$.



45. **The devil's curve (Gabriel Cramer, 1750)** Find the slopes of the devil's curve $y^4 - 4y^2 = x^4 - 9x^2$ at the four indicated points.



46. **The folium of Descartes** (See Figure 3.28.)

- Find the slope of the folium of Descartes $x^3 + y^3 - 9xy = 0$ at the points $(4, 2)$ and $(2, 4)$.
- At what point other than the origin does the folium have a horizontal tangent?
- Find the coordinates of the point A in Figure 3.28, where the folium has a vertical tangent.

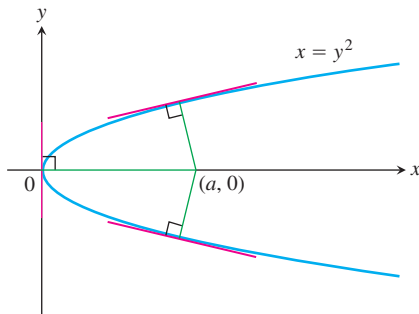
Theory and Examples

47. **Intersecting normal** The line that is normal to the curve $x^2 + 2xy - 3y^2 = 0$ at $(1, 1)$ intersects the curve at what other point?

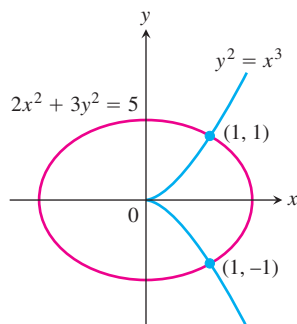
48. **Power rule for rational exponents** Let p and q be integers with $q > 0$. If $y = x^{p/q}$, differentiate the equivalent equation $y^q = x^p$ implicitly and show that, for $y \neq 0$,

$$\frac{d}{dx} x^{p/q} = \frac{p}{q} x^{(p/q)-1}.$$

49. **Normals to a parabola** Show that if it is possible to draw three normals from the point $(a, 0)$ to the parabola $x = y^2$ shown in the accompanying diagram, then a must be greater than $1/2$. One of the normals is the x -axis. For what value of a are the other two normals perpendicular?



50. Is there anything special about the tangents to the curves $y^2 = x^3$ and $2x^2 + 3y^2 = 5$ at the points $(1, \pm 1)$? Give reasons for your answer.

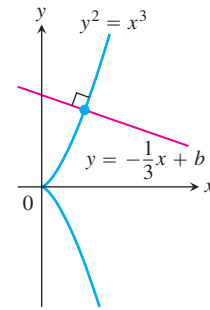


51. Verify that the following pairs of curves meet orthogonally.

a. $x^2 + y^2 = 4$, $x^2 = 3y^2$

b. $x = 1 - y^2$, $x = \frac{1}{3}y^2$

52. The graph of $y^2 = x^3$ is called a **semicubical parabola** and is shown in the accompanying figure. Determine the constant b so that the line $y = -\frac{1}{3}x + b$ meets this graph orthogonally.



T In Exercises 53 and 54, find both dy/dx (treating y as a differentiable function of x) and dx/dy (treating x as a differentiable function of y). How do dy/dx and dx/dy seem to be related? Explain the relationship geometrically in terms of the graphs.

53. $xy^3 + x^2y = 6$

54. $x^3 + y^2 = \sin^2 y$

COMPUTER EXPLORATIONS

Use a CAS to perform the following steps in Exercises 55–62.

- Plot the equation with the implicit plotter of a CAS. Check to see that the given point P satisfies the equation.
- Using implicit differentiation, find a formula for the derivative dy/dx and evaluate it at the given point P .
- Use the slope found in part (b) to find an equation for the tangent line to the curve at P . Then plot the implicit curve and tangent line together on a single graph.

55. $x^3 - xy + y^3 = 7$, $P(2, 1)$

56. $x^5 + y^3x + yx^2 + y^4 = 4$, $P(1, 1)$

57. $y^2 + y = \frac{2+x}{1-x}$, $P(0, 1)$

58. $y^3 + \cos xy = x^2$, $P(1, 0)$

59. $x + \tan\left(\frac{y}{x}\right) = 2$, $P\left(1, \frac{\pi}{4}\right)$

60. $xy^3 + \tan(x+y) = 1$, $P\left(\frac{\pi}{4}, 0\right)$

61. $2y^2 + (xy)^{1/3} = x^2 + 2$, $P(1, 1)$

62. $x\sqrt{1+2y} + y = x^2$, $P(1, 0)$

3.8 Derivatives of Inverse Functions and Logarithms

In Section 1.6 we saw how the inverse of a function undoes, or inverts, the effect of that function. We defined there the natural logarithm function $f^{-1}(x) = \ln x$ as the inverse of the natural exponential function $f(x) = e^x$. This is one of the most important function-inverse pairs in mathematics and science. We learned how to differentiate the exponential function in Section 3.3. Here we learn a rule for differentiating the inverse of a differentiable function and we apply the rule to find the derivative of the natural logarithm function.

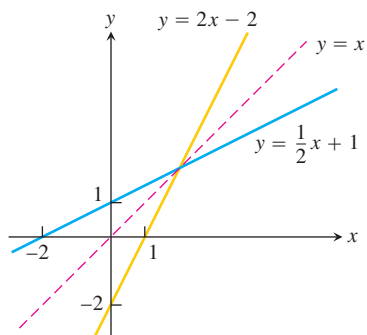


FIGURE 3.34 Graphing a line and its inverse together shows the graphs' symmetry with respect to the line $y = x$. The slopes are reciprocals of each other.

Derivatives of Inverses of Differentiable Functions

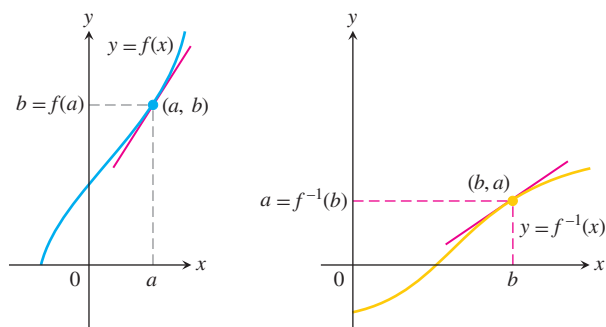
We calculated the inverse of the function $f(x) = (1/2)x + 1$ as $f^{-1}(x) = 2x - 2$ in Example 3 of Section 1.6. Figure 3.34 shows again the graphs of both functions. If we calculate their derivatives, we see that

$$\frac{d}{dx} f(x) = \frac{d}{dx} \left(\frac{1}{2}x + 1 \right) = \frac{1}{2}$$

$$\frac{d}{dx} f^{-1}(x) = \frac{d}{dx} (2x - 2) = 2.$$

The derivatives are reciprocals of one another, so the slope of one line is the reciprocal of the slope of its inverse line. (See Figure 3.34.)

This is not a special case. Reflecting any nonhorizontal or nonvertical line across the line $y = x$ always inverts the line's slope. If the original line has slope $m \neq 0$, the reflected line has slope $1/m$.



The slopes are reciprocal: $(f^{-1})'(b) = \frac{1}{f'(a)}$ or $(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$

FIGURE 3.35 The graphs of inverse functions have reciprocal slopes at corresponding points.

The reciprocal relationship between the slopes of f and f^{-1} holds for other functions as well, but we must be careful to compare slopes at corresponding points. If the slope of $y = f(x)$ at the point $(a, f(a))$ is $f'(a)$ and $f'(a) \neq 0$, then the slope of $y = f^{-1}(x)$ at the point $(f(a), a)$ is the reciprocal $1/f'(a)$ (Figure 3.35). If we set $b = f(a)$, then

$$(f^{-1})'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}.$$

If $y = f(x)$ has a horizontal tangent line at $(a, f(a))$ then the inverse function f^{-1} has a vertical tangent line at $(f(a), a)$, and this infinite slope implies that f^{-1} is not differentiable at $f(a)$. Theorem 3 gives the conditions under which f^{-1} is differentiable in its domain (which is the same as the range of f).

THEOREM 3—The Derivative Rule for Inverses If f has an interval I as domain and $f'(x)$ exists and is never zero on I , then f^{-1} is differentiable at every point in its domain (the range of f). The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))} \quad (1)$$

or

$$\left. \frac{df^{-1}}{dx} \right|_{x=b} = \frac{1}{\left. \frac{df}{dx} \right|_{x=f^{-1}(b)}}$$

Theorem 3 makes two assertions. The first of these has to do with the conditions under which f^{-1} is differentiable; the second assertion is a formula for the derivative of f^{-1} when it exists. While we omit the proof of the first assertion, the second one is proved in the following way:

$$f(f^{-1}(x)) = x \quad \text{Inverse function relationship}$$

$$\frac{d}{dx} f(f^{-1}(x)) = 1 \quad \text{Differentiating both sides}$$

$$f'(f^{-1}(x)) \cdot \frac{d}{dx} f^{-1}(x) = 1 \quad \text{Chain Rule}$$

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}. \quad \text{Solving for the derivative}$$

EXAMPLE 1 The function $f(x) = x^2, x \geq 0$ and its inverse $f^{-1}(x) = \sqrt{x}$ have derivatives $f'(x) = 2x$ and $(f^{-1})'(x) = 1/(2\sqrt{x})$.

Let's verify that Theorem 3 gives the same formula for the derivative of $f^{-1}(x)$:

$$\begin{aligned} (f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} \\ &= \frac{1}{2(f^{-1}(x))} && \begin{array}{l} f'(x) = 2x \text{ with } x \text{ replaced} \\ \text{by } f^{-1}(x) \end{array} \\ &= \frac{1}{2(\sqrt{x})}. \end{aligned}$$

Theorem 3 gives a derivative that agrees with the known derivative of the square root function.

Let's examine Theorem 3 at a specific point. We pick $x = 2$ (the number a) and $f(2) = 4$ (the value b). Theorem 3 says that the derivative of f at 2, $f'(2) = 4$, and the derivative of f^{-1} at $f(2)$, $(f^{-1})'(4)$, are reciprocals. It states that

$$(f^{-1})'(4) = \frac{1}{f'(f^{-1}(4))} = \frac{1}{f'(2)} = \frac{1}{2x} \Big|_{x=2} = \frac{1}{4}.$$

See Figure 3.36. ■

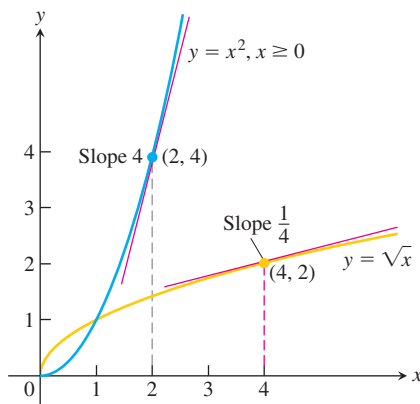


FIGURE 3.36 The derivative of $f^{-1}(x) = \sqrt{x}$ at the point $(4, 2)$ is the reciprocal of the derivative of $f(x) = x^2$ at $(2, 4)$ (Example 1).

We will use the procedure illustrated in Example 1 to calculate formulas for the derivatives of many inverse functions throughout this chapter. Equation (1) sometimes enables us to find specific values of df^{-1}/dx without knowing a formula for f^{-1} .

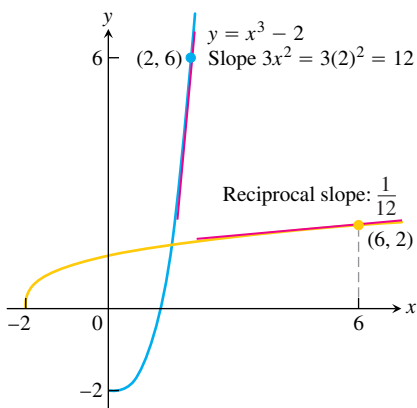


FIGURE 3.37 The derivative of $f(x) = x^3 - 2$ at $x = 2$ tells us the derivative of f^{-1} at $x = 6$ (Example 2).

EXAMPLE 2 Let $f(x) = x^3 - 2$. Find the value of df^{-1}/dx at $x = 6 = f(2)$ without finding a formula for $f^{-1}(x)$.

Solution We apply Theorem 3 to obtain the value of the derivative of f^{-1} at $x = 6$:

$$\begin{aligned} \left. \frac{df}{dx} \right|_{x=2} &= 3x^2 \Big|_{x=2} = 12 \\ \left. \frac{df^{-1}}{dx} \right|_{x=f(2)} &= \frac{1}{\left. \frac{df}{dx} \right|_{x=2}} = \frac{1}{12}. \end{aligned} \quad \text{Eq. (1)}$$

See Figure 3.37. ■

Derivative of the Natural Logarithm Function

Since we know the exponential function $f(x) = e^x$ is differentiable everywhere, we can apply Theorem 3 to find the derivative of its inverse $f^{-1}(x) = \ln x$:

$$\begin{aligned} (f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} && \text{Theorem 3} \\ &= \frac{1}{e^{f^{-1}(x)}} && f'(u) = e^u \\ &= \frac{1}{e^{\ln x}} \\ &= \frac{1}{x}. && \text{Inverse function relationship} \end{aligned}$$

Alternate Derivation Instead of applying Theorem 3 directly, we can find the derivative of $y = \ln x$ using implicit differentiation, as follows:

$$\begin{aligned} y &= \ln x \\ e^y &= x && \text{Inverse function relationship} \\ \frac{d}{dx}(e^y) &= \frac{d}{dx}(x) && \text{Differentiate implicitly} \\ e^y \frac{dy}{dx} &= 1 && \text{Chain Rule} \\ \frac{dy}{dx} &= \frac{1}{e^y} = \frac{1}{x}. && e^y = x \end{aligned}$$

No matter which derivation we use, the derivative of $y = \ln x$ with respect to x is

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, \quad x > 0.$$

The Chain Rule extends this formula for positive functions $u(x)$:

$$\frac{d}{dx} \ln u = \frac{d}{du} \ln u \cdot \frac{du}{dx}$$

$$\frac{d}{dx} \ln u = \frac{1}{u} \frac{du}{dx}, \quad u > 0. \quad (2)$$

EXAMPLE 3 We use Equation (2) to find derivatives.

(a) $\frac{d}{dx} \ln 2x = \frac{1}{2x} \frac{d}{dx} (2x) = \frac{1}{2x} (2) = \frac{1}{x}, \quad x > 0$

(b) Equation (2) with $u = x^2 + 3$ gives

$$\frac{d}{dx} \ln(x^2 + 3) = \frac{1}{x^2 + 3} \cdot \frac{d}{dx} (x^2 + 3) = \frac{1}{x^2 + 3} \cdot 2x = \frac{2x}{x^2 + 3}. \quad \blacksquare$$

Notice the remarkable occurrence in Example 3a. The function $y = \ln 2x$ has the same derivative as the function $y = \ln x$. This is true of $y = \ln bx$ for any constant b , provided that $bx > 0$:

$$\frac{d}{dx} \ln bx = \frac{1}{bx} \cdot \frac{d}{dx} (bx) = \frac{1}{bx} (b) = \frac{1}{x}. \quad (3)$$

If $x < 0$ and $b < 0$, then $bx > 0$ and Equation (3) still applies. In particular, if $x < 0$ and $b = -1$ we get

$$\frac{d}{dx} \ln(-x) = \frac{1}{x} \quad \text{for } x < 0.$$

Since $|x| = x$ when $x > 0$ and $|x| = -x$ when $x < 0$, we have the following important result.

$$\frac{d}{dx} \ln |x| = \frac{1}{x}, \quad x \neq 0 \quad (4)$$

EXAMPLE 4 A line with slope m passes through the origin and is tangent to the graph of $y = \ln x$. What is the value of m ?

Solution Suppose the point of tangency occurs at the unknown point $x = a > 0$. Then we know that the point $(a, \ln a)$ lies on the graph and that the tangent line at that point has slope $m = 1/a$ (Figure 3.38). Since the tangent line passes through the origin, its slope is

$$m = \frac{\ln a - 0}{a - 0} = \frac{\ln a}{a}.$$

Setting these two formulas for m equal to each other, we have

$$\frac{\ln a}{a} = \frac{1}{a}$$

$$\ln a = 1$$

$$e^{\ln a} = e^1$$

$$a = e$$

$$m = \frac{1}{e}. \quad \blacksquare$$

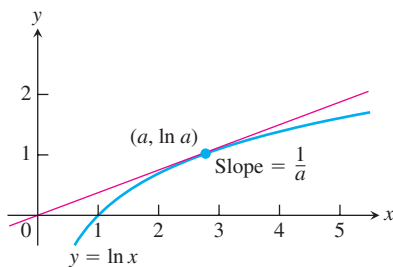


FIGURE 3.38 The tangent line intersects the curve at some point $(a, \ln a)$, where the slope of the curve is $1/a$ (Example 4).

The Derivatives of a^u and $\log_a u$

We start with the equation $a^x = e^{\ln(a^x)} = e^{x \ln a}$, which was established in Section 1.6:

$$\begin{aligned} \frac{d}{dx} a^x &= \frac{d}{dx} e^{x \ln a} = e^{x \ln a} \cdot \frac{d}{dx} (x \ln a) & \frac{d}{dx} e^u &= e^u \frac{du}{dx} \\ &= a^x \ln a. \end{aligned}$$

If $a > 0$, then

$$\frac{d}{dx} a^x = a^x \ln a.$$

This equation shows why e^x is the exponential function preferred in calculus. If $a = e$, then $\ln a = 1$ and the derivative of a^x simplifies to

$$\frac{d}{dx} e^x = e^x \ln e = e^x.$$

With the Chain Rule, we get a more general form for the derivative of a general exponential function.

If $a > 0$ and u is a differentiable function of x , then a^u is a differentiable function of x and

$$\frac{d}{dx} a^u = a^u \ln a \frac{du}{dx}. \quad (5)$$

EXAMPLE 5 We illustrate using Equation (5).

(a) $\frac{d}{dx} 3^x = 3^x \ln 3$ Eq. (5) with $a = 3, u = x$

(b) $\frac{d}{dx} 3^{-x} = 3^{-x}(\ln 3) \frac{d}{dx}(-x) = -3^{-x} \ln 3$ Eq. (5) with $a = 3, u = -x$

(c) $\frac{d}{dx} 3^{\sin x} = 3^{\sin x}(\ln 3) \frac{d}{dx}(\sin x) = 3^{\sin x}(\ln 3) \cos x$..., $u = \sin x$ ■

In Section 3.3 we looked at the derivative $f'(0)$ for the exponential functions $f(x) = a^x$ at various values of the base a . The number $f'(0)$ is the limit, $\lim_{h \rightarrow 0} (a^h - 1)/h$, and gives the slope of the graph of a^x when it crosses the y -axis at the point $(0, 1)$. We now see that the value of this slope is

$$\lim_{h \rightarrow 0} \frac{a^h - 1}{h} = \ln a. \quad (6)$$

In particular, when $a = e$ we obtain

$$\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = \ln e = 1.$$

However, we have not fully justified that these limits actually exist. While all of the arguments given in deriving the derivatives of the exponential and logarithmic functions are correct, they do assume the existence of these limits. In Chapter 7 we will give another development of the theory of logarithmic and exponential functions which fully justifies that both limits do in fact exist and have the values derived above.

To find the derivative of $\log_a u$ for an arbitrary base ($a > 0, a \neq 1$), we start with the change-of-base formula for logarithms (reviewed in Section 1.6) and express $\log_a u$ in terms of natural logarithms,

$$\log_a x = \frac{\ln x}{\ln a}.$$

Taking derivatives, we have

$$\begin{aligned}\frac{d}{dx} \log_a x &= \frac{d}{dx} \left(\frac{\ln x}{\ln a} \right) \\ &= \frac{1}{\ln a} \cdot \frac{d}{dx} \ln x && \ln a \text{ is a constant.} \\ &= \frac{1}{\ln a} \cdot \frac{1}{x} \\ &= \frac{1}{x \ln a}.\end{aligned}$$

If u is a differentiable function of x and $u > 0$, the Chain Rule gives the following formula.

For $a > 0$ and $a \neq 1$,

$$\frac{d}{dx} \log_a u = \frac{1}{u \ln a} \frac{du}{dx}. \quad (7)$$

Logarithmic Differentiation

The derivatives of positive functions given by formulas that involve products, quotients, and powers can often be found more quickly if we take the natural logarithm of both sides before differentiating. This enables us to use the laws of logarithms to simplify the formulas before differentiating. The process, called **logarithmic differentiation**, is illustrated in the next example.

EXAMPLE 6 Find dy/dx if

$$y = \frac{(x^2 + 1)(x + 3)^{1/2}}{x - 1}, \quad x > 1.$$

Solution We take the natural logarithm of both sides and simplify the result with the algebraic properties of logarithms from Theorem 1 in Section 1.6:

$$\begin{aligned}\ln y &= \ln \frac{(x^2 + 1)(x + 3)^{1/2}}{x - 1} \\ &= \ln((x^2 + 1)(x + 3)^{1/2}) - \ln(x - 1) && \text{Rule 2} \\ &= \ln(x^2 + 1) + \ln(x + 3)^{1/2} - \ln(x - 1) && \text{Rule 1} \\ &= \ln(x^2 + 1) + \frac{1}{2} \ln(x + 3) - \ln(x - 1). && \text{Rule 4}\end{aligned}$$

We then take derivatives of both sides with respect to x , using Equation (2) on the left:

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{x^2 + 1} \cdot 2x + \frac{1}{2} \cdot \frac{1}{x + 3} - \frac{1}{x - 1}.$$

Next we solve for dy/dx :

$$\frac{dy}{dx} = y \left(\frac{2x}{x^2 + 1} + \frac{1}{2x + 6} - \frac{1}{x - 1} \right).$$

Finally, we substitute for y :

$$\frac{dy}{dx} = \frac{(x^2 + 1)(x + 3)^{1/2}}{x - 1} \left(\frac{2x}{x^2 + 1} + \frac{1}{2x + 6} - \frac{1}{x - 1} \right).$$

Proof of the Power Rule (General Version)

The definition of the general exponential function enables us to make sense of raising any positive number to a real power n , rational or irrational. That is, we can define the power function $y = x^n$ for any exponent n .

DEFINITION For any $x > 0$ and for any real number n ,

$$x^n = e^{n \ln x}.$$

Because the logarithm and exponential functions are inverses of each other, the definition gives

$$\ln x^n = n \ln x, \quad \text{for all real numbers } n.$$

That is, the Power Rule for the natural logarithm holds for *all* real exponents n , not just for rational exponents.

The definition of the power function also enables us to establish the derivative Power Rule for any real power n , as stated in Section 3.3.

General Power Rule for Derivatives

For $x > 0$ and any real number n ,

$$\frac{d}{dx} x^n = nx^{n-1}.$$

If $x \leq 0$, then the formula holds whenever the derivative, x^n , and x^{n-1} all exist.

Proof Differentiating x^n with respect to x gives

$$\begin{aligned} \frac{d}{dx} x^n &= \frac{d}{dx} e^{n \ln x} && \text{Definition of } x^n, x > 0 \\ &= e^{n \ln x} \cdot \frac{d}{dx} (n \ln x) && \text{Chain Rule for } e^u \\ &= x^n \cdot \frac{n}{x} && \text{Definition and derivative of } \ln x \\ &= nx^{n-1}. && x^n \cdot x^{-1} = x^{n-1} \end{aligned}$$

In short, whenever $x > 0$,

$$\frac{d}{dx} x^n = nx^{n-1}.$$

For $x < 0$, if $y = x^n$, y' , and x^{n-1} all exist, then

$$\ln |y| = \ln |x|^n = n \ln |x|.$$

Using implicit differentiation (which *assumes* the existence of the derivative y') and Equation (4), we have

$$\frac{y'}{y} = \frac{n}{x}.$$

Solving for the derivative,

$$y' = n \frac{y}{x} = n \frac{x^n}{x} = nx^{n-1}.$$

It can be shown directly from the definition of the derivative that the derivative equals 0 when $x = 0$ and $n \geq 1$. This completes the proof of the general version of the Power Rule for all values of x . ■

EXAMPLE 7 Differentiate $f(x) = x^x$, $x > 0$.

Solution We note that $f(x) = x^x = e^{x \ln x}$, so differentiation gives

$$\begin{aligned} f'(x) &= \frac{d}{dx}(e^{x \ln x}) \\ &= e^{x \ln x} \frac{d}{dx}(x \ln x) && \frac{d}{dx} e^u, u = x \ln x \\ &= e^{x \ln x} \left(\ln x + x \cdot \frac{1}{x} \right) \\ &= x^x (\ln x + 1). && x > 0 \end{aligned}$$

The Number e Expressed as a Limit

In Section 1.5 we defined the number e as the base value for which the exponential function $y = a^x$ has slope 1 when it crosses the y -axis at $(0, 1)$. Thus e is the constant that satisfies the equation

$$\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = \ln e = 1. \quad \text{Slope equals } \ln e \text{ from Eq. (6)}$$

We also stated that e could be calculated as $\lim_{y \rightarrow \infty} (1 + 1/y)^y$, or by substituting $y = 1/x$, as $\lim_{x \rightarrow 0} (1 + x)^{1/x}$. We now prove this result.

THEOREM 4—The Number e as a Limit The number e can be calculated as the limit

$$e = \lim_{x \rightarrow 0} (1 + x)^{1/x}.$$

Proof If $f(x) = \ln x$, then $f'(x) = 1/x$, so $f'(1) = 1$. But, by the definition of derivative,

$$\begin{aligned} f'(1) &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = \lim_{x \rightarrow 0} \frac{f(1+x) - f(1)}{x} \\ &= \lim_{x \rightarrow 0} \frac{\ln(1+x) - \ln 1}{x} = \lim_{x \rightarrow 0} \frac{1}{x} \ln(1+x) && \ln 1 = 0 \\ &= \lim_{x \rightarrow 0} \ln(1+x)^{1/x} = \ln \left[\lim_{x \rightarrow 0} (1+x)^{1/x} \right]. && \ln \text{ is continuous,} \\ & && \text{Theorem 10 in} \\ & && \text{Chapter 2} \end{aligned}$$

Because $f'(1) = 1$, we have

$$\ln \left[\lim_{x \rightarrow 0} (1 + x)^{1/x} \right] = 1.$$

Therefore, exponentiating both sides we get

$$\lim_{x \rightarrow 0} (1 + x)^{1/x} = e. \quad \blacksquare$$

Approximating the limit in Theorem 4 by taking x very small gives approximations to e . Its value is $e \approx 2.718281828459045$ to 15 decimal places.

Exercises 3.8

Derivatives of Inverse Functions

In Exercises 1–4:

- Find $f^{-1}(x)$.
 - Graph f and f^{-1} together.
 - Evaluate df/dx at $x = a$ and df^{-1}/dx at $x = f(a)$ to show that at these points $df^{-1}/dx = 1/(df/dx)$.
- $f(x) = 2x + 3$, $a = -1$ $f(x) = (1/5)x + 7$, $a = -1$
 - $f(x) = 5 - 4x$, $a = 1/2$ $f(x) = 2x^2$, $x \geq 0$, $a = 5$
 - a. Show that $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$ are inverses of one another.
 - Graph f and g over an x -interval large enough to show the graphs intersecting at $(1, 1)$ and $(-1, -1)$. Be sure the picture shows the required symmetry about the line $y = x$.
 - Find the slopes of the tangents to the graphs of f and g at $(1, 1)$ and $(-1, -1)$ (four tangents in all).
 - What lines are tangent to the curves at the origin?
- a. Show that $h(x) = x^3/4$ and $k(x) = (4x)^{1/3}$ are inverses of one another.
 - Graph h and k over an x -interval large enough to show the graphs intersecting at $(2, 2)$ and $(-2, -2)$. Be sure the picture shows the required symmetry about the line $y = x$.
 - Find the slopes of the tangents to the graphs at h and k at $(2, 2)$ and $(-2, -2)$.
 - What lines are tangent to the curves at the origin?
- Let $f(x) = x^3 - 3x^2 - 1$, $x \geq 2$. Find the value of df^{-1}/dx at the point $x = -1 = f(3)$.
 - Let $f(x) = x^2 - 4x - 5$, $x > 2$. Find the value of df^{-1}/dx at the point $x = 0 = f(5)$.
 - Suppose that the differentiable function $y = f(x)$ has an inverse and that the graph of f passes through the point $(2, 4)$ and has a slope of $1/3$ there. Find the value of df^{-1}/dx at $x = 4$.
 - Suppose that the differentiable function $y = g(x)$ has an inverse and that the graph of g passes through the origin with slope 2. Find the slope of the graph of g^{-1} at the origin.

Derivatives of Logarithms

In Exercises 11–40, find the derivative of y with respect to x , t , or θ , as appropriate.

11. $y = \ln 3x$

12. $y = \ln kx$, k constant

13. $y = \ln(t^2)$

15. $y = \ln \frac{3}{x}$

17. $y = \ln(\theta + 1)$

19. $y = \ln x^3$

21. $y = t(\ln t)^2$

23. $y = \frac{x^4}{4} \ln x - \frac{x^4}{16}$

25. $y = \frac{\ln t}{t}$

27. $y = \frac{\ln x}{1 + \ln x}$

29. $y = \ln(\ln x)$

31. $y = \theta(\sin(\ln \theta) + \cos(\ln \theta))$

32. $y = \ln(\sec \theta + \tan \theta)$

33. $y = \ln \frac{1}{x\sqrt{x+1}}$

35. $y = \frac{1 + \ln t}{1 - \ln t}$

37. $y = \ln(\sec(\ln \theta))$

39. $y = \ln \left(\frac{(x^2 + 1)^5}{\sqrt{1 - x}} \right)$

14. $y = \ln(t^{3/2})$

16. $y = \ln \frac{10}{x}$

18. $y = \ln(2\theta + 2)$

20. $y = (\ln x)^3$

22. $y = t\sqrt{\ln t}$

24. $y = (x^2 \ln x)^4$

26. $y = \frac{1 + \ln t}{t}$

28. $y = \frac{x \ln x}{1 + \ln x}$

30. $y = \ln(\ln(\ln x))$

34. $y = \frac{1}{2} \ln \frac{1+x}{1-x}$

36. $y = \sqrt{\ln \sqrt{t}}$

38. $y = \ln \left(\frac{\sqrt{\sin \theta \cos \theta}}{1 + 2 \ln \theta} \right)$

40. $y = \ln \sqrt{\frac{(x+1)^5}{(x+2)^{20}}}$

Logarithmic Differentiation

In Exercises 41–54, use logarithmic differentiation to find the derivative of y with respect to the given independent variable.

41. $y = \sqrt{x(x+1)}$

42. $y = \sqrt{(x^2 + 1)(x - 1)^2}$

43. $y = \sqrt{\frac{t}{t+1}}$

44. $y = \sqrt{\frac{1}{t(t+1)}}$

45. $y = \sqrt{\theta + 3 \sin \theta}$

46. $y = (\tan \theta) \sqrt{2\theta + 1}$

47. $y = t(t+1)(t+2)$

48. $y = \frac{1}{t(t+1)(t+2)}$

49. $y = \frac{\theta + 5}{\theta \cos \theta}$

50. $y = \frac{\theta \sin \theta}{\sqrt{\sec \theta}}$

51. $y = \frac{x\sqrt{x^2 + 1}}{(x+1)^{2/3}}$

52. $y = \sqrt{\frac{(x+1)^{10}}{(2x+1)^5}}$

53. $y = \sqrt[3]{\frac{x(x-2)}{x^2+1}}$

54. $y = \sqrt[3]{\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)}}$

Finding Derivatives

In Exercises 55–62, find the derivative of y with respect to x , t , or θ , as appropriate.

55. $y = \ln(\cos^2 \theta)$

56. $y = \ln(3\theta e^{-\theta})$

57. $y = \ln(3te^{-t})$

58. $y = \ln(2e^{-t} \sin t)$

59. $y = \ln\left(\frac{e^\theta}{1+e^\theta}\right)$

60. $y = \ln\left(\frac{\sqrt{\theta}}{1+\sqrt{\theta}}\right)$

61. $y = e^{(\cos t + \ln t)}$

62. $y = e^{\sin t}(\ln t^2 + 1)$

In Exercises 63–66, find dy/dx .

63. $\ln y = e^y \sin x$

64. $\ln xy = e^{x+y}$

65. $x^y = y^x$

66. $\tan y = e^x + \ln x$

In Exercises 67–88, find the derivative of y with respect to the given independent variable.

67. $y = 2^x$

68. $y = 3^{-x}$

69. $y = 5^{\sqrt{s}}$

70. $y = 2^{(s^2)}$

71. $y = x^\pi$

72. $y = t^{1-e}$

73. $y = \log_2 5\theta$

74. $y = \log_3(1 + \theta \ln 3)$

75. $y = \log_4 x + \log_4 x^2$

76. $y = \log_{25} e^x - \log_5 \sqrt{x}$

77. $y = \log_2 r \cdot \log_4 r$

78. $y = \log_3 r \cdot \log_9 r$

79. $y = \log_3 \left(\left(\frac{x+1}{x-1} \right)^{\ln 3} \right)$

80. $y = \log_5 \sqrt{\left(\frac{7x}{3x+2} \right)^{\ln 5}}$

81. $y = \theta \sin(\log_7 \theta)$

82. $y = \log_7 \left(\frac{\sin \theta \cos \theta}{e^\theta 2^\theta} \right)$

83. $y = \log_5 e^x$

84. $y = \log_2 \left(\frac{x^2 e^2}{2\sqrt{x+1}} \right)$

85. $y = 3^{\log_2 t}$

86. $y = 3 \log_8(\log_2 t)$

87. $y = \log_2(8t^{\ln 2})$

88. $y = t \log_3(e^{(\sin t)(\ln 3)})$

Logarithmic Differentiation with Exponentials

In Exercises 89–96, use logarithmic differentiation to find the derivative of y with respect to the given independent variable.

89. $y = (x+1)^x$

90. $y = x^{(x+1)}$

91. $y = (\sqrt{t})^t$

92. $y = t^{\sqrt{t}}$

93. $y = (\sin x)^x$

94. $y = x^{\sin x}$

95. $y = x^{\ln x}$

96. $y = (\ln x)^{\ln x}$

Theory and Applications

97. If we write $g(x)$ for $f^{-1}(x)$, Equation (1) can be written as

$$g'(f(a)) = \frac{1}{f'(a)}, \quad \text{or} \quad g'(f(a)) \cdot f'(a) = 1.$$

If we then write x for a , we get

$$g'(f(x)) \cdot f'(x) = 1.$$

The latter equation may remind you of the Chain Rule, and indeed there is a connection.

Assume that f and g are differentiable functions that are inverses of one another, so that $(g \circ f)(x) = x$. Differentiate both

sides of this equation with respect to x , using the Chain Rule to express $(g \circ f)'(x)$ as a product of derivatives of g and f . What do you find? (This is not a proof of Theorem 3 because we assume here the theorem's conclusion that $g = f^{-1}$ is differentiable.)

98. Show that $\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x$ for any $x > 0$.

99. If $y = A \sin(\ln x) + B \cos(\ln x)$, where A and B are constants, show that

$$x^2 y'' + xy' + y = 0.$$

100. Using mathematical induction, show that

$$\frac{d^n}{dx^n} \ln x = (-1)^{n-1} \frac{(n-1)!}{x^n}.$$

COMPUTER EXPLORATIONS

In Exercises 101–108, you will explore some functions and their inverses together with their derivatives and tangent line approximations at specified points. Perform the following steps using your CAS:

- Plot the function $y = f(x)$ together with its derivative over the given interval. Explain why you know that f is one-to-one over the interval.
- Solve the equation $y = f(x)$ for x as a function of y , and name the resulting inverse function g .
- Find the equation for the tangent line to f at the specified point $(x_0, f(x_0))$.
- Find the equation for the tangent line to g at the point $(f(x_0), x_0)$ located symmetrically across the 45° line $y = x$ (which is the graph of the identity function). Use Theorem 3 to find the slope of this tangent line.
- Plot the functions f and g , the identity, the two tangent lines, and the line segment joining the points $(x_0, f(x_0))$ and $(f(x_0), x_0)$. Discuss the symmetries you see across the main diagonal.

101. $y = \sqrt{3x-2}$, $\frac{2}{3} \leq x \leq 4$, $x_0 = 3$

102. $y = \frac{3x+2}{2x-11}$, $-2 \leq x \leq 2$, $x_0 = 1/2$

103. $y = \frac{4x}{x^2+1}$, $-1 \leq x \leq 1$, $x_0 = 1/2$

104. $y = \frac{x^3}{x^2+1}$, $-1 \leq x \leq 1$, $x_0 = 1/2$

105. $y = x^3 - 3x^2 - 1$, $2 \leq x \leq 5$, $x_0 = \frac{27}{10}$

106. $y = 2 - x - x^3$, $-2 \leq x \leq 2$, $x_0 = \frac{3}{2}$

107. $y = e^x$, $-3 \leq x \leq 5$, $x_0 = 1$

108. $y = \sin x$, $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, $x_0 = 1$

In Exercises 109 and 110, repeat the steps above to solve for the functions $y = f(x)$ and $x = f^{-1}(y)$ defined implicitly by the given equations over the interval.

109. $y^{1/3} - 1 = (x+2)^3$, $-5 \leq x \leq 5$, $x_0 = -3/2$

110. $\cos y = x^{1/5}$, $0 \leq x \leq 1$, $x_0 = 1/2$

3.9 Inverse Trigonometric Functions

We introduced the six basic inverse trigonometric functions in Section 1.6, but focused there on the arcsine and arccosine functions. Here we complete the study of how all six inverse trigonometric functions are defined, graphed, and evaluated, and how their derivatives are computed.

Inverses of $\tan x$, $\cot x$, $\sec x$, and $\csc x$

The graphs of all six basic inverse trigonometric functions are shown in Figure 3.39. We obtain these graphs by reflecting the graphs of the restricted trigonometric functions (as discussed in Section 1.6) through the line $y = x$. Let's take a closer look at the arctangent, arccotangent, arcsecant, and arccosecant functions.

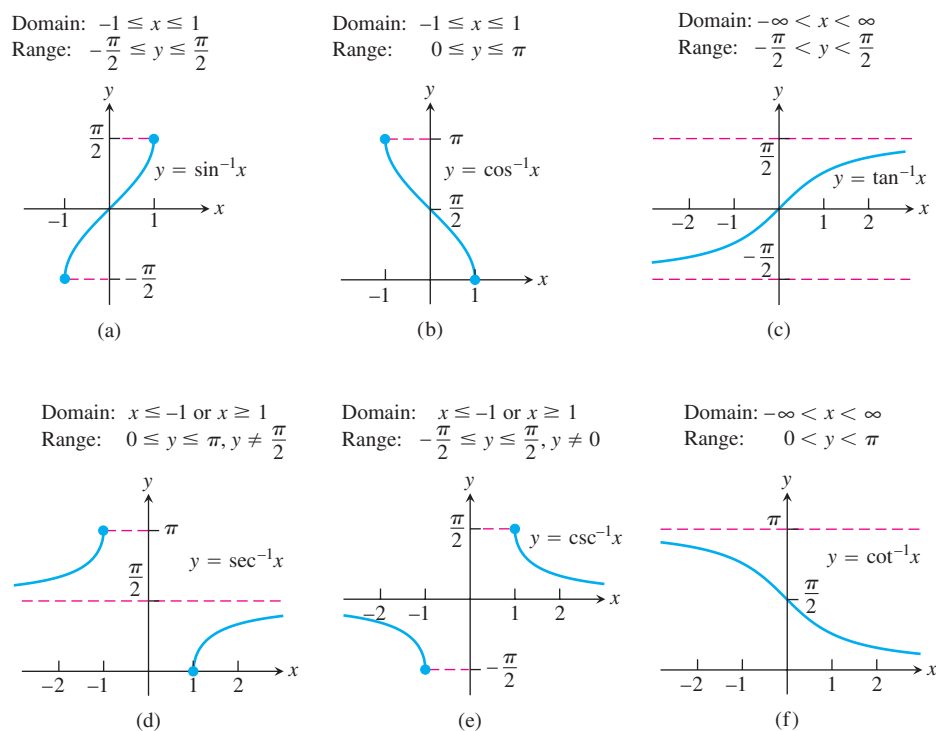


FIGURE 3.39 Graphs of the six basic inverse trigonometric functions.

The arctangent of x is a radian angle whose tangent is x . The arccotangent of x is an angle whose cotangent is x . The angles belong to the restricted domains of the tangent and cotangent functions.

DEFINITION

$y = \tan^{-1}x$ is the number in $(-\pi/2, \pi/2)$ for which $\tan y = x$.

$y = \cot^{-1}x$ is the number in $(0, \pi)$ for which $\cot y = x$.

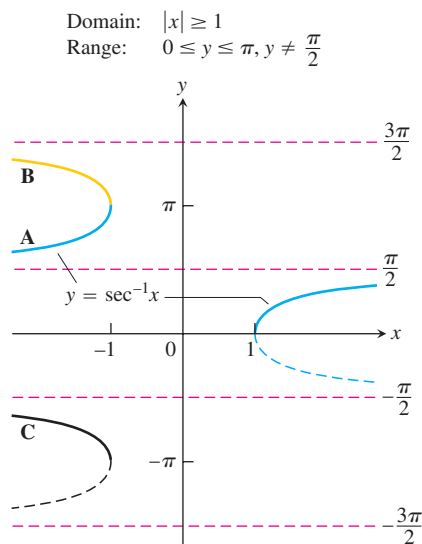


FIGURE 3.40 There are several logical choices for the left-hand branch of $y = \sec^{-1} x$. With choice **A**, $\sec^{-1} x = \cos^{-1}(1/x)$, a useful identity employed by many calculators.

We use open intervals to avoid values where the tangent and cotangent are undefined. The graph of $y = \tan^{-1} x$ is symmetric about the origin because it is a branch of the graph $x = \tan y$ that is symmetric about the origin (Figure 3.39c). Algebraically this means that

$$\tan^{-1}(-x) = -\tan^{-1} x;$$

the arctangent is an odd function. The graph of $y = \cot^{-1} x$ has no such symmetry (Figure 3.39f). Notice from Figure 3.39c that the graph of the arctangent function has two horizontal asymptotes; one at $y = \pi/2$ and the other at $y = -\pi/2$.

The inverses of the restricted forms of $\sec x$ and $\csc x$ are chosen to be the functions graphed in Figures 3.39d and 3.39e.

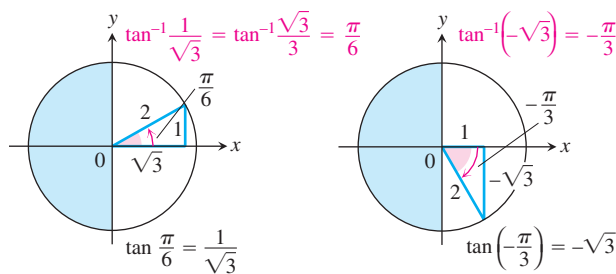
Caution There is no general agreement about how to define $\sec^{-1} x$ for negative values of x . We chose angles in the second quadrant between $\pi/2$ and π . This choice makes $\sec^{-1} x = \cos^{-1}(1/x)$. It also makes $\sec^{-1} x$ an increasing function on each interval of its domain. Some tables choose $\sec^{-1} x$ to lie in $[-\pi, -\pi/2)$ for $x < 0$ and some texts choose it to lie in $[\pi, 3\pi/2)$ (Figure 3.40). These choices simplify the formula for the derivative (our formula needs absolute value signs) but fail to satisfy the computational equation $\sec^{-1} x = \cos^{-1}(1/x)$. From this, we can derive the identity

$$\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2} - \sin^{-1}\left(\frac{1}{x}\right) \tag{1}$$

by applying Equation (5) in Section 1.6.

EXAMPLE 1 The accompanying figures show two values of $\tan^{-1} x$.

x	$\tan^{-1} x$
$\sqrt{3}$	$\pi/3$
1	$\pi/4$
$\sqrt{3}/3$	$\pi/6$
$-\sqrt{3}/3$	$-\pi/6$
-1	$-\pi/4$
$-\sqrt{3}$	$-\pi/3$



The angles come from the first and fourth quadrants because the range of $\tan^{-1} x$ is $(-\pi/2, \pi/2)$. ■

The Derivative of $y = \sin^{-1} u$

We know that the function $x = \sin y$ is differentiable in the interval $-\pi/2 < y < \pi/2$ and that its derivative, the cosine, is positive there. Theorem 3 in Section 3.8 therefore assures us that the inverse function $y = \sin^{-1} x$ is differentiable throughout the interval $-1 < x < 1$. We cannot expect it to be differentiable at $x = 1$ or $x = -1$ because the tangents to the graph are vertical at these points (see Figure 3.41).

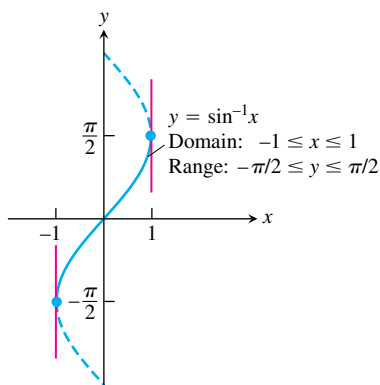


FIGURE 3.41 The graph of $y = \sin^{-1} x$ has vertical tangents at $x = -1$ and $x = 1$.

We find the derivative of $y = \sin^{-1} x$ by applying Theorem 3 with $f(x) = \sin x$ and $f^{-1}(x) = \sin^{-1} x$:

$$\begin{aligned}
 (f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} && \text{Theorem 3} \\
 &= \frac{1}{\cos(\sin^{-1} x)} && f'(u) = \cos u \\
 &= \frac{1}{\sqrt{1 - \sin^2(\sin^{-1} x)}} && \cos u = \sqrt{1 - \sin^2 u} \\
 &= \frac{1}{\sqrt{1 - x^2}}. && \sin(\sin^{-1} x) = x
 \end{aligned}$$

If u is a differentiable function of x with $|u| < 1$, we apply the Chain Rule to get

$$\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1 - u^2}} \frac{du}{dx}, \quad |u| < 1.$$

EXAMPLE 2 Using the Chain Rule, we calculate the derivative

$$\frac{d}{dx}(\sin^{-1} x^2) = \frac{1}{\sqrt{1 - (x^2)^2}} \cdot \frac{d}{dx}(x^2) = \frac{2x}{\sqrt{1 - x^4}}. \quad \blacksquare$$

The Derivative of $y = \tan^{-1} u$

We find the derivative of $y = \tan^{-1} x$ by applying Theorem 3 with $f(x) = \tan x$ and $f^{-1}(x) = \tan^{-1} x$. Theorem 3 can be applied because the derivative of $\tan x$ is positive for $-\pi/2 < x < \pi/2$:

$$\begin{aligned}
 (f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} && \text{Theorem 3} \\
 &= \frac{1}{\sec^2(\tan^{-1} x)} && f'(u) = \sec^2 u \\
 &= \frac{1}{1 + \tan^2(\tan^{-1} x)} && \sec^2 u = 1 + \tan^2 u \\
 &= \frac{1}{1 + x^2}. && \tan(\tan^{-1} x) = x
 \end{aligned}$$

The derivative is defined for all real numbers. If u is a differentiable function of x , we get the Chain Rule form:

$$\frac{d}{dx}(\tan^{-1} u) = \frac{1}{1 + u^2} \frac{du}{dx}.$$

The Derivative of $y = \sec^{-1} u$

Since the derivative of $\sec x$ is positive for $0 < x < \pi/2$ and $\pi/2 < x < \pi$, Theorem 3 says that the inverse function $y = \sec^{-1} x$ is differentiable. Instead of applying the formula

in Theorem 3 directly, we find the derivative of $y = \sec^{-1} x$, $|x| > 1$, using implicit differentiation and the Chain Rule as follows:

$$\begin{aligned}
 y &= \sec^{-1} x \\
 \sec y &= x && \text{Inverse function relationship} \\
 \frac{d}{dx}(\sec y) &= \frac{d}{dx} x && \text{Differentiate both sides.} \\
 \sec y \tan y \frac{dy}{dx} &= 1 && \text{Chain Rule} \\
 \frac{dy}{dx} &= \frac{1}{\sec y \tan y}. && \text{Since } |x| > 1, y \text{ lies in } (0, \pi/2) \cup (\pi/2, \pi) \text{ and } \sec y \tan y \neq 0.
 \end{aligned}$$

To express the result in terms of x , we use the relationships

$$\sec y = x \quad \text{and} \quad \tan y = \pm \sqrt{\sec^2 y - 1} = \pm \sqrt{x^2 - 1}$$

to get

$$\frac{dy}{dx} = \pm \frac{1}{x\sqrt{x^2 - 1}}.$$

Can we do anything about the \pm sign? A glance at Figure 3.42 shows that the slope of the graph $y = \sec^{-1} x$ is always positive. Thus,

$$\frac{d}{dx} \sec^{-1} x = \begin{cases} +\frac{1}{x\sqrt{x^2 - 1}} & \text{if } x > 1 \\ -\frac{1}{x\sqrt{x^2 - 1}} & \text{if } x < -1. \end{cases}$$

With the absolute value symbol, we can write a single expression that eliminates the “ \pm ” ambiguity:

$$\frac{d}{dx} \sec^{-1} x = \frac{1}{|x|\sqrt{x^2 - 1}}.$$

If u is a differentiable function of x with $|u| > 1$, we have the formula

$$\frac{d}{dx}(\sec^{-1} u) = \frac{1}{|u|\sqrt{u^2 - 1}} \frac{du}{dx}, \quad |u| > 1.$$

EXAMPLE 3 Using the Chain Rule and derivative of the arcsecant function, we find

$$\begin{aligned}
 \frac{d}{dx} \sec^{-1}(5x^4) &= \frac{1}{|5x^4|\sqrt{(5x^4)^2 - 1}} \frac{d}{dx}(5x^4) \\
 &= \frac{1}{5x^4\sqrt{25x^8 - 1}} (20x^3) && 5x^4 > 1 > 0 \\
 &= \frac{4}{x\sqrt{25x^8 - 1}}.
 \end{aligned}$$

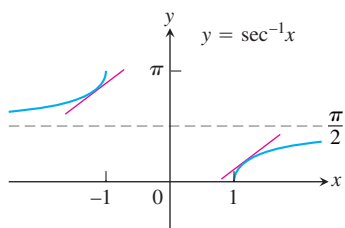


FIGURE 3.42 The slope of the curve $y = \sec^{-1} x$ is positive for both $x < -1$ and $x > 1$.

Derivatives of the Other Three Inverse Trigonometric Functions

We could use the same techniques to find the derivatives of the other three inverse trigonometric functions—arccosine, arccotangent, and arccosecant—but there is an easier way, thanks to the following identities.

Inverse Function–Inverse Cofunction Identities

$$\cos^{-1} x = \pi/2 - \sin^{-1} x$$

$$\cot^{-1} x = \pi/2 - \tan^{-1} x$$

$$\csc^{-1} x = \pi/2 - \sec^{-1} x$$

We saw the first of these identities in Equation (5) of Section 1.6. The others are derived in a similar way. It follows easily that the derivatives of the inverse cofunctions are the negatives of the derivatives of the corresponding inverse functions. For example, the derivative of $\cos^{-1} x$ is calculated as follows:

$$\begin{aligned} \frac{d}{dx}(\cos^{-1} x) &= \frac{d}{dx}\left(\frac{\pi}{2} - \sin^{-1} x\right) && \text{Identity} \\ &= -\frac{d}{dx}(\sin^{-1} x) \\ &= -\frac{1}{\sqrt{1-x^2}}. && \text{Derivative of arcsine} \end{aligned}$$

The derivatives of the inverse trigonometric functions are summarized in Table 3.1.

TABLE 3.1 Derivatives of the inverse trigonometric functions

1. $\frac{d(\sin^{-1} u)}{dx} = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}, \quad |u| < 1$
2. $\frac{d(\cos^{-1} u)}{dx} = -\frac{1}{\sqrt{1-u^2}} \frac{du}{dx}, \quad |u| < 1$
3. $\frac{d(\tan^{-1} u)}{dx} = \frac{1}{1+u^2} \frac{du}{dx}$
4. $\frac{d(\cot^{-1} u)}{dx} = -\frac{1}{1+u^2} \frac{du}{dx}$
5. $\frac{d(\sec^{-1} u)}{dx} = \frac{1}{|u|\sqrt{u^2-1}} \frac{du}{dx}, \quad |u| > 1$
6. $\frac{d(\csc^{-1} u)}{dx} = -\frac{1}{|u|\sqrt{u^2-1}} \frac{du}{dx}, \quad |u| > 1$

Exercises 3.9

Common Values

Use reference triangles like those in Example 1 to find the angles in Exercises 1–8.

- a. $\tan^{-1} 1$ b. $\tan^{-1}(-\sqrt{3})$ c. $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$
- a. $\tan^{-1}(-1)$ b. $\tan^{-1}\sqrt{3}$ c. $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$
- a. $\sin^{-1}\left(\frac{-1}{2}\right)$ b. $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ c. $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)$
- a. $\sin^{-1}\left(\frac{1}{2}\right)$ b. $\sin^{-1}\left(\frac{-1}{\sqrt{2}}\right)$ c. $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$
- a. $\cos^{-1}\left(\frac{1}{2}\right)$ b. $\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$ c. $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$
- a. $\csc^{-1}\sqrt{2}$ b. $\csc^{-1}\left(\frac{-2}{\sqrt{3}}\right)$ c. $\csc^{-1} 2$
- a. $\sec^{-1}(-\sqrt{2})$ b. $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ c. $\sec^{-1}(-2)$
- a. $\cot^{-1}(-1)$ b. $\cot^{-1}(\sqrt{3})$ c. $\cot^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

Evaluations

Find the values in Exercises 9–12.

- $\sin\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$
- $\sec\left(\cos^{-1}\frac{1}{2}\right)$
- $\tan\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$
- $\cot\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$

Limits

Find the limits in Exercises 13–20. (If in doubt, look at the function's graph.)

- $\lim_{x \rightarrow 1^-} \sin^{-1} x$
- $\lim_{x \rightarrow -1^+} \cos^{-1} x$
- $\lim_{x \rightarrow \infty} \tan^{-1} x$
- $\lim_{x \rightarrow -\infty} \tan^{-1} x$
- $\lim_{x \rightarrow \infty} \sec^{-1} x$
- $\lim_{x \rightarrow -\infty} \sec^{-1} x$
- $\lim_{x \rightarrow \infty} \csc^{-1} x$
- $\lim_{x \rightarrow -\infty} \csc^{-1} x$

Finding Derivatives

In Exercises 21–42, find the derivative of y with respect to the appropriate variable.

- $y = \cos^{-1}(x^2)$
- $y = \cos^{-1}(1/x)$
- $y = \sin^{-1}\sqrt{2}t$
- $y = \sin^{-1}(1-t)$
- $y = \sec^{-1}(2s+1)$
- $y = \sec^{-1}5s$
- $y = \csc^{-1}(x^2+1), x > 0$
- $y = \csc^{-1}\frac{x}{2}$
- $y = \sec^{-1}\frac{1}{t}, 0 < t < 1$
- $y = \sin^{-1}\frac{3}{t^2}$
- $y = \cot^{-1}\sqrt{t}$
- $y = \cot^{-1}\sqrt{t-1}$
- $y = \ln(\tan^{-1}x)$
- $y = \tan^{-1}(\ln x)$

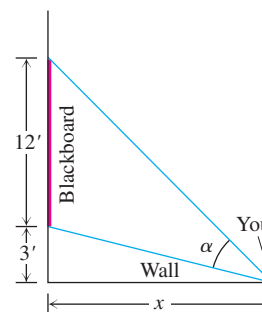
- $y = \csc^{-1}(e^t)$
- $y = \cos^{-1}(e^{-t})$
- $y = s\sqrt{1-s^2} + \cos^{-1}s$
- $y = \sqrt{s^2-1} - \sec^{-1}s$
- $y = \tan^{-1}\sqrt{x^2-1} + \csc^{-1}x, x > 1$
- $y = \cot^{-1}\frac{1}{x} - \tan^{-1}x$
- $y = x\sin^{-1}x + \sqrt{1-x^2}$
- $y = \ln(x^2+4) - x\tan^{-1}\left(\frac{x}{2}\right)$

Theory and Examples

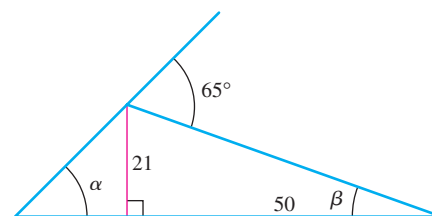
43. You are sitting in a classroom next to the wall looking at the blackboard at the front of the room. The blackboard is 12 ft long and starts 3 ft from the wall you are sitting next to. Show that your viewing angle is

$$\alpha = \cot^{-1}\frac{x}{15} - \cot^{-1}\frac{x}{3}$$

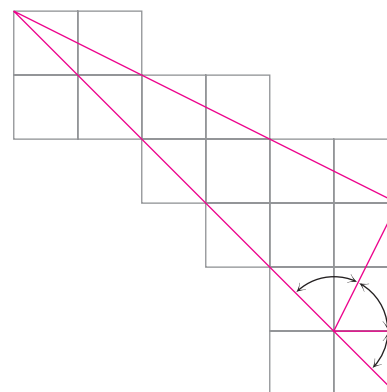
if you are x ft from the front wall.



44. Find the angle α .

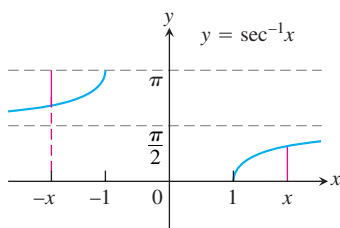


45. Here is an informal proof that $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$. Explain what is going on.



46. Two derivations of the identity $\sec^{-1}(-x) = \pi - \sec^{-1}x$

- a. (Geometric) Here is a pictorial proof that $\sec^{-1}(-x) = \pi - \sec^{-1}x$. See if you can tell what is going on.



- b. (Algebraic) Derive the identity $\sec^{-1}(-x) = \pi - \sec^{-1}x$ by combining the following two equations from the text:

$$\cos^{-1}(-x) = \pi - \cos^{-1}x \quad \text{Eq. (4), Section 1.6}$$

$$\sec^{-1}x = \cos^{-1}(1/x) \quad \text{Eq. (1)}$$

Which of the expressions in Exercises 47–50 are defined, and which are not? Give reasons for your answers.

47. a. $\tan^{-1}2$ b. $\cos^{-1}2$
 48. a. $\csc^{-1}(1/2)$ b. $\csc^{-1}2$
 49. a. $\sec^{-1}0$ b. $\sin^{-1}\sqrt{2}$
 50. a. $\cot^{-1}(-1/2)$ b. $\cos^{-1}(-5)$
 51. Use the identity

$$\csc^{-1}u = \frac{\pi}{2} - \sec^{-1}u$$

to derive the formula for the derivative of $\csc^{-1}u$ in Table 3.1 from the formula for the derivative of $\sec^{-1}u$.

52. Derive the formula

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

for the derivative of $y = \tan^{-1}x$ by differentiating both sides of the equivalent equation $\tan y = x$.

53. Use the Derivative Rule in Section 3.8, Theorem 3, to derive

$$\frac{d}{dx} \sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}, \quad |x| > 1.$$

54. Use the identity

$$\cot^{-1}u = \frac{\pi}{2} - \tan^{-1}u$$

to derive the formula for the derivative of $\cot^{-1}u$ in Table 3.1 from the formula for the derivative of $\tan^{-1}u$.

55. What is special about the functions

$$f(x) = \sin^{-1} \frac{x-1}{x+1}, \quad x \geq 0, \quad \text{and} \quad g(x) = 2 \tan^{-1} \sqrt{x}?$$

Explain.

56. What is special about the functions

$$f(x) = \sin^{-1} \frac{1}{\sqrt{x^2+1}} \quad \text{and} \quad g(x) = \tan^{-1} \frac{1}{x}?$$

Explain.

- T** 57. Find the values of

a. $\sec^{-1}1.5$ b. $\csc^{-1}(-1.5)$ c. $\cot^{-1}2$

- T** 58. Find the values of

a. $\sec^{-1}(-3)$ b. $\csc^{-1}1.7$ c. $\cot^{-1}(-2)$

- T** In Exercises 59–61, find the domain and range of each composite function. Then graph the composites on separate screens. Do the graphs make sense in each case? Give reasons for your answers. Comment on any differences you see.

59. a. $y = \tan^{-1}(\tan x)$ b. $y = \tan(\tan^{-1}x)$
 60. a. $y = \sin^{-1}(\sin x)$ b. $y = \sin(\sin^{-1}x)$
 61. a. $y = \cos^{-1}(\cos x)$ b. $y = \cos(\cos^{-1}x)$

- T** Use your graphing utility for Exercises 62–66.

62. Graph $y = \sec(\sec^{-1}x) = \sec(\cos^{-1}(1/x))$. Explain what you see.
 63. **Newton's serpentine** Graph Newton's serpentine, $y = 4x/(x^2+1)$. Then graph $y = 2 \sin(2 \tan^{-1}x)$ in the same graphing window. What do you see? Explain.
 64. Graph the rational function $y = (2-x^2)/x^2$. Then graph $y = \cos(2 \sec^{-1}x)$ in the same graphing window. What do you see? Explain.
 65. Graph $f(x) = \sin^{-1}x$ together with its first two derivatives. Comment on the behavior of f and the shape of its graph in relation to the signs and values of f' and f'' .
 66. Graph $f(x) = \tan^{-1}x$ together with its first two derivatives. Comment on the behavior of f and the shape of its graph in relation to the signs and values of f' and f'' .

3.10 | Related Rates

In this section we look at problems that ask for the rate at which some variable changes when it is known how the rate of some other related variable (or perhaps several variables) changes. The problem of finding a rate of change from other known rates of change is called a *related rates problem*.

Related Rates Equations

Suppose we are pumping air into a spherical balloon. Both the volume and radius of the balloon are increasing over time. If V is the volume and r is the radius of the balloon at an instant of time, then

$$V = \frac{4}{3} \pi r^3.$$

Using the Chain Rule, we differentiate both sides with respect to t to find an equation relating the rates of change of V and r ,

$$\frac{dV}{dt} = \frac{dV}{dr} \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

So if we know the radius r of the balloon and the rate dV/dt at which the volume is increasing at a given instant of time, then we can solve this last equation for dr/dt to find how fast the radius is increasing at that instant. Note that it is easier to directly measure the rate of increase of the volume (the rate at which air is being pumped into the balloon) than it is to measure the increase in the radius. The related rates equation allows us to calculate dr/dt from dV/dt .

Very often the key to relating the variables in a related rates problem is drawing a picture that shows the geometric relations between them, as illustrated in the following example.

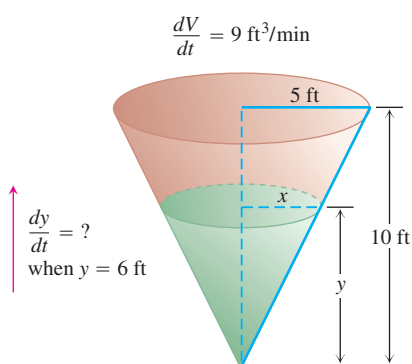


FIGURE 3.43 The geometry of the conical tank and the rate at which water fills the tank determine how fast the water level rises (Example 1).

EXAMPLE 1 Water runs into a conical tank at the rate of $9 \text{ ft}^3/\text{min}$. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?

Solution Figure 3.43 shows a partially filled conical tank. The variables in the problem are

V = volume (ft^3) of the water in the tank at time t (min)

x = radius (ft) of the surface of the water at time t

y = depth (ft) of the water in the tank at time t .

We assume that V , x , and y are differentiable functions of t . The constants are the dimensions of the tank. We are asked for dy/dt when

$$y = 6 \text{ ft} \quad \text{and} \quad \frac{dV}{dt} = 9 \text{ ft}^3/\text{min}.$$

The water forms a cone with volume

$$V = \frac{1}{3} \pi x^2 y.$$

This equation involves x as well as V and y . Because no information is given about x and dx/dt at the time in question, we need to eliminate x . The similar triangles in Figure 3.43 give us a way to express x in terms of y :

$$\frac{x}{y} = \frac{5}{10} \quad \text{or} \quad x = \frac{y}{2}.$$

Therefore, find

$$V = \frac{1}{3} \pi \left(\frac{y}{2}\right)^2 y = \frac{\pi}{12} y^3$$

to give the derivative

$$\frac{dV}{dt} = \frac{\pi}{12} \cdot 3y^2 \frac{dy}{dt} = \frac{\pi}{4} y^2 \frac{dy}{dt}.$$

Finally, use $y = 6$ and $dV/dt = 9$ to solve for dy/dt .

$$9 = \frac{\pi}{4} (6)^2 \frac{dy}{dt}$$

$$\frac{dy}{dt} = \frac{1}{\pi} \approx 0.32$$

At the moment in question, the water level is rising at about 0.32 ft/min. ■

Related Rates Problem Strategy

1. Draw a picture and name the variables and constants. Use t for time. Assume that all variables are differentiable functions of t .
2. Write down the numerical information (in terms of the symbols you have chosen).
3. Write down what you are asked to find (usually a rate, expressed as a derivative).
4. Write an equation that relates the variables. You may have to combine two or more equations to get a single equation that relates the variable whose rate you want to the variables whose rates you know.
5. Differentiate with respect to t . Then express the rate you want in terms of the rates and variables whose values you know.
6. Evaluate. Use known values to find the unknown rate.

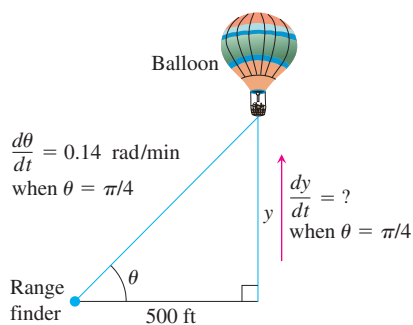


FIGURE 3.44 The rate of change of the balloon's height is related to the rate of change of the angle the range finder makes with the ground (Example 2).

EXAMPLE 2 A hot air balloon rising straight up from a level field is tracked by a range finder 500 ft from the liftoff point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?

Solution We answer the question in six steps.

1. Draw a picture and name the variables and constants (Figure 3.44). The variables in the picture are
 θ = the angle in radians the range finder makes with the ground.
 y = the height in feet of the balloon.

We let t represent time in minutes and assume that θ and y are differentiable functions of t .

The one constant in the picture is the distance from the range finder to the liftoff point (500 ft). There is no need to give it a special symbol.

2. Write down the additional numerical information.

$$\frac{d\theta}{dt} = 0.14 \text{ rad/min} \quad \text{when} \quad \theta = \frac{\pi}{4}$$

3. Write down what we are to find. We want dy/dt when $\theta = \pi/4$.
4. Write an equation that relates the variables y and θ .

$$\frac{y}{500} = \tan \theta \quad \text{or} \quad y = 500 \tan \theta$$

5. Differentiate with respect to t using the Chain Rule. The result tells how dy/dt (which we want) is related to $d\theta/dt$ (which we know).

$$\frac{dy}{dt} = 500 (\sec^2 \theta) \frac{d\theta}{dt}$$

6. Evaluate with $\theta = \pi/4$ and $d\theta/dt = 0.14$ to find dy/dt .

$$\frac{dy}{dt} = 500(\sqrt{2})^2(0.14) = 140 \quad \sec \frac{\pi}{4} = \sqrt{2}$$

At the moment in question, the balloon is rising at the rate of 140 ft/min. ■

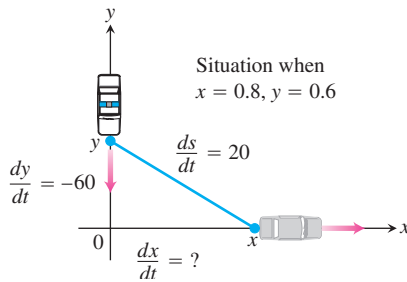


FIGURE 3.45 The speed of the car is related to the speed of the police cruiser and the rate of change of the distance between them (Example 3).

EXAMPLE 3 A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 mi north of the intersection and the car is 0.8 mi to the east, the police determine with radar that the distance between them and the car is increasing at 20 mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?

Solution We picture the car and cruiser in the coordinate plane, using the positive x -axis as the eastbound highway and the positive y -axis as the southbound highway (Figure 3.45). We let t represent time and set

$$\begin{aligned}x &= \text{position of car at time } t \\y &= \text{position of cruiser at time } t \\s &= \text{distance between car and cruiser at time } t.\end{aligned}$$

We assume that x , y , and s are differentiable functions of t .

We want to find dx/dt when

$$x = 0.8 \text{ mi}, \quad y = 0.6 \text{ mi}, \quad \frac{dy}{dt} = -60 \text{ mph}, \quad \frac{ds}{dt} = 20 \text{ mph}.$$

Note that dy/dt is negative because y is decreasing.

We differentiate the distance equation

$$s^2 = x^2 + y^2$$

(we could also use $s = \sqrt{x^2 + y^2}$), and obtain

$$\begin{aligned}2s \frac{ds}{dt} &= 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \\ \frac{ds}{dt} &= \frac{1}{s} \left(x \frac{dx}{dt} + y \frac{dy}{dt} \right) \\ &= \frac{1}{\sqrt{x^2 + y^2}} \left(x \frac{dx}{dt} + y \frac{dy}{dt} \right).\end{aligned}$$

Finally, we use $x = 0.8$, $y = 0.6$, $dy/dt = -60$, $ds/dt = 20$, and solve for dx/dt .

$$\begin{aligned}20 &= \frac{1}{\sqrt{(0.8)^2 + (0.6)^2}} \left(0.8 \frac{dx}{dt} + (0.6)(-60) \right) \\ \frac{dx}{dt} &= \frac{20\sqrt{(0.8)^2 + (0.6)^2} + (0.6)(60)}{0.8} = 70\end{aligned}$$

At the moment in question, the car's speed is 70 mph. ■

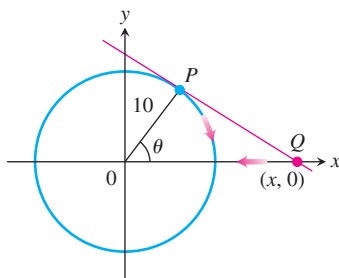


FIGURE 3.46 The particle P travels clockwise along the circle (Example 4).

EXAMPLE 4 A particle P moves clockwise at a constant rate along a circle of radius 10 ft centered at the origin. The particle's initial position is $(0, 10)$ on the y -axis and its final destination is the point $(10, 0)$ on the x -axis. Once the particle is in motion, the tangent line at P intersects the x -axis at a point Q (which moves over time). If it takes the particle 30 sec to travel from start to finish, how fast is the point Q moving along the x -axis when it is 20 ft from the center of the circle?

Solution We picture the situation in the coordinate plane with the circle centered at the origin (see Figure 3.46). We let t represent time and let θ denote the angle from the x -axis to the radial line joining the origin to P . Since the particle travels from start to finish in 30 sec, it is traveling along the circle at a constant rate of $\pi/2$ radians in $1/2$ min, or π rad/min. In other words, $d\theta/dt = -\pi$, with t being measured in minutes. The negative sign appears because θ is decreasing over time.

Setting $x(t)$ to be the distance at time t from the point Q to the origin, we want to find dx/dt when

$$x = 20 \text{ ft} \quad \text{and} \quad \frac{d\theta}{dt} = -\pi \text{ rad/min.}$$

To relate the variables x and θ , we see from Figure 3.46 that $x \cos \theta = 10$, or $x = 10 \sec \theta$. Differentiation of this last equation gives

$$\frac{dx}{dt} = 10 \sec \theta \tan \theta \frac{d\theta}{dt} = -10\pi \sec \theta \tan \theta.$$

Note that dx/dt is negative because x is decreasing (Q is moving towards the origin).

When $x = 20$, $\cos \theta = 1/2$ and $\sec \theta = 2$. Also, $\tan \theta = \sqrt{\sec^2 \theta - 1} = \sqrt{3}$. It follows that

$$\frac{dx}{dt} = (-10\pi)(2)(\sqrt{3}) = -20\sqrt{3}\pi.$$

At the moment in question, the point Q is moving towards the origin at the speed of $20\sqrt{3}\pi \approx 108.8$ ft/min. ■

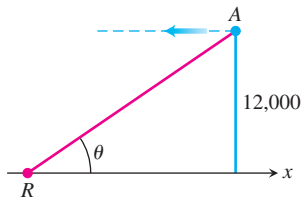


FIGURE 3.47 Jet airliner A traveling at constant altitude toward radar station R (Example 5).

EXAMPLE 5 A jet airliner is flying at a constant altitude of 12,000 ft above sea level as it approaches a Pacific island. The aircraft comes within the direct line of sight of a radar station located on the island, and the radar indicates the initial angle between sea level and its line of sight to the aircraft is 30° . How fast (in miles per hour) is the aircraft approaching the island when first detected by the radar instrument if it is turning upward (counterclockwise) at the rate of $2/3$ deg/sec in order to keep the aircraft within its direct line of sight?

Solution The aircraft A and radar station R are pictured in the coordinate plane, using the positive x -axis as the horizontal distance at sea level from R to A , and the positive y -axis as the vertical altitude above sea level. We let t represent time and observe that $y = 12,000$ is a constant. The general situation and line-of-sight angle θ are depicted in Figure 3.47. We want to find dx/dt when $\theta = \pi/6$ rad and $d\theta/dt = 2/3$ deg/sec.

From Figure 3.47, we see that

$$\frac{12,000}{x} = \tan \theta \quad \text{or} \quad x = 12,000 \cot \theta.$$

Using miles instead of feet for our distance units, the last equation translates to

$$x = \frac{12,000}{5280} \cot \theta.$$

Differentiation with respect to t gives

$$\frac{dx}{dt} = -\frac{1200}{528} \csc^2 \theta \frac{d\theta}{dt}.$$

When $\theta = \pi/6$, $\sin^2 \theta = 1/4$, so $\csc^2 \theta = 4$. Converting $d\theta/dt = 2/3$ deg/sec to radians per hour, we find

$$\frac{d\theta}{dt} = \frac{2}{3} \left(\frac{\pi}{180} \right) (3600) \text{ rad/hr.} \quad 1 \text{ hr} = 3600 \text{ sec, } 1 \text{ deg} = \pi/180 \text{ rad}$$

Substitution into the equation for dx/dt then gives

$$\frac{dx}{dt} = \left(-\frac{1200}{528} \right) (4) \left(\frac{2}{3} \right) \left(\frac{\pi}{180} \right) (3600) \approx -380.$$

The negative sign appears because the distance x is decreasing, so the aircraft is approaching the island at a speed of approximately 380 mi/hr when first detected by the radar. ■

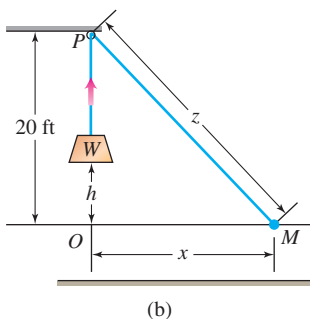
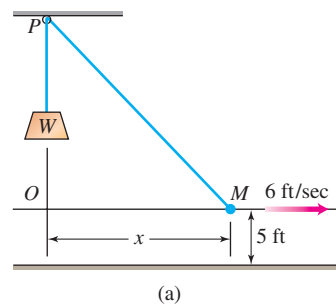


FIGURE 3.48 A worker at M walks to the right pulling the weight W upwards as the rope moves through the pulley P (Example 6).

EXAMPLE 6 Figure 3.48(a) shows a rope running through a pulley at P and bearing a weight W at one end. The other end is held 5 ft above the ground in the hand M of a worker. Suppose the pulley is 25 ft above ground, the rope is 45 ft long, and the worker is walking rapidly away from the vertical line PW at the rate of 6 ft/sec. How fast is the weight being raised when the worker's hand is 21 ft away from PW ?

Solution We let OM be the horizontal line of length x ft from a point O directly below the pulley to the worker's hand M at any instant of time (Figure 3.48). Let h be the height of the weight W above O , and let z denote the length of rope from the pulley P to the worker's hand. We want to know dh/dt when $x = 21$ given that $dx/dt = 6$. Note that the height of P above O is 20 ft because O is 5 ft above the ground. We assume the angle at O is a right angle.

At any instant of time t we have the following relationships (see Figure 3.48b):

$$20 - h + z = 45 \quad \text{Total length of rope is 45 ft.}$$

$$20^2 + x^2 = z^2. \quad \text{Angle at } O \text{ is a right angle.}$$

If we solve for $z = 25 + h$ in the first equation, and substitute into the second equation, we have

$$20^2 + x^2 = (25 + h)^2. \quad (1)$$

Differentiating both sides with respect to t gives

$$2x \frac{dx}{dt} = 2(25 + h) \frac{dh}{dt},$$

and solving this last equation for dh/dt we find

$$\frac{dh}{dt} = \frac{x}{25 + h} \frac{dx}{dt}. \quad (2)$$

Since we know dx/dt , it remains only to find $25 + h$ at the instant when $x = 21$. From Equation (1),

$$20^2 + 21^2 = (25 + h)^2$$

so that

$$(25 + h)^2 = 841, \quad \text{or} \quad 25 + h = 29.$$

Equation (2) now gives

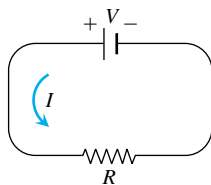
$$\frac{dh}{dt} = \frac{21}{29} \cdot 6 = \frac{126}{29} \approx 4.3 \text{ ft/sec}$$

as the rate at which the weight is being raised when $x = 21$ ft. ■

Exercises 3.10

- Area** Suppose that the radius r and area $A = \pi r^2$ of a circle are differentiable functions of t . Write an equation that relates dA/dt to dr/dt .
- Surface area** Suppose that the radius r and surface area $S = 4\pi r^2$ of a sphere are differentiable functions of t . Write an equation that relates dS/dt to dr/dt .
- Assume that $y = 5x$ and $dx/dt = 2$. Find dy/dt .
- Assume that $2x + 3y = 12$ and $dy/dt = -2$. Find dx/dt .
- If $y = x^2$ and $dx/dt = 3$, then what is dy/dt when $x = -1$?
- If $x = y^3 - y$ and $dy/dt = 5$, then what is dx/dt when $y = 2$?
- If $x^2 + y^2 = 25$ and $dx/dt = -2$, then what is dy/dt when $x = 3$ and $y = -4$?
- If $x^2 y^3 = 4/27$ and $dy/dt = 1/2$, then what is dx/dt when $x = 2$?
- If $L = \sqrt{x^2 + y^2}$, $dx/dt = -1$, and $dy/dt = 3$, find dL/dt when $x = 5$ and $y = 12$.
- If $r + s^2 + v^3 = 12$, $dr/dt = 4$, and $ds/dt = -3$, find dv/dt when $r = 3$ and $s = 1$.

11. If the original 24 m edge length x of a cube decreases at the rate of 5 m/min, when $x = 3$ m at what rate does the cube's
- surface area change?
 - volume change?
12. A cube's surface area increases at the rate of $72 \text{ in}^2/\text{sec}$. At what rate is the cube's volume changing when the edge length is $x = 3$ in?
13. **Volume** The radius r and height h of a right circular cylinder are related to the cylinder's volume V by the formula $V = \pi r^2 h$.
- How is dV/dt related to dh/dt if r is constant?
 - How is dV/dt related to dr/dt if h is constant?
 - How is dV/dt related to dr/dt and dh/dt if neither r nor h is constant?
14. **Volume** The radius r and height h of a right circular cone are related to the cone's volume V by the equation $V = (1/3)\pi r^2 h$.
- How is dV/dt related to dh/dt if r is constant?
 - How is dV/dt related to dr/dt if h is constant?
 - How is dV/dt related to dr/dt and dh/dt if neither r nor h is constant?
15. **Changing voltage** The voltage V (volts), current I (amperes), and resistance R (ohms) of an electric circuit like the one shown here are related by the equation $V = IR$. Suppose that V is increasing at the rate of 1 volt/sec while I is decreasing at the rate of $1/3$ amp/sec. Let t denote time in seconds.



- What is the value of dV/dt ?
 - What is the value of dI/dt ?
 - What equation relates dR/dt to dV/dt and dI/dt ?
 - Find the rate at which R is changing when $V = 12$ volts and $I = 2$ amp. Is R increasing, or decreasing?
16. **Electrical power** The power P (watts) of an electric circuit is related to the circuit's resistance R (ohms) and current I (amperes) by the equation $P = RI^2$.
- How are dP/dt , dR/dt , and dI/dt related if none of P , R , and I are constant?
 - How is dR/dt related to dI/dt if P is constant?
17. **Distance** Let x and y be differentiable functions of t and let $s = \sqrt{x^2 + y^2}$ be the distance between the points $(x, 0)$ and $(0, y)$ in the xy -plane.
- How is ds/dt related to dx/dt if y is constant?
 - How is ds/dt related to dx/dt and dy/dt if neither x nor y is constant?
 - How is dx/dt related to dy/dt if s is constant?
18. **Diagonals** If x , y , and z are lengths of the edges of a rectangular box, the common length of the box's diagonals is $s = \sqrt{x^2 + y^2 + z^2}$.

- Assuming that x , y , and z are differentiable functions of t , how is ds/dt related to dx/dt , dy/dt , and dz/dt ?
- How is ds/dt related to dy/dt and dz/dt if x is constant?
- How are dx/dt , dy/dt , and dz/dt related if s is constant?

19. **Area** The area A of a triangle with sides of lengths a and b enclosing an angle of measure θ is

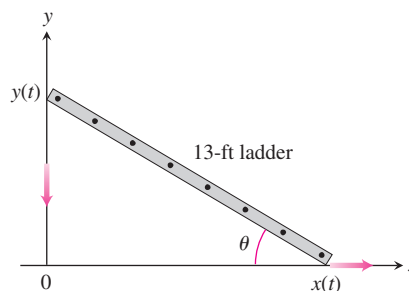
$$A = \frac{1}{2} ab \sin \theta.$$

- How is dA/dt related to $d\theta/dt$ if a and b are constant?
 - How is dA/dt related to $d\theta/dt$ and da/dt if only b is constant?
 - How is dA/dt related to $d\theta/dt$, da/dt , and db/dt if none of a , b , and θ are constant?
20. **Heating a plate** When a circular plate of metal is heated in an oven, its radius increases at the rate of 0.01 cm/min. At what rate is the plate's area increasing when the radius is 50 cm?
21. **Changing dimensions in a rectangle** The length l of a rectangle is decreasing at the rate of 2 cm/sec while the width w is increasing at the rate of 2 cm/sec. When $l = 12$ cm and $w = 5$ cm, find the rates of change of (a) the area, (b) the perimeter, and (c) the lengths of the diagonals of the rectangle. Which of these quantities are decreasing, and which are increasing?
22. **Changing dimensions in a rectangular box** Suppose that the edge lengths x , y , and z of a closed rectangular box are changing at the following rates:

$$\frac{dx}{dt} = 1 \text{ m/sec}, \quad \frac{dy}{dt} = -2 \text{ m/sec}, \quad \frac{dz}{dt} = 1 \text{ m/sec}.$$

Find the rates at which the box's (a) volume, (b) surface area, and (c) diagonal length $s = \sqrt{x^2 + y^2 + z^2}$ are changing at the instant when $x = 4$, $y = 3$, and $z = 2$.

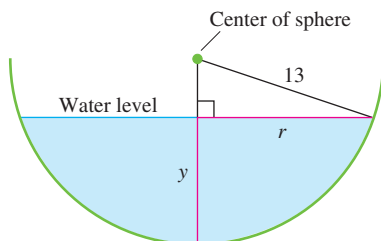
23. **A sliding ladder** A 13-ft ladder is leaning against a house when its base starts to slide away (see accompanying figure). By the time the base is 12 ft from the house, the base is moving at the rate of 5 ft/sec.
- How fast is the top of the ladder sliding down the wall then?
 - At what rate is the area of the triangle formed by the ladder, wall, and ground changing then?
 - At what rate is the angle θ between the ladder and the ground changing then?



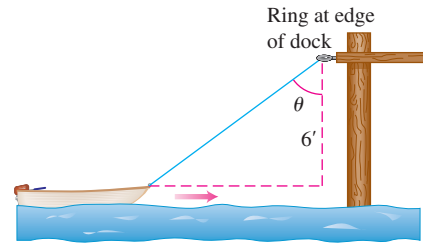
24. **Commercial air traffic** Two commercial airplanes are flying at an altitude of 40,000 ft along straight-line courses that intersect at right angles. Plane A is approaching the intersection point at a speed of 442 knots (nautical miles per hour; a nautical mile is 2000 yd). Plane B is approaching the intersection at 481 knots. At what rate is the distance between the planes changing when A is 5

nautical miles from the intersection point and B is 12 nautical miles from the intersection point?

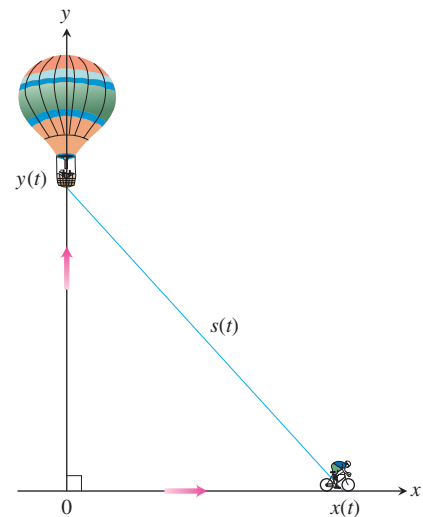
25. **Flying a kite** A girl flies a kite at a height of 300 ft, the wind carrying the kite horizontally away from her at a rate of 25 ft/sec. How fast must she let out the string when the kite is 500 ft away from her?
26. **Boring a cylinder** The mechanics at Lincoln Automotive are reboring a 6-in.-deep cylinder to fit a new piston. The machine they are using increases the cylinder's radius one thousandth of an inch every 3 min. How rapidly is the cylinder volume increasing when the bore (diameter) is 3.800 in.?
27. **A growing sand pile** Sand falls from a conveyor belt at the rate of $10 \text{ m}^3/\text{min}$ onto the top of a conical pile. The height of the pile is always three-eighths of the base diameter. How fast are the (a) height and (b) radius changing when the pile is 4 m high? Answer in centimeters per minute.
28. **A draining conical reservoir** Water is flowing at the rate of $50 \text{ m}^3/\text{min}$ from a shallow concrete conical reservoir (vertex down) of base radius 45 m and height 6 m.
- How fast (centimeters per minute) is the water level falling when the water is 5 m deep?
 - How fast is the radius of the water's surface changing then? Answer in centimeters per minute.
29. **A draining hemispherical reservoir** Water is flowing at the rate of $6 \text{ m}^3/\text{min}$ from a reservoir shaped like a hemispherical bowl of radius 13 m, shown here in profile. Answer the following questions, given that the volume of water in a hemispherical bowl of radius R is $V = (\pi/3)y^2(3R - y)$ when the water is y meters deep.



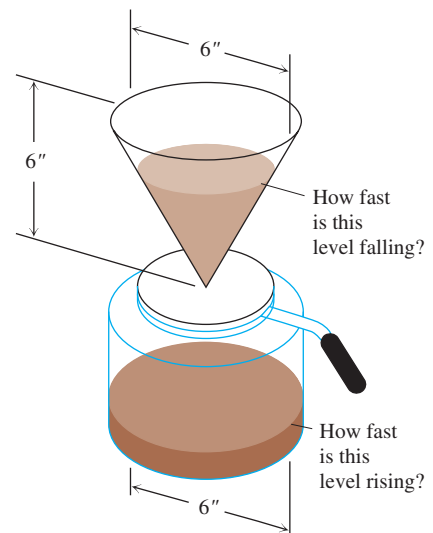
- At what rate is the water level changing when the water is 8 m deep?
 - What is the radius r of the water's surface when the water is y m deep?
 - At what rate is the radius r changing when the water is 8 m deep?
30. **A growing raindrop** Suppose that a drop of mist is a perfect sphere and that, through condensation, the drop picks up moisture at a rate proportional to its surface area. Show that under these circumstances the drop's radius increases at a constant rate.
31. **The radius of an inflating balloon** A spherical balloon is inflated with helium at the rate of $100\pi \text{ ft}^3/\text{min}$. How fast is the balloon's radius increasing at the instant the radius is 5 ft? How fast is the surface area increasing?
32. **Hauling in a dinghy** A dinghy is pulled toward a dock by a rope from the bow through a ring on the dock 6 ft above the bow. The rope is hauled in at the rate of 2 ft/sec.
- How fast is the boat approaching the dock when 10 ft of rope are out?
 - At what rate is the angle θ changing at this instant (see the figure)?



33. **A balloon and a bicycle** A balloon is rising vertically above a level, straight road at a constant rate of 1 ft/sec. Just when the balloon is 65 ft above the ground, a bicycle moving at a constant rate of 17 ft/sec passes under it. How fast is the distance $s(t)$ between the bicycle and balloon increasing 3 sec later?



34. **Making coffee** Coffee is draining from a conical filter into a cylindrical coffeepot at the rate of $10 \text{ in}^3/\text{min}$.
- How fast is the level in the pot rising when the coffee in the cone is 5 in. deep?
 - How fast is the level in the cone falling then?



- 35. Cardiac output** In the late 1860s, Adolf Fick, a professor of physiology in the Faculty of Medicine in Würzburg, Germany, developed one of the methods we use today for measuring how much blood your heart pumps in a minute. Your cardiac output as you read this sentence is probably about 7 L/min. At rest it is likely to be a bit under 6 L/min. If you are a trained marathon runner running a marathon, your cardiac output can be as high as 30 L/min.

Your cardiac output can be calculated with the formula

$$y = \frac{Q}{D},$$

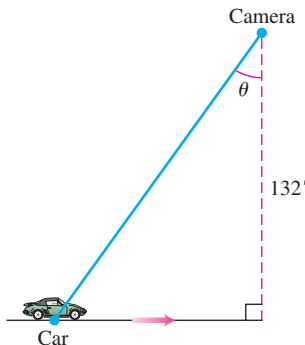
where Q is the number of milliliters of CO_2 you exhale in a minute and D is the difference between the CO_2 concentration (ml/L) in the blood pumped to the lungs and the CO_2 concentration in the blood returning from the lungs. With $Q = 233$ ml/min and $D = 97 - 56 = 41$ ml/L,

$$y = \frac{233 \text{ ml/min}}{41 \text{ ml/L}} \approx 5.68 \text{ L/min},$$

fairly close to the 6 L/min that most people have at basal (resting) conditions. (Data courtesy of J. Kenneth Herd, M.D., Quillan College of Medicine, East Tennessee State University.)

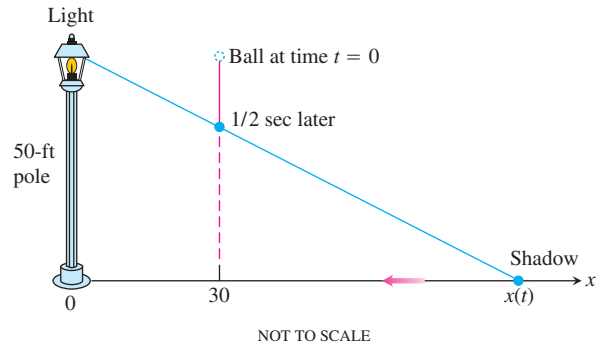
Suppose that when $Q = 233$ and $D = 41$, we also know that D is decreasing at the rate of 2 units a minute but that Q remains unchanged. What is happening to the cardiac output?

- 36. Moving along a parabola** A particle moves along the parabola $y = x^2$ in the first quadrant in such a way that its x -coordinate (measured in meters) increases at a steady 10 m/sec. How fast is the angle of inclination θ of the line joining the particle to the origin changing when $x = 3$ m?
- 37. Motion in the plane** The coordinates of a particle in the metric xy -plane are differentiable functions of time t with $dx/dt = -1$ m/sec and $dy/dt = -5$ m/sec. How fast is the particle's distance from the origin changing as it passes through the point $(5, 12)$?
- 38. Videotaping a moving car** You are videotaping a race from a stand 132 ft from the track, following a car that is moving at 180 mi/h (264 ft/sec), as shown in the accompanying figure. How fast will your camera angle θ be changing when the car is right in front of you? A half second later?

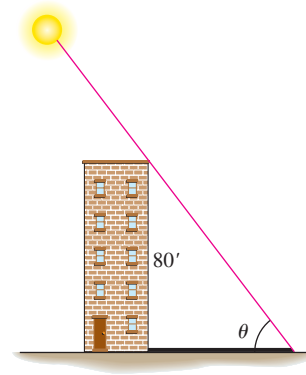


- 39. A moving shadow** A light shines from the top of a pole 50 ft high. A ball is dropped from the same height from a point 30 ft

away from the light. (See accompanying figure.) How fast is the shadow of the ball moving along the ground 1/2 sec later? (Assume the ball falls a distance $s = 16t^2$ ft in t sec.)

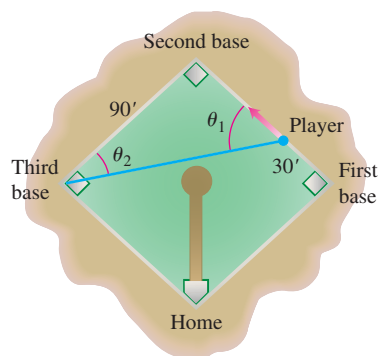


- 40. A building's shadow** On a morning of a day when the sun will pass directly overhead, the shadow of an 80-ft building on level ground is 60 ft long. At the moment in question, the angle θ the sun makes with the ground is increasing at the rate of $0.27^\circ/\text{min}$. At what rate is the shadow decreasing? (Remember to use radians. Express your answer in inches per minute, to the nearest tenth.)



- 41. A melting ice layer** A spherical iron ball 8 in. in diameter is coated with a layer of ice of uniform thickness. If the ice melts at the rate of $10 \text{ in}^3/\text{min}$, how fast is the thickness of the ice decreasing when it is 2 in. thick? How fast is the outer surface area of ice decreasing?
- 42. Highway patrol** A highway patrol plane flies 3 mi above a level, straight road at a steady 120 mi/h. The pilot sees an oncoming car and with radar determines that at the instant the line-of-sight distance from plane to car is 5 mi, the line-of-sight distance is decreasing at the rate of 160 mi/h. Find the car's speed along the highway.
- 43. Baseball players** A baseball diamond is a square 90 ft on a side. A player runs from first base to second at a rate of 16 ft/sec.
- At what rate is the player's distance from third base changing when the player is 30 ft from first base?
 - At what rates are angles θ_1 and θ_2 (see the figure) changing at that time?

- c. The player slides into second base at the rate of 15 ft/sec. At what rates are angles θ_1 and θ_2 changing as the player touches base?



44. **Ships** Two ships are steaming straight away from a point O along routes that make a 120° angle. Ship A moves at 14 knots (nautical miles per hour; a nautical mile is 2000 yd). Ship B moves at 21 knots. How fast are the ships moving apart when $OA = 5$ and $OB = 3$ nautical miles?

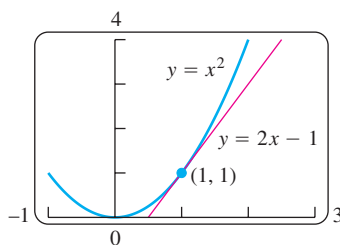
3.11 Linearization and Differentials

Sometimes we can approximate complicated functions with simpler ones that give the accuracy we want for specific applications and are easier to work with. The approximating functions discussed in this section are called *linearizations*, and they are based on tangent lines. Other approximating functions, such as polynomials, are discussed in Chapter 10.

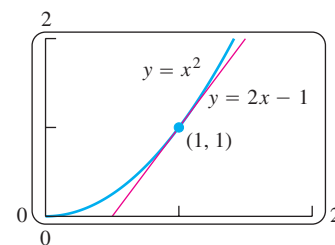
We introduce new variables dx and dy , called *differentials*, and define them in a way that makes Leibniz's notation for the derivative dy/dx a true ratio. We use dy to estimate error in measurement, which then provides for a precise proof of the Chain Rule (Section 3.6).

Linearization

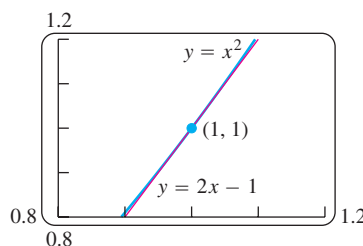
As you can see in Figure 3.49, the tangent to the curve $y = x^2$ lies close to the curve near the point of tangency. For a brief interval to either side, the y -values along the tangent line



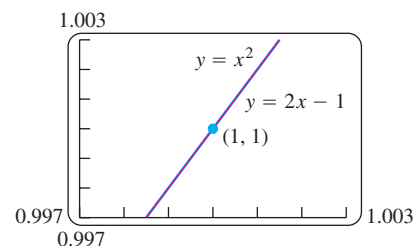
$y = x^2$ and its tangent $y = 2x - 1$ at $(1, 1)$.



Tangent and curve very close near $(1, 1)$.



Tangent and curve very close throughout entire x -interval shown.



Tangent and curve closer still. Computer screen cannot distinguish tangent from curve on this x -interval.

FIGURE 3.49 The more we magnify the graph of a function near a point where the function is differentiable, the flatter the graph becomes and the more it resembles its tangent.

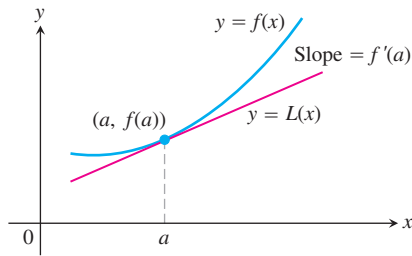


FIGURE 3.50 The tangent to the curve $y = f(x)$ at $x = a$ is the line $L(x) = f(a) + f'(a)(x - a)$.

give good approximations to the y -values on the curve. We observe this phenomenon by zooming in on the two graphs at the point of tangency or by looking at tables of values for the difference between $f(x)$ and its tangent line near the x -coordinate of the point of tangency. The phenomenon is true not just for parabolas; every differentiable curve behaves locally like its tangent line.

In general, the tangent to $y = f(x)$ at a point $x = a$, where f is differentiable (Figure 3.50), passes through the point $(a, f(a))$, so its point-slope equation is

$$y = f(a) + f'(a)(x - a).$$

Thus, this tangent line is the graph of the linear function

$$L(x) = f(a) + f'(a)(x - a).$$

For as long as this line remains close to the graph of f , $L(x)$ gives a good approximation to $f(x)$.

DEFINITIONS If f is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x - a)$$

is the **linearization** of f at a . The approximation

$$f(x) \approx L(x)$$

of f by L is the **standard linear approximation** of f at a . The point $x = a$ is the **center** of the approximation.

EXAMPLE 1 Find the linearization of $f(x) = \sqrt{1 + x}$ at $x = 0$ (Figure 3.51).

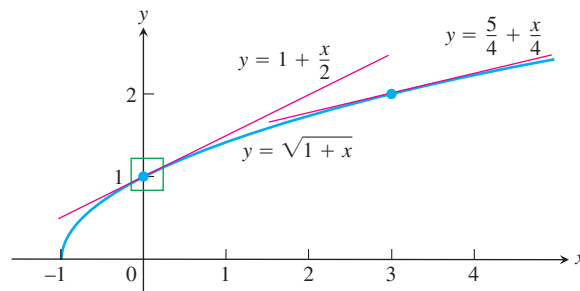


FIGURE 3.51 The graph of $y = \sqrt{1 + x}$ and its linearizations at $x = 0$ and $x = 3$. Figure 3.52 shows a magnified view of the small window about 1 on the y -axis.

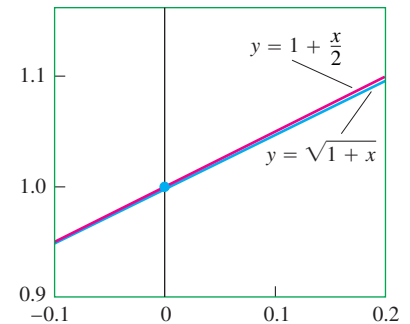


FIGURE 3.52 Magnified view of the window in Figure 3.51.

Solution Since

$$f'(x) = \frac{1}{2}(1 + x)^{-1/2},$$

we have $f(0) = 1$ and $f'(0) = 1/2$, giving the linearization

$$L(x) = f(a) + f'(a)(x - a) = 1 + \frac{1}{2}(x - 0) = 1 + \frac{x}{2}.$$

See Figure 3.52. ■

The following table shows how accurate the approximation $\sqrt{1 + x} \approx 1 + (x/2)$ from Example 1 is for some values of x near 0. As we move away from zero, we lose

accuracy. For example, for $x = 2$, the linearization gives 2 as the approximation for $\sqrt{3}$, which is not even accurate to one decimal place.

Approximation	True value	True value – approximation
$\sqrt{1.2} \approx 1 + \frac{0.2}{2} = 1.10$	1.095445	$<10^{-2}$
$\sqrt{1.05} \approx 1 + \frac{0.05}{2} = 1.025$	1.024695	$<10^{-3}$
$\sqrt{1.005} \approx 1 + \frac{0.005}{2} = 1.00250$	1.002497	$<10^{-5}$

Do not be misled by the preceding calculations into thinking that whatever we do with a linearization is better done with a calculator. In practice, we would never use a linearization to find a particular square root. The utility of a linearization is its ability to replace a complicated formula by a simpler one over an entire interval of values. If we have to work with $\sqrt{1+x}$ for x close to 0 and can tolerate the small amount of error involved, we can work with $1 + (x/2)$ instead. Of course, we then need to know how much error there is. We further examine the estimation of error in Chapter 10.

A linear approximation normally loses accuracy away from its center. As Figure 3.51 suggests, the approximation $\sqrt{1+x} \approx 1 + (x/2)$ will probably be too crude to be useful near $x = 3$. There, we need the linearization at $x = 3$.

EXAMPLE 2 Find the linearization of $f(x) = \sqrt{1+x}$ at $x = 3$.

Solution We evaluate the equation defining $L(x)$ at $a = 3$. With

$$f(3) = 2, \quad f'(3) = \frac{1}{2}(1+x)^{-1/2} \Big|_{x=3} = \frac{1}{4},$$

we have

$$L(x) = 2 + \frac{1}{4}(x-3) = \frac{5}{4} + \frac{x}{4}. \quad \blacksquare$$

At $x = 3.2$, the linearization in Example 2 gives

$$\sqrt{1+x} = \sqrt{1+3.2} \approx \frac{5}{4} + \frac{3.2}{4} = 1.250 + 0.800 = 2.050,$$

which differs from the true value $\sqrt{4.2} \approx 2.04939$ by less than one one-thousandth. The linearization in Example 1 gives

$$\sqrt{1+x} = \sqrt{1+3.2} \approx 1 + \frac{3.2}{2} = 1 + 1.6 = 2.6,$$

a result that is off by more than 25%.

EXAMPLE 3 Find the linearization of $f(x) = \cos x$ at $x = \pi/2$ (Figure 3.53).

Solution Since $f(\pi/2) = \cos(\pi/2) = 0$, $f'(x) = -\sin x$, and $f'(\pi/2) = -\sin(\pi/2) = -1$, we find the linearization at $a = \pi/2$ to be

$$\begin{aligned} L(x) &= f(a) + f'(a)(x-a) \\ &= 0 + (-1)\left(x - \frac{\pi}{2}\right) \\ &= -x + \frac{\pi}{2}. \end{aligned} \quad \blacksquare$$

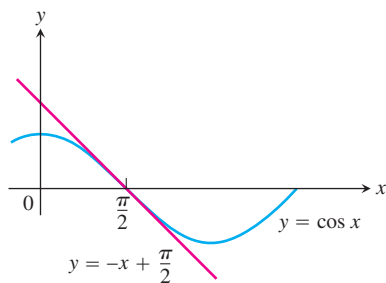


FIGURE 3.53 The graph of $f(x) = \cos x$ and its linearization at $x = \pi/2$. Near $x = \pi/2$, $\cos x \approx -x + (\pi/2)$ (Example 3).

An important linear approximation for roots and powers is

$$(1 + x)^k \approx 1 + kx \quad (x \text{ near } 0; \text{ any number } k)$$

(Exercise 15). This approximation, good for values of x sufficiently close to zero, has broad application. For example, when x is small,

$$\sqrt{1 + x} \approx 1 + \frac{1}{2}x \quad k = 1/2$$

$$\frac{1}{1 - x} = (1 - x)^{-1} \approx 1 + (-1)(-x) = 1 + x \quad k = -1; \text{ replace } x \text{ by } -x.$$

$$\sqrt[3]{1 + 5x^4} = (1 + 5x^4)^{1/3} \approx 1 + \frac{1}{3}(5x^4) = 1 + \frac{5}{3}x^4 \quad k = 1/3; \text{ replace } x \text{ by } 5x^4.$$

$$\frac{1}{\sqrt{1 - x^2}} = (1 - x^2)^{-1/2} \approx 1 + \left(-\frac{1}{2}\right)(-x^2) = 1 + \frac{1}{2}x^2 \quad k = -1/2; \text{ replace } x \text{ by } -x^2.$$

Differentials

We sometimes use the Leibniz notation dy/dx to represent the derivative of y with respect to x . Contrary to its appearance, it is not a ratio. We now introduce two new variables dx and dy with the property that when their ratio exists, it is equal to the derivative.

DEFINITION Let $y = f(x)$ be a differentiable function. The **differential dx** is an independent variable. The **differential dy** is

$$dy = f'(x) dx.$$

Unlike the independent variable dx , the variable dy is always a dependent variable. It depends on both x and dx . If dx is given a specific value and x is a particular number in the domain of the function f , then these values determine the numerical value of dy .

EXAMPLE 4

- (a) Find dy if $y = x^5 + 37x$.
 (b) Find the value of dy when $x = 1$ and $dx = 0.2$.

Solution

- (a) $dy = (5x^4 + 37) dx$
 (b) Substituting $x = 1$ and $dx = 0.2$ in the expression for dy , we have

$$dy = (5 \cdot 1^4 + 37)0.2 = 8.4. \quad \blacksquare$$

The geometric meaning of differentials is shown in Figure 3.54. Let $x = a$ and set $dx = \Delta x$. The corresponding change in $y = f(x)$ is

$$\Delta y = f(a + dx) - f(a).$$

The corresponding change in the tangent line L is

$$\begin{aligned} \Delta L &= L(a + dx) - L(a) \\ &= \underbrace{f(a) + f'(a)[(a + dx) - a]}_{L(a + dx)} - \underbrace{f(a)}_{L(a)} \\ &= f'(a) dx. \end{aligned}$$

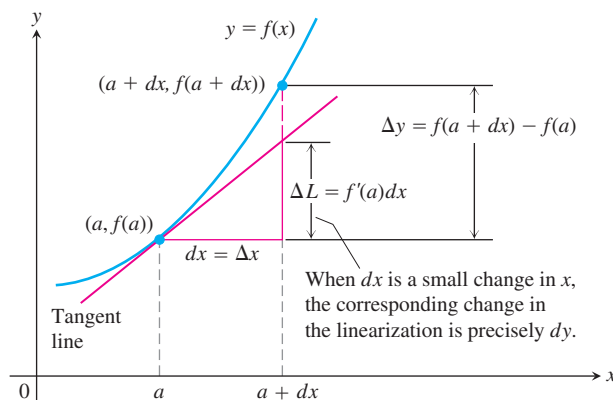


FIGURE 3.54 Geometrically, the differential dy is the change ΔL in the linearization of f when $x = a$ changes by an amount $dx = \Delta x$.

That is, the change in the linearization of f is precisely the value of the differential dy when $x = a$ and $dx = \Delta x$. Therefore, dy represents the amount the tangent line rises or falls when x changes by an amount $dx = \Delta x$.

If $dx \neq 0$, then the quotient of the differential dy by the differential dx is equal to the derivative $f'(x)$ because

$$dy \div dx = \frac{f'(x) dx}{dx} = f'(x) = \frac{dy}{dx}.$$

We sometimes write

$$df = f'(x) dx$$

in place of $dy = f'(x) dx$, calling df the **differential of f** . For instance, if $f(x) = 3x^2 - 6$, then

$$df = d(3x^2 - 6) = 6x dx.$$

Every differentiation formula like

$$\frac{d(u + v)}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \text{or} \quad \frac{d(\sin u)}{dx} = \cos u \frac{du}{dx}$$

has a corresponding differential form like

$$d(u + v) = du + dv \quad \text{or} \quad d(\sin u) = \cos u du.$$

EXAMPLE 5 We can use the Chain Rule and other differentiation rules to find differentials of functions.

(a) $d(\tan 2x) = \sec^2(2x) d(2x) = 2 \sec^2 2x dx$

(b) $d\left(\frac{x}{x+1}\right) = \frac{(x+1) dx - x d(x+1)}{(x+1)^2} = \frac{x dx + dx - x dx}{(x+1)^2} = \frac{dx}{(x+1)^2}$ ■

Estimating with Differentials

Suppose we know the value of a differentiable function $f(x)$ at a point a and want to estimate how much this value will change if we move to a nearby point $a + dx$. If $dx = \Delta x$ is small, then we can see from Figure 3.54 that Δy is approximately equal to the differential dy . Since

$$f(a + dx) = f(a) + \Delta y, \quad \Delta x = dx$$

the differential approximation gives

$$f(a + dx) \approx f(a) + dy$$

when $dx = \Delta x$. Thus the approximation $\Delta y \approx dy$ can be used to estimate $f(a + dx)$ when $f(a)$ is known and dx is small.

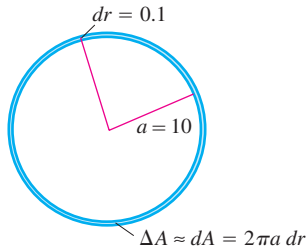


FIGURE 3.55 When dr is small compared with a , the differential dA gives the estimate $A(a + dr) = \pi a^2 + dA$ (Example 6).

EXAMPLE 6 The radius r of a circle increases from $a = 10$ m to 10.1 m (Figure 3.55). Use dA to estimate the increase in the circle's area A . Estimate the area of the enlarged circle and compare your estimate to the true area found by direct calculation.

Solution Since $A = \pi r^2$, the estimated increase is

$$dA = A'(a) dr = 2\pi a dr = 2\pi(10)(0.1) = 2\pi \text{ m}^2.$$

Thus, since $A(r + \Delta r) \approx A(r) + dA$, we have

$$\begin{aligned} A(10 + 0.1) &\approx A(10) + 2\pi \\ &= \pi(10)^2 + 2\pi = 102\pi. \end{aligned}$$

The area of a circle of radius 10.1 m is approximately $102\pi \text{ m}^2$.

The true area is

$$\begin{aligned} A(10.1) &= \pi(10.1)^2 \\ &= 102.01\pi \text{ m}^2. \end{aligned}$$

The error in our estimate is $0.01\pi \text{ m}^2$, which is the difference $\Delta A - dA$. ■

Error in Differential Approximation

Let $f(x)$ be differentiable at $x = a$ and suppose that $dx = \Delta x$ is an increment of x . We have two ways to describe the change in f as x changes from a to $a + \Delta x$:

The true change: $\Delta f = f(a + \Delta x) - f(a)$

The differential estimate: $df = f'(a) \Delta x$.

How well does df approximate Δf ?

We measure the approximation error by subtracting df from Δf :

$$\begin{aligned} \text{Approximation error} &= \Delta f - df \\ &= \Delta f - f'(a)\Delta x \\ &= \underbrace{f(a + \Delta x) - f(a)}_{\Delta f} - f'(a)\Delta x \\ &= \underbrace{\left(\frac{f(a + \Delta x) - f(a)}{\Delta x} - f'(a) \right)}_{\text{Call this part } \epsilon} \cdot \Delta x \\ &= \epsilon \cdot \Delta x. \end{aligned}$$

As $\Delta x \rightarrow 0$, the difference quotient

$$\frac{f(a + \Delta x) - f(a)}{\Delta x}$$

approaches $f'(a)$ (remember the definition of $f'(a)$), so the quantity in parentheses becomes a very small number (which is why we called it ϵ). In fact, $\epsilon \rightarrow 0$ as $\Delta x \rightarrow 0$. When Δx is small, the approximation error $\epsilon \Delta x$ is smaller still.

$$\underbrace{\Delta f}_{\text{change}} = \underbrace{f'(a)\Delta x}_{\text{estimated change}} + \underbrace{\epsilon \Delta x}_{\text{error}}$$

Although we do not know the exact size of the error, it is the product $\epsilon \cdot \Delta x$ of two small quantities that both approach zero as $\Delta x \rightarrow 0$. For many common functions, whenever Δx is small, the error is still smaller.

Change in $y = f(x)$ near $x = a$

If $y = f(x)$ is differentiable at $x = a$ and x changes from a to $a + \Delta x$, the change Δy in f is given by

$$\Delta y = f'(a) \Delta x + \epsilon \Delta x \quad (1)$$

in which $\epsilon \rightarrow 0$ as $\Delta x \rightarrow 0$.

In Example 6 we found that

$$\Delta A = \pi(10.1)^2 - \pi(10)^2 = (102.01 - 100)\pi = \underbrace{(2\pi)}_{dA} + \underbrace{(0.01\pi)}_{\text{error}} \text{ m}^2$$

so the approximation error is $\Delta A - dA = \epsilon \Delta r = 0.01\pi$ and $\epsilon = 0.01\pi/\Delta r = 0.01\pi/0.1 = 0.1\pi$ m.

Proof of the Chain Rule

Equation (1) enables us to prove the Chain Rule correctly. Our goal is to show that if $f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x , then the composite $y = f(g(x))$ is a differentiable function of x . Since a function is differentiable if and only if it has a derivative at each point in its domain, we must show that whenever g is differentiable at x_0 and f is differentiable at $g(x_0)$, then the composite is differentiable at x_0 and the derivative of the composite satisfies the equation

$$\left. \frac{dy}{dx} \right|_{x=x_0} = f'(g(x_0)) \cdot g'(x_0).$$

Let Δx be an increment in x and let Δu and Δy be the corresponding increments in u and y . Applying Equation (1) we have

$$\Delta u = g'(x_0)\Delta x + \epsilon_1 \Delta x = (g'(x_0) + \epsilon_1)\Delta x,$$

where $\epsilon_1 \rightarrow 0$ as $\Delta x \rightarrow 0$. Similarly,

$$\Delta y = f'(u_0)\Delta u + \epsilon_2 \Delta u = (f'(u_0) + \epsilon_2)\Delta u,$$

where $\epsilon_2 \rightarrow 0$ as $\Delta u \rightarrow 0$. Notice also that $\Delta u \rightarrow 0$ as $\Delta x \rightarrow 0$. Combining the equations for Δu and Δy gives

$$\Delta y = (f'(u_0) + \epsilon_2)(g'(x_0) + \epsilon_1)\Delta x,$$

so

$$\frac{\Delta y}{\Delta x} = f'(u_0)g'(x_0) + \epsilon_2 g'(x_0) + f'(u_0)\epsilon_1 + \epsilon_2\epsilon_1.$$

Since ϵ_1 and ϵ_2 go to zero as Δx goes to zero, three of the four terms on the right vanish in the limit, leaving

$$\left. \frac{dy}{dx} \right|_{x=x_0} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = f'(u_0)g'(x_0) = f'(g(x_0)) \cdot g'(x_0). \quad \blacksquare$$

Sensitivity to Change

The equation $df = f'(x) dx$ tells how *sensitive* the output of f is to a change in input at different values of x . The larger the value of f' at x , the greater the effect of a given change dx . As we move from a to a nearby point $a + dx$, we can describe the change in f in three ways:

	True	Estimated
Absolute change	$\Delta f = f(a + dx) - f(a)$	$df = f'(a) dx$
Relative change	$\frac{\Delta f}{f(a)}$	$\frac{df}{f(a)}$
Percentage change	$\frac{\Delta f}{f(a)} \times 100$	$\frac{df}{f(a)} \times 100$

EXAMPLE 7 You want to calculate the depth of a well from the equation $s = 16t^2$ by timing how long it takes a heavy stone you drop to splash into the water below. How sensitive will your calculations be to a 0.1-sec error in measuring the time?

Solution The size of ds in the equation

$$ds = 32t dt$$

depends on how big t is. If $t = 2$ sec, the change caused by $dt = 0.1$ is about

$$ds = 32(2)(0.1) = 6.4 \text{ ft.}$$

Three seconds later at $t = 5$ sec, the change caused by the same dt is

$$ds = 32(5)(0.1) = 16 \text{ ft.}$$

For a fixed error in the time measurement, the error in using ds to estimate the depth is larger when the time it takes until the stone splashes into the water is longer. ■

EXAMPLE 8 In the late 1830s, French physiologist Jean Poiseuille (“pwa-ZOY”) discovered the formula we use today to predict how much the radius of a partially clogged artery decreases the normal volume of flow. His formula,

$$V = kr^4,$$

says that the volume V of fluid flowing through a small pipe or tube in a unit of time at a fixed pressure is a constant times the fourth power of the tube’s radius r . How does a 10% decrease in r affect V ? (See Figure 3.56.)

Solution The differentials of r and V are related by the equation

$$dV = \frac{dV}{dr} dr = 4kr^3 dr.$$

The relative change in V is

$$\frac{dV}{V} = \frac{4kr^3 dr}{kr^4} = 4 \frac{dr}{r}.$$

The relative change in V is 4 times the relative change in r ; so a 10% decrease in r will result in a 40% decrease in the flow. ■

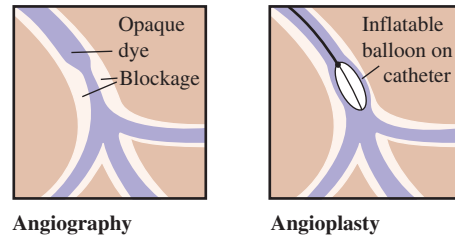


FIGURE 3.56 To unblock a clogged artery, an opaque dye is injected into it to make the inside visible under X-rays. Then a balloon-tipped catheter is inflated inside the artery to widen it at the blockage site.

EXAMPLE 9 Newton's second law,

$$F = \frac{d}{dt}(mv) = m \frac{dv}{dt} = ma,$$

is stated with the assumption that mass is constant, but we know this is not strictly true because the mass of a body increases with velocity. In Einstein's corrected formula, mass has the value

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}},$$

where the "rest mass" m_0 represents the mass of a body that is not moving and c is the speed of light, which is about 300,000 km/sec. Use the approximation

$$\frac{1}{\sqrt{1 - x^2}} \approx 1 + \frac{1}{2}x^2 \quad (2)$$

to estimate the increase Δm in mass resulting from the added velocity v .

Solution When v is very small compared with c , v^2/c^2 is close to zero and it is safe to use the approximation

$$\frac{1}{\sqrt{1 - v^2/c^2}} \approx 1 + \frac{1}{2} \left(\frac{v^2}{c^2} \right) \quad \text{Eq. (2) with } x = \frac{v}{c}$$

to obtain

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \approx m_0 \left[1 + \frac{1}{2} \left(\frac{v^2}{c^2} \right) \right] = m_0 + \frac{1}{2} m_0 v^2 \left(\frac{1}{c^2} \right),$$

or

$$m \approx m_0 + \frac{1}{2} m_0 v^2 \left(\frac{1}{c^2} \right). \quad (3)$$

Equation (3) expresses the increase in mass that results from the added velocity v . ■

Converting Mass to Energy

Equation (3) derived in Example 9 has an important interpretation. In Newtonian physics, $(1/2)m_0 v^2$ is the kinetic energy (KE) of the body, and if we rewrite Equation (3) in the form

$$(m - m_0)c^2 \approx \frac{1}{2} m_0 v^2,$$

we see that

$$(m - m_0)c^2 \approx \frac{1}{2} m_0 v^2 = \frac{1}{2} m_0 v^2 - \frac{1}{2} m_0 (0)^2 = \Delta(\text{KE}),$$

or

$$(\Delta m)c^2 \approx \Delta(\text{KE}).$$

So the change in kinetic energy $\Delta(\text{KE})$ in going from velocity 0 to velocity v is approximately equal to $(\Delta m)c^2$, the change in mass times the square of the speed of light. Using $c \approx 3 \times 10^8$ m/sec, we see that a small change in mass can create a large change in energy.

Exercises 3.11

Finding Linearizations

In Exercises 1–5, find the linearization $L(x)$ of $f(x)$ at $x = a$.

1. $f(x) = x^3 - 2x + 3, \quad a = 2$

2. $f(x) = \sqrt{x^2 + 9}, \quad a = -4$

3. $f(x) = x + \frac{1}{x}, \quad a = 1$

4. $f(x) = \sqrt[3]{x}, \quad a = -8$

5. $f(x) = \tan x, \quad a = \pi$

6. **Common linear approximations at $x = 0$** Find the linearizations of the following functions at $x = 0$.

(a) $\sin x$ (b) $\cos x$ (c) $\tan x$ (d) e^x (e) $\ln(1 + x)$

Linearization for Approximation

In Exercises 7–14, find a linearization at a suitably chosen integer near x_0 at which the given function and its derivative are easy to evaluate.

7. $f(x) = x^2 + 2x, \quad x_0 = 0.1$

8. $f(x) = x^{-1}, \quad x_0 = 0.9$

9. $f(x) = 2x^2 + 3x - 3, \quad x_0 = -0.9$

10. $f(x) = 1 + x, \quad x_0 = 8.1$

11. $f(x) = \sqrt[3]{x}, \quad x_0 = 8.5$

12. $f(x) = \frac{x}{x+1}, \quad x_0 = 1.3$

13. $f(x) = e^{-x}, \quad x_0 = -0.1$

14. $f(x) = \sin^{-1} x, \quad x_0 = \pi/12$

15. Show that the linearization of $f(x) = (1 + x)^k$ at $x = 0$ is $L(x) = 1 + kx$.

16. Use the linear approximation $(1 + x)^k \approx 1 + kx$ to find an approximation for the function $f(x)$ for values of x near zero.

a. $f(x) = (1 - x)^6$ b. $f(x) = \frac{2}{1 - x}$

c. $f(x) = \frac{1}{\sqrt{1 + x}}$ d. $f(x) = \sqrt{2 + x^2}$

e. $f(x) = (4 + 3x)^{1/3}$ f. $f(x) = \sqrt[3]{\left(1 - \frac{1}{2 + x}\right)^2}$

17. **Faster than a calculator** Use the approximation $(1 + x)^k \approx 1 + kx$ to estimate the following.

a. $(1.0002)^{50}$ b. $\sqrt[3]{1.009}$

18. Find the linearization of $f(x) = \sqrt{x+1} + \sin x$ at $x = 0$. How is it related to the individual linearizations of $\sqrt{x+1}$ and $\sin x$ at $x = 0$?

Derivatives in Differential Form

In Exercises 19–38, find dy .

19. $y = x^3 - 3\sqrt{x}$

20. $y = x\sqrt{1 - x^2}$

21. $y = \frac{2x}{1 + x^2}$

22. $y = \frac{2\sqrt{x}}{3(1 + \sqrt{x})}$

23. $2y^{3/2} + xy - x = 0$

24. $xy^2 - 4x^{3/2} - y = 0$

25. $y = \sin(5\sqrt{x})$

26. $y = \cos(x^2)$

27. $y = 4 \tan(x^3/3)$

28. $y = \sec(x^2 - 1)$

29. $y = 3 \csc(1 - 2\sqrt{x})$

30. $y = 2 \cot\left(\frac{1}{\sqrt{x}}\right)$

31. $y = e^{\sqrt{x}}$

32. $y = xe^{-x}$

33. $y = \ln(1 + x^2)$

34. $y = \ln\left(\frac{x+1}{\sqrt{x-1}}\right)$

35. $y = \tan^{-1}(e^{x^2})$

36. $y = \cot^{-1}\left(\frac{1}{x^2}\right) + \cos^{-1} 2x$

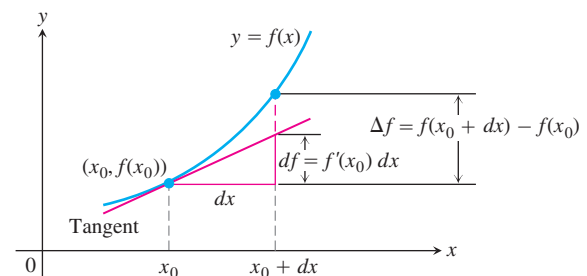
37. $y = \sec^{-1}(e^{-x})$

38. $y = e^{\tan^{-1} \sqrt{x^2+1}}$

Approximation Error

In Exercises 39–44, each function $f(x)$ changes value when x changes from x_0 to $x_0 + dx$. Find

- the change $\Delta f = f(x_0 + dx) - f(x_0)$;
- the value of the estimate $df = f'(x_0) dx$; and
- the approximation error $|\Delta f - df|$.



39. $f(x) = x^2 + 2x$, $x_0 = 1$, $dx = 0.1$
 40. $f(x) = 2x^2 + 4x - 3$, $x_0 = -1$, $dx = 0.1$
 41. $f(x) = x^3 - x$, $x_0 = 1$, $dx = 0.1$
 42. $f(x) = x^4$, $x_0 = 1$, $dx = 0.1$
 43. $f(x) = x^{-1}$, $x_0 = 0.5$, $dx = 0.1$
 44. $f(x) = x^3 - 2x + 3$, $x_0 = 2$, $dx = 0.1$

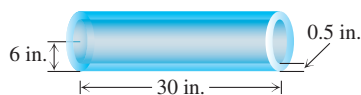
Differential Estimates of Change

In Exercises 45–50, write a differential formula that estimates the given change in volume or surface area.

45. The change in the volume $V = (4/3)\pi r^3$ of a sphere when the radius changes from r_0 to $r_0 + dr$
 46. The change in the volume $V = x^3$ of a cube when the edge lengths change from x_0 to $x_0 + dx$
 47. The change in the surface area $S = 6x^2$ of a cube when the edge lengths change from x_0 to $x_0 + dx$
 48. The change in the lateral surface area $S = \pi r \sqrt{r^2 + h^2}$ of a right circular cone when the radius changes from r_0 to $r_0 + dr$ and the height does not change
 49. The change in the volume $V = \pi r^2 h$ of a right circular cylinder when the radius changes from r_0 to $r_0 + dr$ and the height does not change
 50. The change in the lateral surface area $S = 2\pi r h$ of a right circular cylinder when the height changes from h_0 to $h_0 + dh$ and the radius does not change

Applications

51. The radius of a circle is increased from 2.00 to 2.02 m.
 a. Estimate the resulting change in area.
 b. Express the estimate as a percentage of the circle's original area.
 52. The diameter of a tree was 10 in. During the following year, the circumference increased 2 in. About how much did the tree's diameter increase? The tree's cross-section area?
 53. **Estimating volume** Estimate the volume of material in a cylindrical shell with length 30 in., radius 6 in., and shell thickness 0.5 in.



54. **Estimating height of a building** A surveyor, standing 30 ft from the base of a building, measures the angle of elevation to the top of the building to be 75° . How accurately must the angle be measured for the percentage error in estimating the height of the building to be less than 4%?
 55. **Tolerance** The radius r of a circle is measured with an error of at most 2%. What is the maximum corresponding percentage error in computing the circle's
 a. circumference? b. area?
 56. **Tolerance** The edge x of a cube is measured with an error of at most 0.5%. What is the maximum corresponding percentage error in computing the cube's
 a. surface area? b. volume?

57. **Tolerance** The height and radius of a right circular cylinder are equal, so the cylinder's volume is $V = \pi h^3$. The volume is to be calculated with an error of no more than 1% of the true value. Find approximately the greatest error that can be tolerated in the measurement of h , expressed as a percentage of h .

58. Tolerance

- a. About how accurately must the interior diameter of a 10-m-high cylindrical storage tank be measured to calculate the tank's volume to within 1% of its true value?
 b. About how accurately must the tank's exterior diameter be measured to calculate the amount of paint it will take to paint the side of the tank to within 5% of the true amount?
 59. The diameter of a sphere is measured as 100 ± 1 cm and the volume is calculated from this measurement. Estimate the percentage error in the volume calculation.
 60. Estimate the allowable percentage error in measuring the diameter D of a sphere if the volume is to be calculated correctly to within 3%.
 61. **The effect of flight maneuvers on the heart** The amount of work done by the heart's main pumping chamber, the left ventricle, is given by the equation

$$W = PV + \frac{V\delta v^2}{2g},$$

where W is the work per unit time, P is the average blood pressure, V is the volume of blood pumped out during the unit of time, δ ("delta") is the weight density of the blood, v is the average velocity of the exiting blood, and g is the acceleration of gravity.

When P , V , δ , and v remain constant, W becomes a function of g , and the equation takes the simplified form

$$W = a + \frac{b}{g} \quad (a, b \text{ constant}).$$

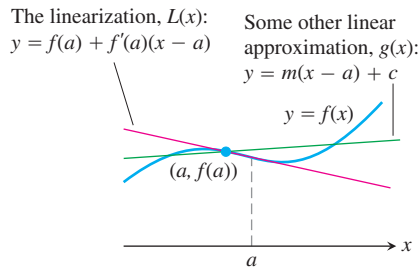
As a member of NASA's medical team, you want to know how sensitive W is to apparent changes in g caused by flight maneuvers, and this depends on the initial value of g . As part of your investigation, you decide to compare the effect on W of a given change dg on the moon, where $g = 5.2$ ft/sec², with the effect the same change dg would have on Earth, where $g = 32$ ft/sec². Use the simplified equation above to find the ratio of dW_{moon} to dW_{Earth} .

62. **Measuring acceleration of gravity** When the length L of a clock pendulum is held constant by controlling its temperature, the pendulum's period T depends on the acceleration of gravity g . The period will therefore vary slightly as the clock is moved from place to place on the earth's surface, depending on the change in g . By keeping track of ΔT , we can estimate the variation in g from the equation $T = 2\pi(L/g)^{1/2}$ that relates T , g , and L .
 a. With L held constant and g as the independent variable, calculate dT and use it to answer parts (b) and (c).
 b. If g increases, will T increase or decrease? Will a pendulum clock speed up or slow down? Explain.
 c. A clock with a 100-cm pendulum is moved from a location where $g = 980$ cm/sec² to a new location. This increases the period by $dT = 0.001$ sec. Find dg and estimate the value of g at the new location.
 63. **The linearization is the best linear approximation** Suppose that $y = f(x)$ is differentiable at $x = a$ and that $g(x) = m(x - a) + c$ is a linear function in which m and c are constants.

If the error $E(x) = f(x) - g(x)$ were small enough near $x = a$, we might think of using g as a linear approximation of f instead of the linearization $L(x) = f(a) + f'(a)(x - a)$. Show that if we impose on g the conditions

1. $E(a) = 0$ The approximation error is zero at $x = a$.
2. $\lim_{x \rightarrow a} \frac{E(x)}{x - a} = 0$ The error is negligible when compared with $x - a$.

then $g(x) = f(a) + f'(a)(x - a)$. Thus, the linearization $L(x)$ gives the only linear approximation whose error is both zero at $x = a$ and negligible in comparison with $x - a$.



64. Quadratic approximations

- a. Let $Q(x) = b_0 + b_1(x - a) + b_2(x - a)^2$ be a quadratic approximation to $f(x)$ at $x = a$ with the properties:
 - i) $Q(a) = f(a)$
 - ii) $Q'(a) = f'(a)$
 - iii) $Q''(a) = f''(a)$.

Determine the coefficients $b_0, b_1,$ and b_2 .

- b. Find the quadratic approximation to $f(x) = 1/(1 - x)$ at $x = 0$.
- T** c. Graph $f(x) = 1/(1 - x)$ and its quadratic approximation at $x = 0$. Then zoom in on the two graphs at the point $(0, 1)$. Comment on what you see.
- T** d. Find the quadratic approximation to $g(x) = 1/x$ at $x = 1$. Graph g and its quadratic approximation together. Comment on what you see.
- T** e. Find the quadratic approximation to $h(x) = \sqrt{1 + x}$ at $x = 0$. Graph h and its quadratic approximation together. Comment on what you see.

- f. What are the linearizations of $f, g,$ and h at the respective points in parts (b), (d), and (e)?

65. The linearization of 2^x

- a. Find the linearization of $f(x) = 2^x$ at $x = 0$. Then round its coefficients to two decimal places.
- T** b. Graph the linearization and function together for $-3 \leq x \leq 3$ and $-1 \leq x \leq 1$.

66. The linearization of $\log_3 x$

- a. Find the linearization of $f(x) = \log_3 x$ at $x = 3$. Then round its coefficients to two decimal places.
- T** b. Graph the linearization and function together in the window $0 \leq x \leq 8$ and $2 \leq x \leq 4$.

COMPUTER EXPLORATIONS

In Exercises 67–72, use a CAS to estimate the magnitude of the error in using the linearization in place of the function over a specified interval I . Perform the following steps:

- a. Plot the function f over I .
- b. Find the linearization L of the function at the point a .
- c. Plot f and L together on a single graph.
- d. Plot the absolute error $|f(x) - L(x)|$ over I and find its maximum value.
- e. From your graph in part (d), estimate as large a $\delta > 0$ as you can, satisfying

$$|x - a| < \delta \quad \Rightarrow \quad |f(x) - L(x)| < \epsilon$$

for $\epsilon = 0.5, 0.1,$ and 0.01 . Then check graphically to see if your δ -estimate holds true.

67. $f(x) = x^3 + x^2 - 2x, [-1, 2], a = 1$
68. $f(x) = \frac{x - 1}{4x^2 + 1}, \left[-\frac{3}{4}, 1\right], a = \frac{1}{2}$
69. $f(x) = x^{2/3}(x - 2), [-2, 3], a = 2$
70. $f(x) = \sqrt{x} - \sin x, [0, 2\pi], a = 2$
71. $f(x) = x2^x, [0, 2], a = 1$
72. $f(x) = \sqrt{x} \sin^{-1} x, [0, 1], a = \frac{1}{2}$

Chapter 3 Questions to Guide Your Review

1. What is the derivative of a function f ? How is its domain related to the domain of f ? Give examples.
2. What role does the derivative play in defining slopes, tangents, and rates of change?
3. How can you sometimes graph the derivative of a function when all you have is a table of the function's values?
4. What does it mean for a function to be differentiable on an open interval? On a closed interval?
5. How are derivatives and one-sided derivatives related?
6. Describe geometrically when a function typically does *not* have a derivative at a point.
7. How is a function's differentiability at a point related to its continuity there, if at all?
8. What rules do you know for calculating derivatives? Give some examples.

9. Explain how the three formulas
- $\frac{d}{dx}(x^n) = nx^{n-1}$
 - $\frac{d}{dx}(cu) = c \frac{du}{dx}$
 - $\frac{d}{dx}(u_1 + u_2 + \cdots + u_n) = \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_n}{dx}$
- enable us to differentiate any polynomial.
- What formula do we need, in addition to the three listed in Question 9, to differentiate rational functions?
 - What is a second derivative? A third derivative? How many derivatives do the functions you know have? Give examples.
 - What is the derivative of the exponential function e^x ? How does the domain of the derivative compare with the domain of the function?
 - What is the relationship between a function's average and instantaneous rates of change? Give an example.
 - How do derivatives arise in the study of motion? What can you learn about a body's motion along a line by examining the derivatives of the body's position function? Give examples.
 - How can derivatives arise in economics?
 - Give examples of still other applications of derivatives.
 - What do the limits $\lim_{h \rightarrow 0}((\sin h)/h)$ and $\lim_{h \rightarrow 0}((\cos h - 1)/h)$ have to do with the derivatives of the sine and cosine functions? What *are* the derivatives of these functions?
 - Once you know the derivatives of $\sin x$ and $\cos x$, how can you find the derivatives of $\tan x$, $\cot x$, $\sec x$, and $\csc x$? What *are* the derivatives of these functions?
 - At what points are the six basic trigonometric functions continuous? How do you know?
 - What is the rule for calculating the derivative of a composite of two differentiable functions? How is such a derivative evaluated? Give examples.
 - If u is a differentiable function of x , how do you find $(d/dx)(u^n)$ if n is an integer? If n is a real number? Give examples.
 - What is implicit differentiation? When do you need it? Give examples.
 - What is the derivative of the natural logarithm function $\ln x$? How does the domain of the derivative compare with the domain of the function?
 - What is the derivative of the exponential function a^x , $a > 0$ and $a \neq 1$? What is the geometric significance of the limit of $(a^h - 1)/h$ as $h \rightarrow 0$? What is the limit when a is the number e ?
 - What is the derivative of $\log_a x$? Are there any restrictions on a ?
 - What is logarithmic differentiation? Give an example.
 - How can you write any real power of x as a power of e ? Are there any restrictions on x ? How does this lead to the Power Rule for differentiating arbitrary real powers?
 - What is one way of expressing the special number e as a limit? What is an approximate numerical value of e correct to 7 decimal places?
 - What are the derivatives of the inverse trigonometric functions? How do the domains of the derivatives compare with the domains of the functions?
 - How do related rates problems arise? Give examples.
 - Outline a strategy for solving related rates problems. Illustrate with an example.
 - What is the linearization $L(x)$ of a function $f(x)$ at a point $x = a$? What is required of f at a for the linearization to exist? How are linearizations used? Give examples.
 - If x moves from a to a nearby value $a + dx$, how do you estimate the corresponding change in the value of a differentiable function $f(x)$? How do you estimate the relative change? The percentage change? Give an example.

Chapter 3 Practice Exercises

Derivatives of Functions

Find the derivatives of the functions in Exercises 1–64.

- $y = x^5 - 0.125x^2 + 0.25x$
- $y = 3 - 0.7x^3 + 0.3x^7$
- $y = x^3 - 3(x^2 + \pi^2)$
- $y = x^7 + \sqrt{7}x - \frac{1}{\pi + 1}$
- $y = (x + 1)^2(x^2 + 2x)$
- $y = (2x - 5)(4 - x)^{-1}$
- $y = (\theta^2 + \sec \theta + 1)^3$
- $y = \left(-1 - \frac{\csc \theta}{2} - \frac{\theta^2}{4}\right)^2$
- $s = \frac{\sqrt{t}}{1 + \sqrt{t}}$
- $s = \frac{1}{\sqrt{t} - 1}$
- $y = 2 \tan^2 x - \sec^2 x$
- $y = \frac{1}{\sin^2 x} - \frac{2}{\sin x}$
- $s = \cos^4(1 - 2t)$
- $s = \cot^3\left(\frac{2}{t}\right)$
- $s = (\sec t + \tan t)^5$
- $s = \csc^5(1 - t + 3t^2)$
- $r = \sqrt{2\theta \sin \theta}$
- $r = \sin \sqrt{2\theta}$
- $y = \frac{1}{2}x^2 \csc \frac{2}{x}$
- $y = x^{-1/2} \sec(2x)^2$
- $y = 5 \cot x^2$
- $y = x^2 \sin^2(2x^2)$
- $s = \left(\frac{4t}{t+1}\right)^{-2}$
- $y = \left(\frac{\sqrt{x}}{1+x}\right)^2$
- $y = \sqrt{\frac{x^2+x}{x^2}}$
- $r = 2\theta\sqrt{\cos \theta}$
- $r = \sin(\theta + \sqrt{\theta + 1})$
- $y = 2\sqrt{x} \sin \sqrt{x}$
- $y = \sqrt{x} \csc(x + 1)^3$
- $y = x^2 \cot 5x$
- $y = x^{-2} \sin^2(x^3)$
- $s = \frac{-1}{15(15t - 1)^3}$
- $y = \left(\frac{2\sqrt{x}}{2\sqrt{x} + 1}\right)^2$
- $y = 4x\sqrt{x + \sqrt{x}}$

35. $r = \left(\frac{\sin \theta}{\cos \theta - 1}\right)^2$ 36. $r = \left(\frac{1 + \sin \theta}{1 - \cos \theta}\right)^2$
37. $y = (2x + 1)\sqrt{2x + 1}$ 38. $y = 20(3x - 4)^{1/4}(3x - 4)^{-1/5}$
39. $y = \frac{3}{(5x^2 + \sin 2x)^{3/2}}$ 40. $y = (3 + \cos^3 3x)^{-1/3}$
41. $y = 10e^{-x/5}$ 42. $y = \sqrt{2}e^{\sqrt{2}x}$
43. $y = \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x}$ 44. $y = x^2e^{-2/x}$
45. $y = \ln(\sin^2 \theta)$ 46. $y = \ln(\sec^2 \theta)$
47. $y = \log_2(x^2/2)$ 48. $y = \log_5(3x - 7)$
49. $y = 8^{-t}$ 50. $y = 9^{2t}$
51. $y = 5x^{3.6}$ 52. $y = \sqrt{2x - \sqrt{2}}$
53. $y = (x + 2)^{x+2}$ 54. $y = 2(\ln x)^{x/2}$
55. $y = \sin^{-1}\sqrt{1 - u^2}$, $0 < u < 1$
56. $y = \sin^{-1}\left(\frac{1}{\sqrt{v}}\right)$, $v > 1$
57. $y = \ln \cos^{-1} x$
58. $y = z \cos^{-1} z - \sqrt{1 - z^2}$
59. $y = t \tan^{-1} t - \frac{1}{2} \ln t$
60. $y = (1 + t^2) \cot^{-1} 2t$
61. $y = z \sec^{-1} z - \sqrt{z^2 - 1}$, $z > 1$
62. $y = 2\sqrt{x - 1} \sec^{-1} \sqrt{x}$
63. $y = \csc^{-1}(\sec \theta)$, $0 < \theta < \pi/2$
64. $y = (1 + x^2)e^{\tan^{-1} x}$

Implicit Differentiation

In Exercises 65–78, find dy/dx by implicit differentiation.

65. $xy + 2x + 3y = 1$ 66. $x^2 + xy + y^2 - 5x = 2$
67. $x^3 + 4xy - 3y^{4/3} = 2x$ 68. $5x^{4/5} + 10y^{6/5} = 15$
69. $\sqrt{xy} = 1$ 70. $x^2y^2 = 1$
71. $y^2 = \frac{x}{x + 1}$ 72. $y^2 = \sqrt{\frac{1 + x}{1 - x}}$
73. $e^{x+2y} = 1$ 74. $y^2 = 2e^{-1/x}$
75. $\ln(x/y) = 1$ 76. $x \sin^{-1} y = 1 + x^2$
77. $ye^{\tan^{-1} x} = 2$ 78. $x^y = \sqrt{2}$

In Exercises 79 and 80, find dp/dq .

79. $p^3 + 4pq - 3q^2 = 2$ 80. $q = (5p^2 + 2p)^{-3/2}$

In Exercises 81 and 82, find dr/ds .

81. $r \cos 2s + \sin^2 s = \pi$ 82. $2rs - r - s + s^2 = -3$

83. Find d^2y/dx^2 by implicit differentiation:

- a. $x^3 + y^3 = 1$ b. $y^2 = 1 - \frac{2}{x}$
84. a. By differentiating $x^2 - y^2 = 1$ implicitly, show that $dy/dx = x/y$.
- b. Then show that $d^2y/dx^2 = -1/y^3$.

Numerical Values of Derivatives

85. Suppose that functions $f(x)$ and $g(x)$ and their first derivatives have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	1	1	-3	1/2
1	3	5	1/2	-4

Find the first derivatives of the following combinations at the given value of x .

- a. $6f(x) - g(x)$, $x = 1$ b. $f(x)g^2(x)$, $x = 0$
- c. $\frac{f(x)}{g(x) + 1}$, $x = 1$ d. $f(g(x))$, $x = 0$
- e. $g(f(x))$, $x = 0$ f. $(x + f(x))^{3/2}$, $x = 1$
- g. $f(x + g(x))$, $x = 0$

86. Suppose that the function $f(x)$ and its first derivative have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$f'(x)$
0	9	-2
1	-3	1/5

Find the first derivatives of the following combinations at the given value of x .

- a. $\sqrt{x} f(x)$, $x = 1$ b. $\sqrt{f(x)}$, $x = 0$
- c. $f(\sqrt{x})$, $x = 1$ d. $f(1 - 5 \tan x)$, $x = 0$
- e. $\frac{f(x)}{2 + \cos x}$, $x = 0$ f. $10 \sin\left(\frac{\pi x}{2}\right) f^2(x)$, $x = 1$

87. Find the value of dy/dt at $t = 0$ if $y = 3 \sin 2x$ and $x = t^2 + \pi$.

88. Find the value of ds/du at $u = 2$ if $s = t^2 + 5t$ and $t = (u^2 + 2u)^{1/3}$.

89. Find the value of dw/ds at $s = 0$ if $w = \sin(e^{\sqrt{r}})$ and $r = 3 \sin(s + \pi/6)$.

90. Find the value of dr/dt at $t = 0$ if $r = (\theta^2 + 7)^{1/3}$ and $\theta^2 t + \theta = 1$.

91. If $y^3 + y = 2 \cos x$, find the value of d^2y/dx^2 at the point $(0, 1)$.

92. If $x^{1/3} + y^{1/3} = 4$, find d^2y/dx^2 at the point $(8, 8)$.

Applying the Derivative Definition

In Exercises 93 and 94, find the derivative using the definition.

93. $f(t) = \frac{1}{2t + 1}$

94. $g(x) = 2x^2 + 1$

95. a. Graph the function

$$f(x) = \begin{cases} x^2, & -1 \leq x < 0 \\ -x^2, & 0 \leq x \leq 1. \end{cases}$$

- b. Is f continuous at $x = 0$?
- c. Is f differentiable at $x = 0$?
- Give reasons for your answers.

96. a. Graph the function

$$f(x) = \begin{cases} x, & -1 \leq x < 0 \\ \tan x, & 0 \leq x \leq \pi/4. \end{cases}$$

- b. Is f continuous at $x = 0$?
c. Is f differentiable at $x = 0$?

Give reasons for your answers.

97. a. Graph the function

$$f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 2 - x, & 1 < x \leq 2. \end{cases}$$

- b. Is f continuous at $x = 1$?
c. Is f differentiable at $x = 1$?

Give reasons for your answers.

98. For what value or values of the constant m , if any, is

$$f(x) = \begin{cases} \sin 2x, & x \leq 0 \\ mx, & x > 0 \end{cases}$$

- a. continuous at $x = 0$?
b. differentiable at $x = 0$?

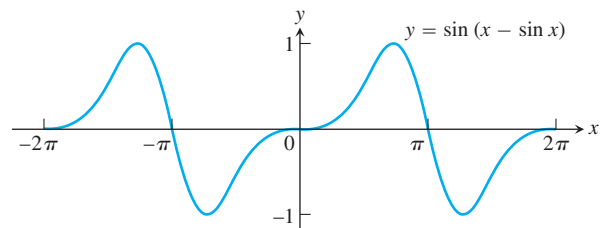
Give reasons for your answers.

Slopes, Tangents, and Normals

99. **Tangents with specified slope** Are there any points on the curve $y = (x/2) + 1/(2x - 4)$ where the slope is $-3/2$? If so, find them.
100. **Tangents with specified slope** Are there any points on the curve $y = x - e^{-x}$ where the slope is 2? If so, find them.
101. **Horizontal tangents** Find the points on the curve $y = 2x^3 - 3x^2 - 12x + 20$ where the tangent is parallel to the x -axis.
102. **Tangent intercepts** Find the x - and y -intercepts of the line that is tangent to the curve $y = x^3$ at the point $(-2, -8)$.
103. **Tangents perpendicular or parallel to lines** Find the points on the curve $y = 2x^3 - 3x^2 - 12x + 20$ where the tangent is
a. perpendicular to the line $y = 1 - (x/24)$.
b. parallel to the line $y = \sqrt{2} - 12x$.
104. **Intersecting tangents** Show that the tangents to the curve $y = (\pi \sin x)/x$ at $x = \pi$ and $x = -\pi$ intersect at right angles.
105. **Normals parallel to a line** Find the points on the curve $y = \tan x$, $-\pi/2 < x < \pi/2$, where the normal is parallel to the line $y = -x/2$. Sketch the curve and normals together, labeling each with its equation.
106. **Tangent and normal lines** Find equations for the tangent and normal to the curve $y = 1 + \cos x$ at the point $(\pi/2, 1)$. Sketch the curve, tangent, and normal together, labeling each with its equation.
107. **Tangent parabola** The parabola $y = x^2 + C$ is to be tangent to the line $y = x$. Find C .
108. **Slope of tangent** Show that the tangent to the curve $y = x^3$ at any point (a, a^3) meets the curve again at a point where the slope is four times the slope at (a, a^3) .
109. **Tangent curve** For what value of c is the curve $y = c/(x + 1)$ tangent to the line through the points $(0, 3)$ and $(5, -2)$?
110. **Normal to a circle** Show that the normal line at any point of the circle $x^2 + y^2 = a^2$ passes through the origin.

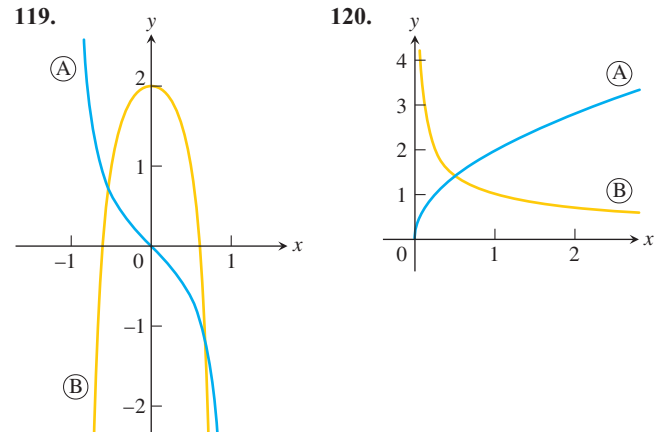
In Exercises 111–116, find equations for the lines that are tangent and normal to the curve at the given point.

111. $x^2 + 2y^2 = 9$, $(1, 2)$
112. $e^x + y^2 = 2$, $(0, 1)$
113. $xy + 2x - 5y = 2$, $(3, 2)$
114. $(y - x)^2 = 2x + 4$, $(6, 2)$
115. $x + \sqrt{xy} = 6$, $(4, 1)$
116. $x^{3/2} + 2y^{3/2} = 17$, $(1, 4)$
117. Find the slope of the curve $x^3y^3 + y^2 = x + y$ at the points $(1, 1)$ and $(1, -1)$.
118. The graph shown suggests that the curve $y = \sin(x - \sin x)$ might have horizontal tangents at the x -axis. Does it? Give reasons for your answer.

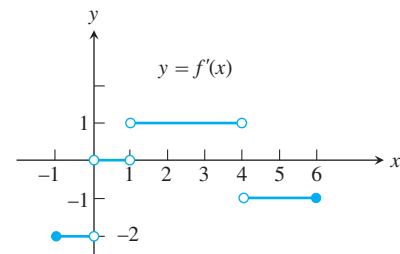


Analyzing Graphs

Each of the figures in Exercises 119 and 120 shows two graphs, the graph of a function $y = f(x)$ together with the graph of its derivative $f'(x)$. Which graph is which? How do you know?



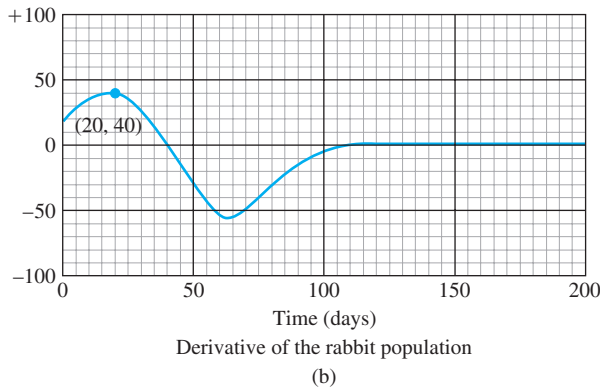
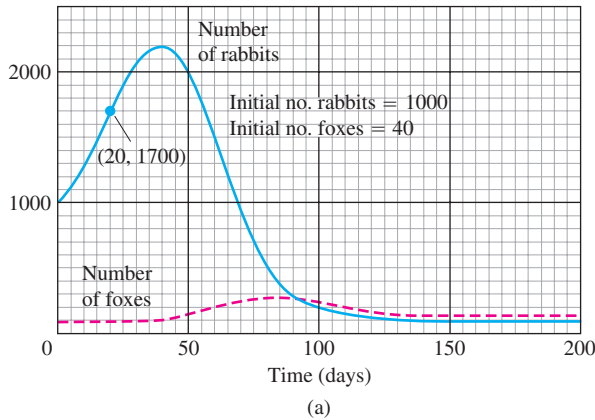
121. Use the following information to graph the function $y = f(x)$ for $-1 \leq x \leq 6$.
- The graph of f is made of line segments joined end to end.
 - The graph starts at the point $(-1, 2)$.
 - The derivative of f , where defined, agrees with the step function shown here.



122. Repeat Exercise 121, supposing that the graph starts at $(-1, 0)$ instead of $(-1, 2)$.

Exercises 123 and 124 are about the accompanying graphs. The graphs in part (a) show the numbers of rabbits and foxes in a small arctic population. They are plotted as functions of time for 200 days. The number of rabbits increases at first, as the rabbits reproduce. But the foxes prey on rabbits and, as the number of foxes increases, the rabbit population levels off and then drops. Part (b) shows the graph of the derivative of the rabbit population, made by plotting slopes.

123. a. What is the value of the derivative of the rabbit population when the number of rabbits is largest? Smallest?
 b. What is the size of the rabbit population when its derivative is largest? Smallest (negative value)?
124. In what units should the slopes of the rabbit and fox population curves be measured?



Trigonometric Limits

Find the limits in Exercises 125–132.

125. $\lim_{x \rightarrow 0} \frac{\sin x}{2x^2 - x}$ 126. $\lim_{x \rightarrow 0} \frac{3x - \tan 7x}{2x}$
 127. $\lim_{r \rightarrow 0} \frac{\sin r}{\tan 2r}$ 128. $\lim_{\theta \rightarrow 0} \frac{\sin(\sin \theta)}{\theta}$
 129. $\lim_{\theta \rightarrow (\pi/2)^-} \frac{4 \tan^2 \theta + \tan \theta + 1}{\tan^2 \theta + 5}$
 130. $\lim_{\theta \rightarrow 0^+} \frac{1 - 2 \cot^2 \theta}{5 \cot^2 \theta - 7 \cot \theta - 8}$
 131. $\lim_{x \rightarrow 0} \frac{x \sin x}{2 - 2 \cos x}$ 132. $\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta^2}$

Show how to extend the functions in Exercises 133 and 134 to be continuous at the origin.

133. $g(x) = \frac{\tan(\tan x)}{\tan x}$ 134. $f(x) = \frac{\tan(\tan x)}{\sin(\sin x)}$

Logarithmic Differentiation

In Exercises 135–140, use logarithmic differentiation to find the derivative of y with respect to the appropriate variable.

135. $y = \frac{2(x^2 + 1)}{\sqrt{\cos 2x}}$ 136. $y = \frac{10\sqrt{3x+4}}{\sqrt{2x-4}}$
 137. $y = \left(\frac{(t+1)(t-1)}{(t-2)(t+3)}\right)^5, t > 2$
 138. $y = \frac{2u2^u}{\sqrt{u^2+1}}$
 139. $y = (\sin \theta)^{\sqrt{\theta}}$ 140. $y = (\ln x)^{1/(\ln x)}$

Related Rates

141. **Right circular cylinder** The total surface area S of a right circular cylinder is related to the base radius r and height h by the equation $S = 2\pi r^2 + 2\pi rh$.

- a. How is dS/dt related to dr/dt if h is constant?
 b. How is dS/dt related to dh/dt if r is constant?
 c. How is dS/dt related to dr/dt and dh/dt if neither r nor h is constant?
 d. How is dr/dt related to dh/dt if S is constant?

142. **Right circular cone** The lateral surface area S of a right circular cone is related to the base radius r and height h by the equation $S = \pi r\sqrt{r^2 + h^2}$.

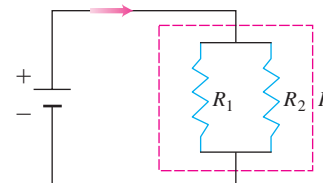
- a. How is dS/dt related to dr/dt if h is constant?
 b. How is dS/dt related to dh/dt if r is constant?
 c. How is dS/dt related to dr/dt and dh/dt if neither r nor h is constant?

143. **Circle's changing area** The radius of a circle is changing at the rate of $-2/\pi$ m/sec. At what rate is the circle's area changing when $r = 10$ m?

144. **Cube's changing edges** The volume of a cube is increasing at the rate of $1200 \text{ cm}^3/\text{min}$ at the instant its edges are 20 cm long. At what rate are the lengths of the edges changing at that instant?

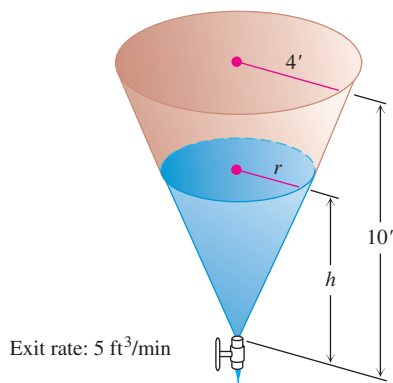
145. **Resistors connected in parallel** If two resistors of R_1 and R_2 ohms are connected in parallel in an electric circuit to make an R -ohm resistor, the value of R can be found from the equation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

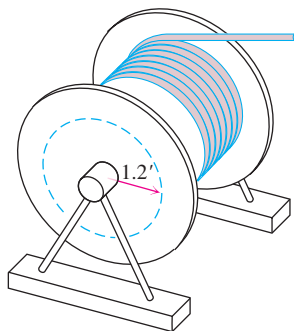


If R_1 is decreasing at the rate of 1 ohm/sec and R_2 is increasing at the rate of 0.5 ohm/sec, at what rate is R changing when $R_1 = 75$ ohms and $R_2 = 50$ ohms?

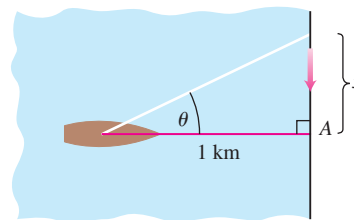
- 146. Impedance in a series circuit** The impedance Z (ohms) in a series circuit is related to the resistance R (ohms) and reactance X (ohms) by the equation $Z = \sqrt{R^2 + X^2}$. If R is increasing at 3 ohms/sec and X is decreasing at 2 ohms/sec, at what rate is Z changing when $R = 10$ ohms and $X = 20$ ohms?
- 147. Speed of moving particle** The coordinates of a particle moving in the metric xy -plane are differentiable functions of time t with $dx/dt = 10$ m/sec and $dy/dt = 5$ m/sec. How fast is the particle moving away from the origin as it passes through the point $(3, -4)$?
- 148. Motion of a particle** A particle moves along the curve $y = x^{3/2}$ in the first quadrant in such a way that its distance from the origin increases at the rate of 11 units per second. Find dx/dt when $x = 3$.
- 149. Draining a tank** Water drains from the conical tank shown in the accompanying figure at the rate of $5 \text{ ft}^3/\text{min}$.
- What is the relation between the variables h and r in the figure?
 - How fast is the water level dropping when $h = 6$ ft?



- 150. Rotating spool** As television cable is pulled from a large spool to be strung from the telephone poles along a street, it unwinds from the spool in layers of constant radius (see accompanying figure). If the truck pulling the cable moves at a steady 6 ft/sec (a touch over 4 mph), use the equation $s = r\theta$ to find how fast (radians per second) the spool is turning when the layer of radius 1.2 ft is being unwound.



- 151. Moving searchlight beam** The figure shows a boat 1 km offshore, sweeping the shore with a searchlight. The light turns at a constant rate, $d\theta/dt = -0.6$ rad/sec.
- How fast is the light moving along the shore when it reaches point A ?
 - How many revolutions per minute is 0.6 rad/sec?



- 152. Points moving on coordinate axes** Points A and B move along the x - and y -axes, respectively, in such a way that the distance r (meters) along the perpendicular from the origin to the line AB remains constant. How fast is OA changing, and is it increasing, or decreasing, when $OB = 2r$ and B is moving toward O at the rate of $0.3r$ m/sec?

Linearization

- 153.** Find the linearizations of

a. $\tan x$ at $x = -\pi/4$ b. $\sec x$ at $x = -\pi/4$.

Graph the curves and linearizations together.

- 154.** We can obtain a useful linear approximation of the function $f(x) = 1/(1 + \tan x)$ at $x = 0$ by combining the approximations

$$\frac{1}{1+x} \approx 1-x \quad \text{and} \quad \tan x \approx x$$

to get

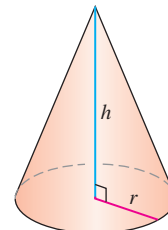
$$\frac{1}{1+\tan x} \approx 1-x.$$

Show that this result is the standard linear approximation of $1/(1 + \tan x)$ at $x = 0$.

- 155.** Find the linearization of $f(x) = \sqrt{1+x} + \sin x - 0.5$ at $x = 0$.
- 156.** Find the linearization of $f(x) = 2/(1-x) + \sqrt{1+x} - 3.1$ at $x = 0$.

Differential Estimates of Change

- 157. Surface area of a cone** Write a formula that estimates the change that occurs in the lateral surface area of a right circular cone when the height changes from h_0 to $h_0 + dh$ and the radius does not change.



$$V = \frac{1}{3}\pi r^2 h$$

$$S = \pi r \sqrt{r^2 + h^2}$$

(Lateral surface area)

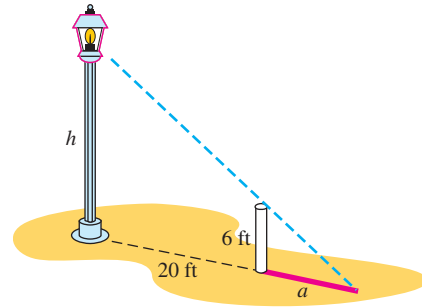
- 158. Controlling error**

- How accurately should you measure the edge of a cube to be reasonably sure of calculating the cube's surface area with an error of no more than 2%?
- Suppose that the edge is measured with the accuracy required in part (a). About how accurately can the cube's

volume be calculated from the edge measurement? To find out, estimate the percentage error in the volume calculation that might result from using the edge measurement.

- 159. Compounding error** The circumference of the equator of a sphere is measured as 10 cm with a possible error of 0.4 cm. This measurement is then used to calculate the radius. The radius is then used to calculate the surface area and volume of the sphere. Estimate the percentage errors in the calculated values of
- the radius.
 - the surface area.
 - the volume.
- 160. Finding height** To find the height of a lamppost (see accompanying figure), you stand a 6 ft pole 20 ft from the lamp and

measure the length a of its shadow, finding it to be 15 ft, give or take an inch. Calculate the height of the lamppost using the value $a = 15$ and estimate the possible error in the result.



Chapter 3 Additional and Advanced Exercises

- 1.** An equation like $\sin^2 \theta + \cos^2 \theta = 1$ is called an **identity** because it holds for all values of θ . An equation like $\sin \theta = 0.5$ is not an identity because it holds only for selected values of θ , not all. If you differentiate both sides of a trigonometric identity in θ with respect to θ , the resulting new equation will also be an identity.

Differentiate the following to show that the resulting equations hold for all θ .

- $\sin 2\theta = 2 \sin \theta \cos \theta$
 - $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$
- 2.** If the identity $\sin(x + a) = \sin x \cos a + \cos x \sin a$ is differentiated with respect to x , is the resulting equation also an identity? Does this principle apply to the equation $x^2 - 2x - 8 = 0$? Explain.
- 3. a.** Find values for the constants a , b , and c that will make

$$f(x) = \cos x \quad \text{and} \quad g(x) = a + bx + cx^2$$

satisfy the conditions

$$f(0) = g(0), \quad f'(0) = g'(0), \quad \text{and} \quad f''(0) = g''(0).$$

- b.** Find values for b and c that will make

$$f(x) = \sin(x + a) \quad \text{and} \quad g(x) = b \sin x + c \cos x$$

satisfy the conditions

$$f(0) = g(0) \quad \text{and} \quad f'(0) = g'(0).$$

- c.** For the determined values of a , b , and c , what happens for the third and fourth derivatives of f and g in each of parts (a) and (b)?

4. Solutions to differential equations

- a.** Show that $y = \sin x$, $y = \cos x$, and $y = a \cos x + b \sin x$ (a and b constants) all satisfy the equation

$$y'' + y = 0.$$

- b.** How would you modify the functions in part (a) to satisfy the equation

$$y'' + 4y = 0?$$

Generalize this result.

- 5. An osculating circle** Find the values of h , k , and a that make the circle $(x - h)^2 + (y - k)^2 = a^2$ tangent to the parabola $y = x^2 + 1$ at the point $(1, 2)$ and that also make the second derivatives d^2y/dx^2 have the same value on both curves there. Circles like this one that are tangent to a curve and have the same second derivative as the curve at the point of tangency are called *osculating circles* (from the Latin *osculari*, meaning “to kiss”). We encounter them again in Chapter 13.

- 6. Marginal revenue** A bus will hold 60 people. The number x of people per trip who use the bus is related to the fare charged (p dollars) by the law $p = [3 - (x/40)]^2$. Write an expression for the total revenue $r(x)$ per trip received by the bus company. What number of people per trip will make the marginal revenue dr/dx equal to zero? What is the corresponding fare? (This fare is the one that maximizes the revenue, so the bus company should probably rethink its fare policy.)

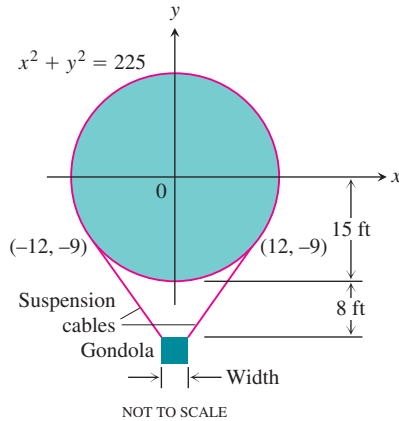
7. Industrial production

- a.** Economists often use the expression “rate of growth” in relative rather than absolute terms. For example, let $u = f(t)$ be the number of people in the labor force at time t in a given industry. (We treat this function as though it were differentiable even though it is an integer-valued step function.)

Let $v = g(t)$ be the average production per person in the labor force at time t . The total production is then $y = uv$. If the labor force is growing at the rate of 4% per year ($du/dt = 0.04u$) and the production per worker is growing at the rate of 5% per year ($dv/dt = 0.05v$), find the rate of growth of the total production, y .

- b. Suppose that the labor force in part (a) is decreasing at the rate of 2% per year while the production per person is increasing at the rate of 3% per year. Is the total production increasing, or is it decreasing, and at what rate?

8. **Designing a gondola** The designer of a 30-ft-diameter spherical hot air balloon wants to suspend the gondola 8 ft below the bottom of the balloon with cables tangent to the surface of the balloon, as shown. Two of the cables are shown running from the top edges of the gondola to their points of tangency, $(-12, -9)$ and $(12, -9)$. How wide should the gondola be?



9. **Pisa by parachute** On August 5, 1988, Mike McCarthy of London jumped from the top of the Tower of Pisa. He then opened his parachute in what he said was a world record low-level parachute jump of 179 ft. Make a rough sketch to show the shape of the graph of his speed during the jump. (Source: *Boston Globe*, Aug. 6, 1988.)
10. **Motion of a particle** The position at time $t \geq 0$ of a particle moving along a coordinate line is

$$s = 10 \cos(t + \pi/4).$$

- What is the particle's starting position ($t = 0$)?
 - What are the points farthest to the left and right of the origin reached by the particle?
 - Find the particle's velocity and acceleration at the points in part (b).
 - When does the particle first reach the origin? What are its velocity, speed, and acceleration then?
11. **Shooting a paper clip** On Earth, you can easily shoot a paper clip 64 ft straight up into the air with a rubber band. In t sec after firing, the paper clip is $s = 64t - 16t^2$ ft above your hand.
- How long does it take the paper clip to reach its maximum height? With what velocity does it leave your hand?
 - On the moon, the same acceleration will send the paper clip to a height of $s = 64t - 2.6t^2$ ft in t sec. About how long will it take the paper clip to reach its maximum height, and how high will it go?
12. **Velocities of two particles** At time t sec, the positions of two particles on a coordinate line are $s_1 = 3t^3 - 12t^2 + 18t + 5$ m and $s_2 = -t^3 + 9t^2 - 12t$ m. When do the particles have the same velocities?

13. **Velocity of a particle** A particle of constant mass m moves along the x -axis. Its velocity v and position x satisfy the equation

$$\frac{1}{2}m(v^2 - v_0^2) = \frac{1}{2}k(x_0^2 - x^2),$$

where k , v_0 , and x_0 are constants. Show that whenever $v \neq 0$,

$$m \frac{dv}{dt} = -kx.$$

14. **Average and instantaneous velocity**

- Show that if the position x of a moving point is given by a quadratic function of t , $x = At^2 + Bt + C$, then the average velocity over any time interval $[t_1, t_2]$ is equal to the instantaneous velocity at the midpoint of the time interval.
 - What is the geometric significance of the result in part (a)?
15. Find all values of the constants m and b for which the function

$$y = \begin{cases} \sin x, & x < \pi \\ mx + b, & x \geq \pi \end{cases}$$

is

- continuous at $x = \pi$.
- differentiable at $x = \pi$.

16. Does the function

$$f(x) = \begin{cases} \frac{1 - \cos x}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

have a derivative at $x = 0$? Explain.

17. a. For what values of a and b will

$$f(x) = \begin{cases} ax, & x < 2 \\ ax^2 - bx + 3, & x \geq 2 \end{cases}$$

be differentiable for all values of x ?

- Discuss the geometry of the resulting graph of f .

18. a. For what values of a and b will

$$g(x) = \begin{cases} ax + b, & x \leq -1 \\ ax^3 + x + 2b, & x > -1 \end{cases}$$

be differentiable for all values of x ?

- Discuss the geometry of the resulting graph of g .

19. **Odd differentiable functions** Is there anything special about the derivative of an odd differentiable function of x ? Give reasons for your answer.

20. **Even differentiable functions** Is there anything special about the derivative of an even differentiable function of x ? Give reasons for your answer.

21. Suppose that the functions f and g are defined throughout an open interval containing the point x_0 , that f is differentiable at x_0 , that $f(x_0) = 0$, and that g is continuous at x_0 . Show that the product fg is differentiable at x_0 . This process shows, for example, that although $|x|$ is not differentiable at $x = 0$, the product $x|x|$ is differentiable at $x = 0$.

22. (Continuation of Exercise 21.) Use the result of Exercise 21 to show that the following functions are differentiable at $x = 0$.

a. $|x| \sin x$ b. $x^{2/3} \sin x$ c. $\sqrt[3]{x}(1 - \cos x)$

d. $h(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$

23. Is the derivative of

$$h(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

continuous at $x = 0$? How about the derivative of $k(x) = xh(x)$? Give reasons for your answers.

24. Suppose that a function f satisfies the following conditions for all real values of x and y :

i) $f(x + y) = f(x) \cdot f(y)$.

ii) $f(x) = 1 + xg(x)$, where $\lim_{x \rightarrow 0} g(x) = 1$.

Show that the derivative $f'(x)$ exists at every value of x and that $f'(x) = f(x)$.

25. **The generalized product rule** Use mathematical induction to prove that if $y = u_1 u_2 \cdots u_n$ is a finite product of differentiable functions, then y is differentiable on their common domain and

$$\frac{dy}{dx} = \frac{du_1}{dx} u_2 \cdots u_n + u_1 \frac{du_2}{dx} \cdots u_n + \cdots + u_1 u_2 \cdots u_{n-1} \frac{du_n}{dx}.$$

26. **Leibniz's rule for higher-order derivatives of products** Leibniz's rule for higher-order derivatives of products of differentiable functions says that

a. $\frac{d^2(uv)}{dx^2} = \frac{d^2u}{dx^2} v + 2 \frac{du}{dx} \frac{dv}{dx} + u \frac{d^2v}{dx^2}$.

b. $\frac{d^3(uv)}{dx^3} = \frac{d^3u}{dx^3} v + 3 \frac{d^2u}{dx^2} \frac{dv}{dx} + 3 \frac{du}{dx} \frac{d^2v}{dx^2} + u \frac{d^3v}{dx^3}$.

c. $\frac{d^n(uv)}{dx^n} = \frac{d^n u}{dx^n} v + n \frac{d^{n-1} u}{dx^{n-1}} \frac{dv}{dx} + \cdots$
 $+ \frac{n(n-1) \cdots (n-k+1)}{k!} \frac{d^{n-k} u}{dx^{n-k}} \frac{d^k v}{dx^k}$
 $+ \cdots + u \frac{d^n v}{dx^n}$.

The equations in parts (a) and (b) are special cases of the equation in part (c). Derive the equation in part (c) by mathematical induction, using

$$\binom{m}{k} + \binom{m}{k+1} = \frac{m!}{k!(m-k)!} + \frac{m!}{(k+1)!(m-k-1)!}.$$

27. **The period of a clock pendulum** The period T of a clock pendulum (time for one full swing and back) is given by the formula $T^2 = 4\pi^2 L/g$, where T is measured in seconds, $g = 32.2$ ft/sec², and L , the length of the pendulum, is measured in feet. Find approximately

- the length of a clock pendulum whose period is $T = 1$ sec.
- the change dT in T if the pendulum in part (a) is lengthened 0.01 ft.
- the amount the clock gains or loses in a day as a result of the period's changing by the amount dT found in part (b).

28. **The melting ice cube** Assume that an ice cube retains its cubical shape as it melts. If we call its edge length s , its volume is $V = s^3$ and its surface area is $6s^2$. We assume that V and s are differentiable functions of time t . We assume also that the cube's volume decreases at a rate that is proportional to its surface area. (This latter assumption seems reasonable enough when we think that the melting takes place at the surface: Changing the amount of surface changes the amount of ice exposed to melt.) In mathematical terms,

$$\frac{dV}{dt} = -k(6s^2), \quad k > 0.$$

The minus sign indicates that the volume is decreasing. We assume that the proportionality factor k is constant. (It probably depends on many things, such as the relative humidity of the surrounding air, the air temperature, and the incidence or absence of sunlight, to name only a few.) Assume a particular set of conditions in which the cube lost 1/4 of its volume during the first hour, and that the volume is V_0 when $t = 0$. How long will it take the ice cube to melt?

Chapter 3 Technology Application Projects

Mathematica/Maple Modules:

Convergence of Secant Slopes to the Derivative Function

You will visualize the secant line between successive points on a curve and observe what happens as the distance between them becomes small. The function, sample points, and secant lines are plotted on a single graph, while a second graph compares the slopes of the secant lines with the derivative function.

Derivatives, Slopes, Tangent Lines, and Making Movies

Parts I–III. You will visualize the derivative at a point, the linearization of a function, and the derivative of a function. You learn how to plot the function and selected tangents on the same graph.

Part IV (Plotting Many Tangents)

Part V (Making Movies). Parts IV and V of the module can be used to animate tangent lines as one moves along the graph of a function.

Convergence of Secant Slopes to the Derivative Function

You will visualize right-hand and left-hand derivatives.

Motion Along a Straight Line: Position \rightarrow Velocity \rightarrow Acceleration

Observe dramatic animated visualizations of the derivative relations among the position, velocity, and acceleration functions. Figures in the text can be animated.



4

APPLICATIONS OF DERIVATIVES

OVERVIEW In this chapter we use derivatives to find extreme values of functions, to determine and analyze the shapes of graphs, and to find numerically where a function equals zero. We also introduce the idea of recovering a function from its derivative. The key to many of these applications is the Mean Value Theorem, which paves the way to integral calculus in Chapter 5.

4.1

Extreme Values of Functions

This section shows how to locate and identify extreme (maximum or minimum) values of a function from its derivative. Once we can do this, we can solve a variety of problems in which we find the optimal (best) way to do something in a given situation (see Section 4.6). Finding maximum and minimum values is one of the most important applications of the derivative.

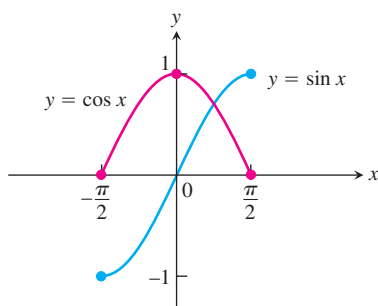


FIGURE 4.1 Absolute extrema for the sine and cosine functions on $[-\pi/2, \pi/2]$. These values can depend on the domain of a function.

DEFINITIONS Let f be a function with domain D . Then f has an **absolute maximum** value on D at a point c if

$$f(x) \leq f(c) \quad \text{for all } x \text{ in } D$$

and an **absolute minimum** value on D at c if

$$f(x) \geq f(c) \quad \text{for all } x \text{ in } D.$$

Maximum and minimum values are called **extreme values** of the function f . Absolute maxima or minima are also referred to as **global** maxima or minima.

For example, on the closed interval $[-\pi/2, \pi/2]$ the function $f(x) = \cos x$ takes on an absolute maximum value of 1 (once) and an absolute minimum value of 0 (twice). On the same interval, the function $g(x) = \sin x$ takes on a maximum value of 1 and a minimum value of -1 (Figure 4.1).

Functions with the same defining rule or formula can have different extrema (maximum or minimum values), depending on the domain. We see this in the following example.