Matrices in Numerical Methods

Lecture Structure:

Topic	Description
1. Introduction to Matrices	Definition of a matrix, basic operations, and their importance in numerical methods and engineering.
2. Solving Systems of Linear Equations Using Matrices	Explanation of how matrices are used to solve systems of linear equations and the advantages of this method.
3. Gaussian Elimination Method	Step-by-step explanation of how to use Gaussian elimination to solve linear systems represented by matrices.
4. Gauss-Jacobi Method	Explanation of the iterative method, its convergence criteria, and its efficiency for large systems. Includes an example with step-by-step calculations over multiple iterations.
5. Assignments	Two assignments to solve the system of equations using both Gaussian Elimination Method and Gauss-Jacobi Method.

1. Introduction to Matrices

A **matrix** is a rectangular array of numbers, symbols, or expressions arranged in rows and columns. Matrices are fundamental in numerical methods because they provide a structured way to represent and manipulate linear equations, which are core to solving many engineering problems. Matrices are used to:

- **Store and organize data**: Such as coefficients in a system of linear equations.
- **Perform transformations**: Such as rotations, scaling, or translations in 2D or 3D space.
- **Solve systems of equations**: Efficiently solve complex systems of equations that arise in engineering problems.

Matrix Notation

A matrix with m rows and n columns is called an $m \times n$ matrix. The elements of the matrix are typically denoted by a_{ij} where i represents the row and j represents the column. For example:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

2. Solving Systems of Linear Equations Using Matrices

One of the most common applications of matrices in numerical methods is solving systems of linear equations. In many engineering problems, we encounter equations of the form:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

which can be written as:

$$Ax = b$$

Where:

- *A* is the matrix of coefficients.
- *x* is the vector of unknowns.
- *b* is the vector of constants.

For example, consider the system of equations:

$$2x + 3y = 8$$

$$4x + y = 7$$

This can be written in matrix form as:

$$\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 7 \end{pmatrix}$$

We can solve this system of equations using matrix operations, such as **Gaussian elimination.**

3. Gaussian Elimination Method

The **Gaussian elimination method** is one of the most widely used techniques for solving systems of linear equations using matrices. It involves performing row operations to transform the matrix into an upper triangular form, from which the solution can be found using back substitution.

Steps in Gaussian Elimination

1. Form the augmented matrix from the system of equations.
2. Perform row operations to eliminate the variables below the main diagonal, transforming the matrix into an upper triangular matrix.
3. Back substitution: Once the system is in upper triangular form, solve for the unknowns starting from the last equation.

Example: Solve for x and y in the following system of equations:

$$2x + 3y = 8$$

$$4x + y = 7$$

1- Write the augmented matrix:

$$\begin{pmatrix} 2 & 3 & | & 8 \\ 4 & 1 & | & 7 \end{pmatrix}$$

2- Perform row operations to eliminate the *x*-term in the second row:

Subtract 2×(Row 1) from Row 2 to eliminate the *x*-term:

$$\begin{pmatrix} 2 & 3 & | & 8 \\ 0 & -5 & | & -9 \end{pmatrix}$$

3- Solve by back substitution:

- From the second row: $-5y = -9 \quad \Rightarrow \quad y = 1.8$
- Substitute $y = 1.8$ into the first row: $2x + 3(1.8) = 8 \quad \Rightarrow \quad 2x + 5.4 = 8 \quad \Rightarrow$ $x = 1.3$

Thus, the solution is $x = 1.3$ and $y = 1.8$.

4- Gauss-Jacobi Method

The **Gauss-Jacobi method** (often called just **Jacobi method**) is an iterative numerical method used to solve systems of linear equations. It is especially useful when dealing with large systems where direct methods like **Gaussian elimination** may be computationally expensive.

The method assumes an initial guess for the solution and iteratively improves that guess based on the equations in the system until it converges to the correct solution (or a sufficiently close approximation).

Jacobi Iterative Formula

Given the system:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

The Jacobi method solves each equation in the system for x_i:

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1, j \neq i}^{n} a_{ij} x_j^{(k)} \right)$$

Where:

- $x_i^{(k+1)}$ is the new value of x_i at iteration $k+1$.
- $x_j^{(k)}$ is the old value of x_j at iteration k.
- a_{ii} is the diagonal element of matrix A.

This process is repeated iteratively until the difference between the old and new values of x becomes sufficiently small (i.e., until convergence).

Steps for Gauss-Jacobi Method

1. Rearrange the system so that the diagonal elements of matrix *A* are dominant (this ensures faster convergence).
2. Make an initial guess for the solution vector *x*.
3. Iteratively update each x_i using the Jacobi formula.
4. Check for convergence by seeing if the difference between the new and old values of *x* is less than a predefined tolerance.
5. Repeat until convergence is achieved.

Example: Solve for *x*, *y*, and *z* in the following system of equations:

$$3x + y + z = 5$$

$$x + 4y + z = 6$$

$$x + y + 5z = 7$$

This can be written in matrix form:

$$\begin{pmatrix} 3 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}$$

We can rearrange the equations as:

1. $x^{(k+1)} = \frac{1}{3}(5 - y^{(k)} - z^{(k)})$
2. $y^{(k+1)} = \frac{1}{4}(6 - x^{(k)} - z^{(k)})$
3. $z^{(k+1)} = \frac{1}{5}(7 - x^{(k)} - y^{(k)})$

From the first equation $3x + y + z = 5$, solve for x:

$$x = \frac{1}{3}(5 - y - z)$$

From the second equation $x + 4y + z = 6$, solve for y:

$$y = \frac{1}{4}(6 - x - z)$$

From the third equation $x + y + 5z = 7$, solve for z:

$$z = \frac{1}{5}(7 - x - y)$$

Let's assume an initial guess for x, y, and z:

$$x^{(0)} = 0, \quad y^{(0)} = 0, \quad z^{(0)} = 0$$

Variable	Initial Guess (0)	Iteration 1	Iteration 2	Iteration 3	Iteration 4
x	$x=0$	$\frac{1}{3}(5-0-0) = 1.6667$	$\frac{1}{3}(5-1.5-1.4) = 0.7$	$\frac{1}{3}(5-0.7333-0.7666) = 1.1667$	$\frac{1}{3}(5-1.1333-1.1133) = 0.9178$
y	$y=0$	$\frac{1}{4}(6-0-0) = 1.5$	$\frac{1}{4}(6-1.6667-1.4) = 0.7333$	$\frac{1}{4}(6-0.7-0.7666) = 1.1333$	$\frac{1}{4}(6-1.1667-1.1133) = 0.93$
z	$z=0$	$\frac{1}{5}(7-0-0) = 1.4$	$\frac{1}{5}(7-1.6667-1.5) = 0.7666$	$\frac{1}{5}(7-0.7-0.7666) = 1.1133$	$\frac{1}{5}(7-1.1667-1.1333) = 0.94$

Thus, the Gauss-Jacobi method produces the following approximate solution after four iterations:

$$x \approx 0.9178, \quad y \approx 0.93, \quad z \approx 0.94$$

Assignment 1: Solve the following system of equations using Gaussian elimination:

$$3x + 2y + z = 10$$

$$2x + 3y + 3z = 18$$

$$x + 2y + 4z = 16$$

Assignment 2: Solve the following system of equations using both Gauss-Jacobi method:

$$10x + 2y + z = 12$$

$$2x + 8y - z = 15$$

$$-x + y + 5z = 10$$