Laplace Transformation

1-1 INTRODUCTION:

Laplace transforms help in solving the differential equations with boundary values without finding the general solution and the values of the arbitrary con Laplace Transform used for analog signals

LAPLACE TRANSFORM

Definition. Let $f(t)$ be function defined for all positive values of t, then.

$$F(s) = \int_0^{\infty} e^{-st} f(t)\, dt$$

The integral exists, is called the **Laplace Transform** of $f(t)$.It is denoted as

$$L[f(t)] = F(s) = \int_0^{\infty} e^{-st} f(t)\, dt$$

L is called the Laplace transform operator

where S is a complex variable S= $\delta + jw$

1.3 IMPORTANT FORMULAE

(1) $L(1) = \frac{1}{s}$

(2) $L(t^n) = \frac{n!}{s^{n+1}}$, when $n = 0, 1, 2, 3....$

(3) $L(e^{at}) = \frac{1}{s-a}$ .....s>0

(4) $L(\cosh at) = \frac{s}{s^2 - a^2}$

(5) $L(\sinh at) = \frac{a}{s^2 - a^2}$

(6) $L \sin(at) = \frac{a}{s^2 + a^2}$

(7) $L(\cos at) = \frac{s}{s^2 + a^2}$

1. $L(1) = \frac{1}{s}$

Proof. $L(1) = \int_0^\infty 1 \cdot e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = -\frac{1}{s}\left[\frac{1}{e^{st}}\right]_0^\infty = -\frac{1}{s}[0-1] = \frac{1}{s}$

Hence $L(1) = \frac{1}{s}$

2. $L(t^n) = \frac{n!}{s^{n+1}}$, where n and s are positive.

Ex. : f(t)=t

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty t \cdot e^{-st}\, dt$$

By using part method ($\int udv = uv - \int vdu$),

Let $u = t$ $dv = e^{-st}\, dt$

$du = dt$ $v = \frac{1}{-s} e^{-st}$

$$= t \cdot \frac{1}{-s}[e^{-st}]_0^\infty - \int_0^\infty \frac{1}{-s} e^{-st}\, dt = \left[\frac{-t}{s} \cdot e^{-st}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\, dt$$

$$= \frac{-t}{s}[e^{-st}]_0^\infty - \left[\frac{1}{s^2} \cdot e^{-st}\right]_0^\infty$$

So, $\mathcal{L}[t] = \frac{1}{s^2}$

Derivetive	Integral
(+) t	e^{-st}
(-) 1	$\frac{e^{-st}}{-s}$
(+) 0	$\frac{e^{-st}}{s^2}$

0 ≤ (variable t) < ∞

3. $L(e^{at}) = \dfrac{1}{s-a}$ where $s > a$

Proof. $L(e^{at}) = \int_0^\infty e^{-st} . e^{at}\, dt = \int_0^\infty e^{-st+at} . dt$

$$= \int_0^\infty e^{(-s+a)t} . dt = \int_0^\infty e^{-(s-a)t} . dt = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = -\frac{1}{s-a}\left[\frac{1}{e^{(s-a)t}}\right]_0^\infty$$

$$= \frac{-1}{(s-a)}(0-1) = \frac{1}{s-a}$$ **Proved**

$e^{ix} = \cos x + i\sin x$	$e^{-ix} = \cos x - i \sin x$
$\sin x = \frac{e^{ix}-e^{-ix}}{2i}$, $\cos x = \frac{e^{ix}+e^{-ix}}{2}$	$\sinh x = \frac{e^{x}-e^{-x}}{2}$, $\cosh x = \frac{e^{x}+e^{-x}}{2}$
sec(x)=1/(cos(x))	cosec x = 1/sin x.
cot x = (cos x) / (sin x).	tan x = sin x/cos x

4. $L(\cosh at) = \dfrac{s}{s^2-a^2}$

Proof. $L(\cosh at) = L\left[\frac{e^{at}+e^{-at}}{2}\right]$ $\left(\because \cosh at = \frac{e^{at}+e^{-at}}{2}\right)$

$$= \frac{1}{2}L(e^{at}) + \frac{1}{2}L(e^{-at})$$

$$= \frac{1}{2}\left[\frac{1}{s-a}+\frac{1}{s+a}\right]$$ $\left[L(e^{at}) = \frac{1}{s-a}\right]$

:.DAWAH.

$$= \frac{1}{2}\left[\frac{s+a+s-a}{s^2-a^2}\right] = \frac{s}{s^2-a^2}$$ **Proved.**

5. $L(\sinh at) = \dfrac{a}{s^2 - a^2}$

Proof. $L(\sinh at) = L\left[\dfrac{1}{2}(e^{at} - e^{-at})\right]$

$$= \frac{1}{2}[L(e^{at}) - L(e^{-at})] = \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{s+a-s+a}{s^2-a^2}\right]$$

$$= \frac{a}{s^2-a^2}$$ **Proved.**

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6. $L(\sin at) = \dfrac{a}{s^2 + a^2}$

Proof. $L(\sin at) = L\left[\dfrac{e^{iat} - e^{-iat}}{2i}\right]$ $\left[\because \sin at = \dfrac{e^{iat} - e^{-iat}}{2i}\right]$

$$= \frac{1}{2i}[L(e^{iat} - e^{-iat})] = \frac{1}{2i}[L(e^{iat}) - L(e^{-iat})]$$

$$= \frac{1}{2i}\left[\frac{1}{s-ia} - \frac{1}{s+ia}\right] = \frac{1}{2i}\frac{s+ia-s+ia}{s^2+a^2}$$

$$= \frac{1}{2i}\frac{2ia}{s^2+a^2} = \frac{a}{s^2+a^2}$$ **Proved**

7. $L(\cos at) = \dfrac{s}{s^2 + a^2}$

Proof. $L(\cos at) = L\left(\dfrac{e^{iat} + e^{-iat}}{2}\right)$ $\left[\because \cos at = \dfrac{e^{iat} + e^{-iat}}{2}\right]$

$$= \frac{1}{2}[L(e^{iat} + e^{-iat})] = \frac{1}{2}[L(e^{iat}) + L(e^{-iat})]$$

$$= \frac{1}{2}\left[\frac{1}{s-ia} + \frac{1}{s+ia}\right] = \frac{1}{2}\frac{s+ia+s-ia}{s^2+a^2}$$ AUC-DAWAH

$$= \frac{s}{s^2+a^2}$$ **Proved**

BY using direct method from important formula find Laplace transform of

1+cos2t SOL// $L|1+\cos 2t| = L|1| + L|\cos 2t| = \dfrac{1}{s} + \dfrac{s}{s^2+2^2} = \dfrac{1}{s} + \dfrac{s}{s^2+4}$

Example 1. *From the first principle, find the Laplace transform of (1 + cos 2 t).*

Solution. Laplace transform of $(1+\cos 2t)$

$$= \int_0^\infty e^{-st}(1+\cos 2t)\,dt = \int_0^\infty e^{-st}\left(1+\frac{e^{2it}+e^{-2it}}{2}\right)dt$$

$$= \frac{1}{2}\int_0^\infty \left[2e^{-st}+e^{(-s+2i)t}+e^{(-s-2i)t}\right]dt = \frac{1}{2}\left[\frac{2e^{-st}}{-s}+\frac{e^{(-s+2i)t}}{-s+2i}+\frac{e^{(-s-2i)t}}{-s-2i}\right]_0^\infty$$

$$= \frac{1}{2}\left[\left(0+\frac{2}{s}\right)+\frac{1}{-s+2i}(0-1)+\frac{1}{-s-2i}(0-1)\right]$$

$$= \frac{1}{2}\left[\frac{2}{s}+\frac{1}{s-2i}+\frac{1}{s+2i}\right] = \frac{1}{2}\left[\frac{2}{s}+\frac{2s}{s^2+4}\right]$$

Example 3:Find (L.T) for the following :

1-f(t)= K+ sin3t - e^{2t}

Solution ///

L|f(t)=L|K|+L| sin 3t| - L|e^{2t}|

$$F(s)=\frac{k}{s}+\frac{3}{s^2+3^2}-\frac{1}{s-2}$$

2- f(t) = cosh 4t +sinh 3t +12

Solution

$$F(s) = \frac{s}{s^2-4^2}+\frac{3}{s^2-3^2}+\frac{12}{s}$$

$$F(s) = \frac{s}{s^2-16}+\frac{3}{s^2-9}+\frac{12}{s}$$

Laplace Transform Table

$F(s) = L\{f(t)\}$	$f(t) = L^{-1}\{F(s)\}$
1	$\frac{1}{s},\ s > 0$
$t^n,\ n > 0$	$\frac{n!}{s^{n+1}},\ s > 0$
$t^n e^{at},\ n > 0$	$\frac{n!}{(s-a)^{n+1}},\ s > a$
e^{at}	$\frac{1}{s-a},\ s > a$
$sin(at)$	$\frac{a}{s^2+a^2},\ s > 0$
$tsin(at)$	$\frac{2as}{(s^2+a^2)^2},\ s > a$
$e^{at}sin(bt)$	$\frac{b}{(s-a)^2+b^2},\ s > a$
$cos(at)$	$\frac{s}{s^2+a^2},\ s > 0$
$tcos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2},\ s > a$
$e^{at}cos(bt)$	$\frac{s-a}{(s^2-a^2)^2+b},\ s > a$

PROPERTIES OF LAPLACE TRANSFORMS

1/ Linearity property:

If c_1 and c_2 any constants while $f_1(t)$ and $f_2(t)$ are functions with Laplace transformation then:

$$\mathcal{L}[c_1 f_1(t) + c_2 f_2(t)] = c_1\, \mathcal{L}[f_1(t)] + c_2\, \mathcal{L}[f_2(t)]$$

2. Scaling Theorem

$$L\{K f(t)\} = K F(s)$$ ... K is constant

3. Real Differentiation

Let F(s) be the Laplace transform of f(t). Then,

$$L\left\{\frac{d\,f(t)}{dt}\right\} = s\,F(s) - f(0^-)$$

where $f(0^-)$ indicates value of f(t) at $t = 0^-$ i.e. just before the instant $t = 0$

The theorem can be extended for n^{th} order derivative as,

$$L\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n F(s) - s^{n-1} f(0^-) - s^{n-2} f'(0^-) - \ldots - f^{(n-1)}(0^-)$$

where $f^{(n-1)}(0^-)$ is the value of $(n-1)^{th}$ derivative of f(t) at $t = 0^-$.

i.e for n = 2, $L\left\{\frac{d^2 f(t)}{dt^2}\right\} = s^2 F(s) - s\,f(0^-) - f'(0^-)$

for n = 3, $L\left\{\frac{d^3 f(t)}{dt^3}\right\} = s^3 F(s) - s^2 f(0^-) - sf'(0^-) - f''(0^-)$ and so on.

$$\mathbf{L\{y''''\} = s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y''(0)}$$

$$\mathcal{L}\left\{f^{(n)}\right\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

4. Real Integration

If F(s) is the Laplace transform of f(t) then,

$$L\left\{\int_0^t f(t)\,dt\right\} = \frac{F(s)}{s}$$

5/Multiplication by (t)

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

$$\mathcal{L}(\sin 3t) = \frac{3}{s^2+3^2}$$
$$\mathcal{L}(\sin 3t) = \frac{3}{s^2+9}$$

$$\mathcal{L}(t \sin 3t) = (-1)^1 \frac{d}{ds}\left[\frac{3}{s^2+9}\right]$$
$$\mathcal{L}(t \sin 3t) = -\left[\frac{-3(2s)}{(s^2+9)^2}\right]$$
$$\mathcal{L}(t \sin 3t) = -\frac{-6s}{(s^2+9)^2}$$
$$\mathcal{L}(t \sin 3t) = \frac{6s}{(s^2+9)^2} \qquad \textit{answer}$$

6. Complex Translation

$$F(s-a) = L\{e^{at} f(t)\}$$ and $$F(s+a) = L\{e^{-at} f(t)\}$$

$$F(s \mp a) = F(s)|_{s = s \mp a}$$

where F(s) is the Laplace transform of f(t).

With the help of this property, we can have the following important results :

(1) $L(e^{at}t^n) = \frac{n!}{(s-a)^{n+1}}$ $\left[L(t^n) = \frac{n!}{s^{n+1}} \right]$

(2) $L(e^{at}\cosh bt) = \frac{s-a}{(s-a)^2-b^2}$

(3) $L(e^{at}\sinh bt) = \frac{b}{(s-a)^2-b^2}$

(4) $L(e^{at}\sin bt) = \frac{b}{(s-a)^2+b^2}$

(5) $L(e^{at}\cos bt) = \frac{s-a}{(s-a)^2+b^2}$

7. Real Translation (Shifting Theorem)

This theorem is useful to obtain the Laplace transform of the shifted or delayed function of time.

If F(s) is the Laplace transform of f(t) then the Laplace transform of the function delayed by time T is,

$$L\{f(t-T)\} = e^{-Ts}F(s)$$

8. Initial Value Theorem

The Laplace transform is very useful to find the initial value of the time function f(t). Thus if F(s) is the Laplace transform of f(t) then,

$$f(0^+) = \lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)$$

9. Final Value Theorem

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

G(s)= $\int_0^\infty f(\tau)\ e^{-s(a+\tau)}\,dt \ = e^{-as}\int_0^\infty e^{-s\tau}.f(\tau)dt = e^{-as}F(s)$

Ex.10: The L.T of f(t- $\frac{\pi}{4}$) when f(t)=t sin 2t.

Sol. $\mathcal{L}[f(t)] = \mathcal{L}[t\sin 2t] = \frac{4s}{(s^2+4)^2}$

naw, . $\mathcal{L}\left[f(t-\frac{\pi}{4})\right] = e^{-\frac{\pi}{4s}} . \frac{4s}{(s^2+4)^2}$

Laplace of t sin at $= \frac{2as}{(s^2+a^2)^2}$

Example 3. *Find the Laplace transform of $\cos^2 t$.*

Solution.

$$\cos 2t = 2\cos^2 t - 1$$

$$\therefore \quad \cos^2 t = \frac{1}{2}[\cos 2t + 1]$$

$$L(\cos^2 t) = L\left[\frac{1}{2}(\cos 2t + 1)\right] = \frac{1}{2}[L(\cos 2t) + L(1)]$$

$$= \frac{1}{2}\left[\frac{s}{s^2 + (2)^2} + \frac{1}{s}\right] = \frac{1}{2}\left[\frac{s}{s^2+4} + \frac{1}{s}\right] \qquad \textbf{Ans.}$$

EXAMPLE

$$f(t) = te^{-t} \cos 4t$$

Let $g(t) = e^{-t} \cos 4t$

Then

$$G(s) = \mathscr{L}\{e^{-t} \cos 4t\}$$

$$= \frac{s+1}{(s+1)^2 + 16}$$

$$= \frac{s+1}{s^2 + 2s + 17}$$

$$\frac{d}{ds}\frac{s+1}{(s+1)^2 + 16} = -\frac{s^2 + 2s - 15}{(s^2 + 2s + 17)^2}$$

So $\mathscr{L}\{t \cdot e^{-t} \cdot \cos 4t\} = \dfrac{s^2 + 2s - 15}{(s^2 + 2s + 17)^2}$

INVERSE LAPLACE TRANSFORMS ILT

IMPORTANT FORMULAE

(1) $L^{-1}\left(\frac{1}{s}\right) = 1$

(2) $L^{-1}\frac{1}{s^n} = \frac{t^{n-1}}{(n-1)!}$

(3) $L^{-1}\frac{1}{s-a} = e^{at}$

(4) $L^{-1}\frac{s}{s^2-a^2} = \cosh at$

(5) $L^{-1}\frac{1}{s^2-a^2} = \frac{1}{a}\sinh at$

(6) $L^{-1}\frac{1}{s^2+a^2} = \frac{1}{a}\sin at$

(7) $L^{-1}\frac{s}{s^2+a^2} = \cos at$

(8) $L^{-1}F(s-a) = e^{at}f(t)$

(9) $L^{-1}\frac{1}{(s-a)^2+b^2} = \frac{1}{b}e^{at}\sin bt$

(10) $L^{-1}\frac{s-a}{(s-a)^2+b^2} = e^{at}\cos bt$

(11) $L^{-1}\frac{1}{(s-a)^2-b^2} = \frac{1}{b}e^{at}\sinh bt$

(12) $L^{-1}\frac{s-a}{(s-a)^2-b^2} = e^{at}\cosh bt$

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(13) $L^{-1}\frac{1}{(s^2+a^2)^2} = \frac{1}{2a^3}(\sin at - at\cos at)$

(14) $L^{-1}\frac{s}{(s^2+a^2)^2} = \frac{1}{2a}t\sin at$

(15) $L^{-1}\frac{s^2-a^2}{(s^2+a^2)^2} = t\cos at$

(16) $L^{-1}(1) = \delta(t)$

(17) $L^{-1}\frac{s^2}{(s^2+a^2)^2} = \frac{1}{2a}[\sin at + at\cos at]$

partial fractions.

case/1

$$\frac{3x^2-5x-52}{(x+2)(x-4)(x+5)} = \frac{A}{x+2} + \frac{B}{x-4} + \frac{C}{x+5}$$

$$\frac{1}{(x+1)(x+1)(x+2)} = \frac{1}{(x+1)^2(x+2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+2}$$

case/2

case/3

$$\frac{1}{(x^2-x+1)(x^2-x+2)}=\frac{Ax+B}{x^2-x+1}+\frac{Cx+D}{x^2-x+2}$$

case/4

$$\frac{1}{(x^2-x+1)^2(x^2-x+2)}=\frac{Ax+B}{x^2-x+1}+\frac{Cx+D}{(x^2-x+1)^2}+\frac{Ex+F}{x^2-x+2}$$

Ex. : If F(s) $=\frac{10s}{s^2+4}-\frac{3}{s^2+16}$. Find f(t).

Sol. F(s)=10.$\frac{s}{s^2+2^2}-3\frac{1}{s^2+4^2}$

$$\frac{1}{s^2+4^2}=\frac{1}{4}\cdot\frac{4}{s^2+4^2}$$

F(t)= $\mathcal{L}^{-1}[10.\frac{s}{s^2+2^2}-3.\frac{1}{4}\frac{4}{s^2+4^2}$ = 10 cos 2t - $\frac{3}{4}\sin 4t$

Ex. : If F(s) $=\frac{2}{s+2}$. Find f(t).

Sol. F(t)= $\mathcal{L}^{-1}[F(s)]=\mathcal{L}^{-1}\left[\frac{2}{s+2}\right]=2.e^{-2t}$

Example : $\mathcal{L}^{-1}\{\frac{3}{s}+\frac{4s+1}{s^2+16}\}$

$$= \mathcal{L}^{-1}\{\tfrac{3}{s}\} + \mathcal{L}^{-1}\{\tfrac{4s}{s^2+4^2}\} + \mathcal{L}^{-1}\{\tfrac{1}{s^2+4^2}\}$$

$$= 3\mathcal{L}^{-1}\{\tfrac{1}{s}\} + 4\mathcal{L}^{-1}\{\tfrac{s}{s^2+4^2}\} + \tfrac{1}{4}\mathcal{L}^{-1}\{\tfrac{4}{s^2+4^2}\}$$

$$= 3 + 4\cos 4t + \tfrac{1}{4}\sin 4t$$

<u>Ex.:</u> If $F(s) = \frac{1}{s^2-2s+5}$. Find f(t)

Sol. $F(s) = \frac{1}{s^2-2s+5} = \frac{1}{s^2-2s+1+4} = \frac{1}{(s-1)^2+2^2}$

$\therefore F(s) = \frac{1}{2} \cdot \frac{2}{(s-1)^2+2^2}$

$\mathcal{L}^{-1}[\frac{1}{2} \cdot \frac{2}{(s-1)^2+2^2}] = \frac{1}{2} e^t \sin 2t$

Example/ Find $\mathscr{L}^{-1}\left\{\frac{2}{s^2+6s+13}\right\}$

$ax^2 + bx + c = 0$

$\left[\frac{b}{2}\right]^2$

Solution

$$\frac{2}{s^2+6s+13} = \frac{2}{(s+3)^2+4} = \left[\frac{2}{s^2+2^2}\right]_{s\to s+3}$$

and, since $2/(s^2+2^2) = \mathscr{L}\{\sin 2t\}$, the shift theorem gives

$$\mathscr{L}^{-1}\left\{\frac{2}{s^2+6s+13}\right\} = e^{-3t}\sin 2t$$

LAPLACE TRANSFORM OF THE DERIVATIVE OF $f(t)$

$L[f'(t)] = s\,L[f(t)] - f(0)$ where $L[f(t)] = F(s)$.

Examples:

$$L[f''(t)] = s[s\,L[f(t)] - f(0)] - f'(0)$$

$$L[f''(t)] = s^2 L[f(t)] - s f(0) - f'(0)$$

$$\mathcal{L}\{y'''\} = s^3 Y(s) - s^2 y(0) - s y'(0) - y''(0)$$

$$L\{y''''\} = s^4 Y(s) - s^3 y(0) - s^2 y'(0) - s y''(0) - y''(0)$$

$$\mathcal{L}\left\{f^{(n)}\right\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

Example: $y'' - 6y' + 5y = 0,\ y(0) = 1,\ y'(0) = -3$

[Step 1] Transform both sides

$L\{y'' - 6y' + 5y\} = L\{0\}$

$s^2 L\{y\} - s\,y(0) - y'(0)) - 6(sL\{y\} - y(0)) + 5L\{y\} = 0$

[Step 2] Simplify to find $Y(s) = L\{y\}$

$s^2 L\{y\} - s -(-3)) - 6(s\,L\{y\} - 1) + 5L\{y\} = 0$

$(s^2 - 6s + 5)\,L\{y\} - s + 9 = 0$

$(s^2 - 6s + 5)\,L\{y\} = s - 9$

[Step 3] Find the inverse transform $y(t)$

Use partial fractions to simplify,

$Y(s) = \frac{s-9}{s^2-6s+5} = \frac{s-9}{(s-1)(s-5)} = \frac{A}{(s-1)} + \frac{B}{(s-5)}$ /*$(s-1)(s-5)$

s-9= A(s-5) +B(s-1)………….. Equating the corresponding coefficients:

$1 = A + B \quad \therefore \quad A = 2$

$-9 = -5A - B \quad \therefore \quad B = -1$

$Y(s) = \frac{2}{(s-1)} + \frac{-1}{(s-5)} \Rightarrow \quad y(t) = 2e^{t} - e^{5t}$ ***Answer***

Example: $y' + 2y = 4t\,e^{2t},\ y(0) = -3.$

$L\{y' + 2y\} = L\{4t\,e^{-2t}\}$

$(sL\{y\} - y(0)) + 2L\{y\} = L\{4t\,e^{-2t}\} = \frac{4}{(s-2)^2}$

$(s + 2)\,L\{y\} + 3 = \frac{4}{(s-2)^2}$

Y(s)(s+2) $=\frac{4}{(s-2)^2}$ -3

Y(s)= $\frac{4}{(s+2)(s-2)^2}$ - $\frac{3}{(s+2)}$ = $\frac{-3s^2+12s-8}{(s+2)(s-2)^2}$ = $\frac{A}{(s+2)}+\frac{B}{(s-2)}+\frac{c}{(s-2)^2}$ /* $(s+2)(s-2)^2$

Y(s) = $-3s^2+12s-8$ = A($(s-2)^2$ +B($s-2$) $(s+2)+C(s+2)$

Let s=0 , -8 = 4A-4B + 2C ....(1)

Let s= 2 , $-3(2)^2+12(2)-8=C\ (2+2)$

-12 + 24 – 8 =4C

4= 4C >>>>C=1` ....(2)

Let s = -2 , $-3(-2)^2+12(-2)-8$ = A($(-2-2)^2$

-12 -24 -8 = 16A

-44= 16A ∴ A= $\frac{-11}{4}$

Substituting in(1) ......B = $\frac{-1}{4}$

$$Y(S)=\frac{-11}{4(s+2)}+\frac{-1}{4(S-2)}+\frac{1}{(s-2)^2}$$

$$y(t)=\mathcal{L}^{-1}\left|\frac{-11}{4(s+2)}\right|+\mathcal{L}^{-1}\left|\frac{-1}{4(S-2)}\right|+\mathcal{L}^{-1}\left|\frac{1}{(s-2)^2}\right|$$

$$y(t)=\frac{-11}{4}e^{-2t}-\frac{1}{4}e^{2t}+te^{2t}$$

NOTE/// $\mathcal{L}^{-1}\left|\frac{s}{(s-a)^2}\right|=\mathcal{L}^{-1}\left|\frac{s-a+a}{(s-a)^2}\right|=\mathcal{L}^{-1}\left|\frac{s-a}{(s-a)^2}\right|+\mathcal{L}^{-1}\left|\frac{a}{(s-a)^2}\right|=[e^{at}+ate^{at}]$

Example. Using Laplace transforms, find the solution of the(IVP) initial value problem?

$$y''-y=e^{3t} \qquad , y(0)=0 , y'(0)=0$$

solution:

$$s^2Y(s)-sy(0)-y'(0)-Y(s)=\frac{1}{s-3}$$

$$s^2Y(s)-Y(s)=\frac{1}{s-3}$$

$$Y(s)[s^2-1]=\frac{1}{s-3}$$

Y(s)= $\frac{1}{(s-3)(s^2-1)}=\frac{A}{s-3}+\frac{BS+C}{s^2-1}$ / * $(s-3)(s^2-1)$

1=A(s^2-1) +($BS+C)(s-3)$

Let s=0 .....1=-A +(-3)C ......(1)

Let s= 3 .....1= 8A .....∴A= $\frac{1}{8}$ FROM (1) ...-3C= $1+\frac{1}{8}=\frac{9}{8}$ ∴ $C=\frac{-3}{8}$

1= As^2 – A +Bs^2-3BS+CS -3C

0=A+B ∴ $B=-A=\frac{-1}{8}$

Y(s) = $\frac{1}{8(s-3)}-\frac{1}{8}\frac{S}{(s^2-1)}-\frac{3}{8}\frac{1}{(s^2-1)}$

$\therefore$ y(t)= $\frac{1}{8}\, e^{3t} - \frac{1}{8}\cosh t\; - \frac{3}{8}\sinh t$

Example

Determine the form of a partial fraction expansion for the rational function

$$F(s) = \frac{2}{s^2+3s+2}.$$

where N(s)=2 and D(s)= s^2 +3s + 2. If the denominator is factored the function may be written

$$F(s) = \frac{2}{s^2+3s+2} = \frac{2}{(s+1)(s+2)}$$

and according to the rules above each of the two linear factors will introduce a single term into the partial fraction

$$F(s) = \frac{A_1}{s+1} + \frac{A_2}{s+2}$$

expansion

SOL//?????

UNIT STEP FUNCTION

$$u\,(t-a) = \begin{cases} 0 & \text{when } t < a \\ 1 & \text{when } t \geq a \end{cases}$$

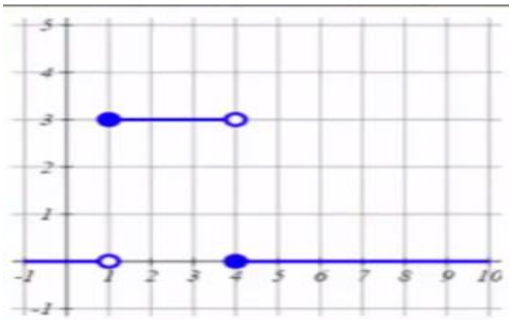
NOTE: Often the unit step function Uc(t) is also denoted as U(t − c), Hc(t), or H(t − c).

THE LAPLACE TRANSFORM OF UNIT STEP FUNCTION:

$$\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}, \qquad s > 0, \qquad c \geq 0$$

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$

Writing a Function Using the Unit Step Function



Heaviside Step Function

$$u_c(t) = u(t-c) = \begin{cases} 0 & if\ t < c \\ 1 & if\ t \geq c \end{cases}$$

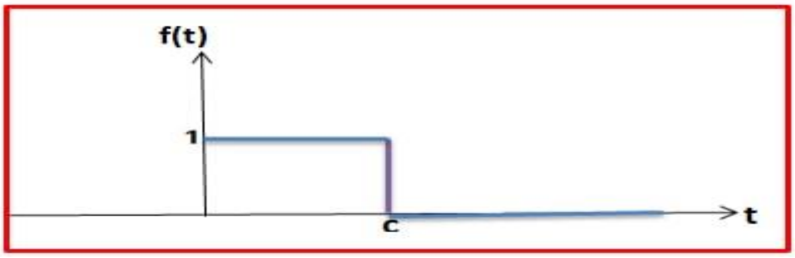
The graph above can be written as y=f(t) $U_{a(t)}$ + g(t) $U_{b(t)}$

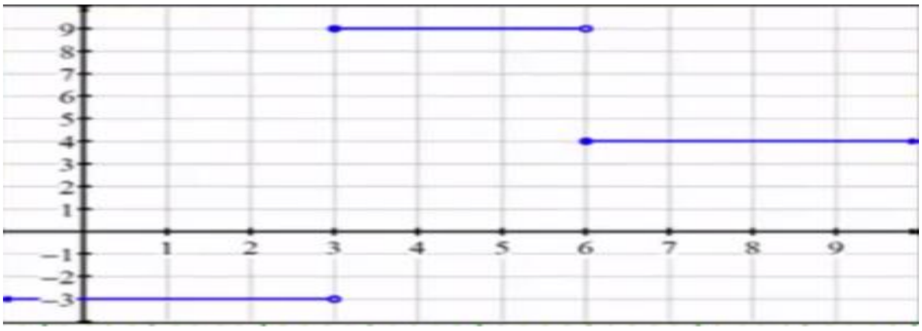
$$Y(t) = \begin{cases} 0 & if\ t < 1 \\ 3\ if & 1 \leq t < 4 \\ 0 & if\ t \geq 4 \end{cases}$$

$Y(t) = 3[U_{1(t)} - U_{4(t)}]$

EXAMPLE : Sketch the function f(t) = 1-u$_c$(t)?

$$f(t) = 1 - \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases} = \begin{cases} 1 & t < c \\ 0 & t \geq c \end{cases}$$



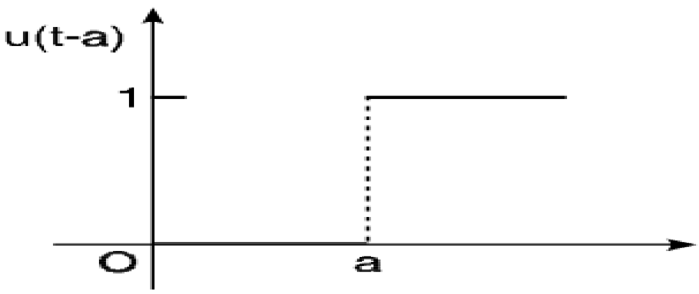


$$Write\ f(t) = \begin{cases} -3 & if\ t < 3 \\ 9 & if\ 3 \leq t < 6 \\ 4 & if\ t \geq 6 \end{cases}$$ in terms of the unit step function.

f(t) = -3[1-$U_{3(t)}$] + 9[$U_{3(t)} - U_{6(t)}$] + $4U_{6(t)}$

f(t)= -3 + $3U_{3(t)} + 9U_{3(t)} - 9U_{6(t)} + 4U_{6(t)}$

f(t) =-3 + $12U_{3(t)} - 5U_{6(t)}$



where $a \geq 0$.

More specifically, the representation of a function

$$g(t)=\begin{cases} g_1(t) & 0<t<t_1 \\ \vdots & \\ g_k(t) & t_{k-1}<t<t_k \end{cases}$$

Is

g(t)= g1(t)+[g2(t)− g1(t)] u(t−t1)+[g3(t)− g2(t)] u(t−t2)+...... +[gk(t)− gk−1(t)] u(t−tk−1).

Example . Express the given function using unit step functions.

$$g(t)=\begin{cases} 0 & 0<t<1 \\ 2 & 1<t<2 \\ 1 & 2<t<3 \\ 3 & 3<t \end{cases}.$$

Solution. We have $g_1(t)=0$, $g_2(t)=2$, $g_3(t)=1$, $g_4(t)=3$. Thus

0+(2-0)u(t-1)+(1-2)u(t-2)+(3-1)u(t-3)

$$g(t)=2\,u(t-1)-u(t-2)+2\,u(t-3).$$

Example . Express

$$g(t)=\begin{cases} 0 & 0<t<2 \\ t+1 & 2<t \end{cases}$$

using unit jump function.

Solution. We have $g(t)=(t+1)\,u(t-2).$

Example. **/Express the following function in terms of units step functions and find its Laplace transform**

$$f(t) = \begin{cases} 8, & t < 2 \\ 6, & t > 2 \end{cases}$$

$$f(t) = \begin{cases} 8+0 & t < 2 \\ 8-2 & t > 2 \end{cases}$$

$$= 8 + \begin{cases} 0 & t < 2 \\ -2 & t > 2 \end{cases} = 8 + (-2)\begin{cases} 0, & t < 2 \\ 1, & t > 2 \end{cases}$$

$$= 8 - 2\,u(t-2)$$

$$\text{L} f(t) = 8\,\text{L}(1) - 2\,\text{L}\,u(t-2) = \frac{8}{s} - 2\frac{e^{-2s}}{s}$$

Example/ Express the following function in terms of unit step function : and find its L.T

$$f(t) = \begin{cases} t-1 & 1 < t < 2 \\ 3-t & 2 < t < 3 \end{cases}$$

Solution. $f(t) = \begin{cases} t-1 & 1 < t < 2 \\ 3-t & 2 < t < 3 \end{cases}$

$$= (t-1)\,[u(t-1) - u(t-2)] + (3-t)\,[u(t-2) - u(t-3)]$$

$$= (t-1)\,u(t-1) - (t-1)\,u(t-2) + (3-t)\,u(t-2) + (t-3)\,u(t-3)$$

$$= (t-1)\,u(t-1) - 2\,(t-2)\,u(t-2) + (t-3)\,u(t-3)$$

$$\text{L} f(t) = \frac{e^{-s}}{s^2} - 2\frac{e^{-2s}}{s^2} + \frac{e^{-3s}}{s^2}$$ **Ans.**

Example *Find the Laplace Transform of* $t^2\,u(t-3)$.

Solution.

$$t^2 . u(t-3) = [(t-3)^2 + 6(t-3) + 9]\,u(t-3)$$

$$= (t-3)^2 . u(t-3) + 6(t-3) . u(t-3) + 9\,u(t-3)$$

$$\text{L}\,t^2 . u(t-3) = \text{L}\,(t-3)^2 . u(t-3) + 6\,\text{L}\,(t-3) . u(t-3) + 9\,\text{L}\,u(t-3)$$

$$= e^{-3s}\left[\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right]$$ **Ans.**

ANOTHER WAY FOR SOLUTION:

$$\mathrm{L}\ t^2 u(t-3) = e^{-3s}\mathrm{L}\,(t+3)^2 = e^{-3s}\mathrm{L}\,[t^2+6t+9]$$

$$= e^{3s}\left[\frac{2}{s^3}+\frac{6}{s^2}+\frac{9}{s}\right] \qquad \textbf{Ans.}$$

Example	Find $f(t)$ given that	$F(s) = \dfrac{s^2+12}{s(s+2)(s+3)}$

SOLUTION

$$\frac{s^2+12}{s(s+2)(s+3)} = \frac{A}{s}+\frac{B}{s+2}+\frac{C}{s+3}$$

$$A = sF(s)\,\Big|_{s=0} = \frac{s^2+12}{(s+2)(s+3)}\Bigg|_{s=0} = \frac{12}{(2)(3)} = 2$$

$$B = (s+2)F(s)\,\Big|_{s=-2} = \frac{s^2+12}{s(s+3)}\Bigg|_{s=-2} = \frac{4+12}{(-2)(1)} = -8$$

$$C = (s+3)F(s)\,\Big|_{s=-3} = \frac{s^2+12}{s(s+2)}\Bigg|_{s=-3} = \frac{9+12}{(-3)(-1)} = 7$$

Thus $A = 2$, $B = -8$, $C = 7$, and Eq. (15.9.1) becomes

$$F(s) = \frac{2}{s} - \frac{8}{s+2} + \frac{7}{s+3}$$

By finding the inverse transform of each term, we obtain

$$f(t) = 2u(t) - 8e^{-2t} + 7e^{-3t}, \qquad t \geq 0.$$

Example: **Express the following function in terms of unit step function : and find its Laplace transform?**

$y'' + y = u(t-3)$?**y(0)=0 ,,,y'(0)=1**

SOL/$s^2Y(s) - sy(0) - y'(0) + Y(s) = \dfrac{e^{-3s}}{s}$

$$s^2Y(s) - 1 + Y(s) = \frac{e^{-3s}}{s}$$

$$(s^2+1)Y(s) = \frac{e^{-3s}}{s} + 1$$

$$Y(s) = \frac{e^{-3s}}{s(s^2+1)} + \frac{1}{(s^2+1)}$$

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$$

=(1-cost) = (1-cos(t-3)

y(t)=u(t-3)[1-cos(t-3)]+sint

Example: Express the following function in terms of unit step function and find its Laplace transform?

$$\mathcal{L}^{-1}\left|\frac{e^{-2s} - 3e^{-4s}}{s+2}\right|$$

SOL//

$$\mathcal{L}^{-1}\left|\frac{e^{-2s}}{s+2}\right| - \mathcal{L}^{-1}\left|\frac{3e^{-4s}}{s+2}\right| = u(t-2)e^{-2(t-2)} - 3u(t-4)e^{-2(t-4)}$$

Example: Express the following function in terms of unit step function and find its Laplace transform?

$$\mathcal{L}^{-1}\left|\frac{3s-15}{2s^2-4s+10}\right|$$

SOL/ $\mathcal{L}^{-1}\left|\frac{3(s-5)}{2(s^2-2s+5)}\right|$

$$\frac{3}{2}\mathcal{L}^{-1}\left|\frac{(s-5)}{(s-1)^2+4)}\right|$$

$$\frac{3}{2}\mathcal{L}^{-1}\left|\frac{(s-1)}{(s-1)^2+4)}\right|+\frac{3}{2}\mathcal{L}^{-1}\left|\frac{-4}{(s-1)^2+4)}\right|$$

$$=\frac{3}{2}e^t cos2t - 3e^t sin2t$$

Delta function δ(t)

What is the Delta Function?

1. δ(x) = 0 for all x ≠ 0.

2. Sifting property: $\int_{-\infty}^{\infty} f(x)\delta(x-a)\,dx$ = f(a)

3. The delta function is used to model “instantaneous” energy transfers.

4. L { δ(t −a)} = e^{-as}

$$\mathcal{L}(\delta(t)) = \int_{0^-}^{\infty} \delta(t)e^{-st}\,dt = e^{0\cdot t} = 1.$$

$$\mathcal{L}(\delta(t-a)) = \int_{0^-}^{\infty} \delta(t-a)e^{-st}\,dt = e^{-sa}$$

$$\int_a^b f(t)\delta(t)\,dt = \begin{cases} f(0) & \text{if } (a,b) \text{ contains } 0 \\ 0 & \text{if } [a,b] \text{ does not contain } 0. \end{cases}$$

$$\int_{-5}^{5} 3\delta(t)\,dt = 3, \quad \int_{-5}^{-3} 3\delta(t)\,dt = 0, \quad \int_{0^-}^{0^+} 3\delta(t)\,dt = 3, \quad \int_{0^+}^{\infty} 3\delta(t)\,dt = 0.$$

Shifting by a

If we shift by a we have, $\int_{-\infty}^{\infty} f(t)\delta(t-a) = f(a)$. More generally:

$$\int_c^d f(t)\delta(t-a)\,dt = \begin{cases} f(a) & \text{if } (c,d) \text{ contains a} \\ 0 & \text{if } [c,d] \text{ does not contain a.} \end{cases}$$

Example . (Practice with δ.) Quickly cover up the answers on the left and try to evaluate each of the integrals on the right.

$$\int_{-1}^{3} \delta(t)2e^{4t^2}\,dt \quad = 2, \quad (\text{evaluate } 2e^{4t^2} \text{ at } t=0)$$

$$\int_{1}^{3} \delta(t)2e^{4t^2}\,dt \quad = 0, \quad (0 \text{ is not in } [1,3])$$

$$\int_{0-}^{3} \delta(t)2e^{4t^2}\,dt \quad = 2, \quad (\text{evaluate } 2e^{4t^2} \text{ at } t=0)$$

$$\int_{0-}^{\infty} \delta(t)2e^{-\tan^2(t^3)}\,dt \quad = 2, \quad (\text{evaluate } 2e^{-\tan^2(t^3)} \text{ at } t=0)$$

$$\int_{-1}^{3} \delta(t-2)2e^{4t^2}\,dt \quad = 2e^{16}, \quad (\text{evaluate } 2e^{2e^{4t^2}} \text{ at } t=2)$$

$$\int_{3}^{5} \delta(t-2)2e^{4t^2}\,dt \quad = 0, \quad (2 \text{ is not in } [3,5])$$

$$\int_{0-}^{3} \delta(t-2)2e^{4t^2}\,dt \quad = 2e^{16} \quad (\text{evaluate } 2e^{2e^{4t^2}} \text{ at } t=2),$$

Example Determine $\mathscr{L}^{-1}\left\{\frac{s^2}{s^2+4}\right\}$.

Solution Since

$$\frac{s^2}{s^2+4} = \frac{s^2+4-4}{s^2+4} = 1 - \frac{4}{s^2+4}$$

$$\mathscr{L}^{-1}\left\{\frac{s^2}{s^2+4}\right\} = \mathscr{L}^{-1}\{1\} - \mathscr{L}^{-1}\left\{\frac{4}{s^2+4}\right\}$$

$$\mathscr{L}^{-1}\left\{\frac{s^2}{s^2+4}\right\} = \delta(t) - 2\sin 2t$$

Two important properties of the delta function are

1. $\delta(t-a)=0$ for $t \neq a$,
2. $\int_{-\infty}^{\infty} \delta(t)=1.$

1–1 EVALUATION OF INTEGRALS

We can evaluate number of integrals having lower limit (0) and upper limit(∞) of Laplace transform.

Example:

$$Evaluate \int_0^{\infty} t\, e^{-3t} \sin t\, dt .$$

$$\int_0^{\infty} t\, e^{-3t} \sin t\, dt = \int_0^{\infty} t\, e^{-st} \sin t\, dt$$

$$= L\,(t \sin t) = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2}$$

WHEN S=3

$$= \frac{2 \times 3}{(3^2+1)^2} = \frac{6}{100} = \frac{3}{50}$$

SOLVE THE EXAMPLES IN SAME SOLUTION ABOV

$\int_0^{\infty} t\, e^{-4t} \sin t\, dt$	**Ans.** $\frac{8}{289}$	$\int_0^{\infty} t\, e^{-2t} \cos t\; dt$	**ans** $\frac{7}{25}$

Example : Find $\mathcal{L}^{-1}\{\frac{1}{s^2+s-2}\}$

Let's check the list of basic transforms . . . **Not found!**

Try **partial fraction decomposition:** $\frac{1}{s^2+s-2} = \frac{A}{s+2} + \frac{B}{s-1}$

$= \frac{As-A+Bs+2B}{(s-1)(s+2)} = \frac{(A+B)s-A+2B}{(s-1)(s+2)} = \frac{0s+1}{(s-1)(s+2)}$

Solving the system $A+B=0$, $-A+2B=1$ gives $A = \frac{-1}{3}$, $B = \frac{1}{3}$.

Now we can find $\mathcal{L}^{-1}\{\frac{1}{s^2+s-2}\}$ using linearity and basic formula

$\mathcal{L}^{-1}\{\frac{1}{s^2+s-2}\} = -\frac{1}{3}\mathcal{L}^{-1}\{\frac{1}{s+2}\} + \frac{1}{3}\mathcal{L}^{-1}\{\frac{1}{s-1}\}$

$-\frac{1}{3}e^{-2t} + \frac{1}{3}e^{t}$.

Example : Find $\mathcal{L}^{-1}\{\frac{2}{(s-5)^3}\}$

Partial fraction decomposition? Nothing further to decompose.

Does the function appear on the list? No. But:

Two similar ones do: $\mathcal{L}^{-1}\{\frac{n!}{s^{n+1}}\} = t^n$, $\mathcal{L}^{-1}\{\frac{1}{s-a}\} = e^{at}$.

$\mathcal{L}^{-1}\{\frac{2}{(s-5)^3}\} = e^{5t}\mathcal{L}^{-1}\{\frac{2}{s^3}\} = t^2e^{5t}$.

Example : Find $\mathcal{L}^{-1}\{\frac{s}{s^2+2s+5}\}$

Partial fractions? The denominator is indecomposable.

Does the function appear on the list? Not quite.

$\frac{s}{s^2+2s+5} = \frac{s}{(s+1)^2+2^2}$ This is similar to: $\frac{s}{s^2+2^2} = \mathcal{L}\{\cos 2t\}$.

Here we use partial fraction

$$\frac{s+1}{(s-2)(s+2)} = \frac{A}{s-2} + \frac{B}{s+2}, \qquad A = 3/4, \quad B = 1/4.$$

PARTIAL FRACTIONS METHOD

Example Find the inverse Laplace transform of

$$F(s) = \frac{11s+7}{s^2-1} = \frac{11s+7}{(s-1)(s+1)} = \frac{A}{s-1} + \frac{B}{s+1}$$

Then A is found by substituting $s = 1$ into

$$\frac{11s+7}{\cancel{(s-1)}(s+1)} = \frac{11+7}{2} = 9$$

and B is found by substituting $s = -1$ into

$$\frac{11s+7}{(s-1)\cancel{(s+1)}} = \frac{-11+7}{-2} = 2.$$

This gives the partial fraction expansion, as before, as

$$F(s) = \frac{9}{s+1} + \frac{2}{s-1}.$$

Example *Find the inverse transforms of* $\frac{1}{s^2-5s+6}$.

Solution. Let us convert the given function into partial fractions.

$$L^{-1}\left[\frac{1}{s^2-5s+6}\right] = L^{-1}\left[\frac{1}{s-3} - \frac{1}{s-2}\right]$$

$$= L^{-1}\left(\frac{1}{s-3}\right) - L^{-1}\left(\frac{1}{s-2}\right) = e^{3t} - e^{2t}$$

1. INVERSE LAPLACE TRANSFORM BY CONVOLUTION

$$L\left\{\int_0^t f_1(x) * f_2(t-x)\,dx\right\} = F_1\cdot(s)\cdot F_2(s) \quad \text{or} \quad \int_0^t f_1(x)\cdot f_2(t-x)\,dx = L^{-1}\, F_1(s)\cdot F_2(s)$$

Example (*Convolution*) Find

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-2)(s-3)}\right\}$$

using the convolution property:

$$\mathcal{L}\left\{\int_0^1 f(t)g(t-\tau)\mathrm{d}\tau\right\} = F(s)G(s).$$

Solution

$$\frac{1}{(s-2)(s-3)} = \frac{1}{(s-2)}\frac{1}{(s-3)}.$$

Therefore, call

$$F(s) = \frac{1}{s-2}, \qquad G(s) = \frac{1}{s-3}$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} = \mathrm{e}^{2t}$$

$$g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} = \mathrm{e}^{3t}.$$

Then by the convolution rule

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-2)(s-3)}\right\} = \int_0^t e^{2\tau} e^{3(t-\tau)}\, d\tau .$$

This integral is an integral over the variable τ. t is a constant as far as the integration process is concerned. We can use the properties of powers to separate out the terms in τ and the terms in t, giving

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-2)(s-3)}\right\} = e^{3t}\int_0^t e^{-\tau}\, d\tau = e^{3t}\left[\frac{e^{-\tau}}{-1}\right]_0^t$$
$$= e^{3t}(-e^{-t}+1)$$

INVERSION FORMULA FOR THE LAPLACE TRANSFORM

$f(x)$ = sum of the residues of $e^{sx} F(s)$ at the poles of $F(s)$.

Example: Obtain the inverse Laplace transform of $\frac{s+1}{s^2+2s}$

Solution: Let $F(s)=\frac{s+1}{s^2+2s}$ ....(1)

$$L^{-1}\left[\frac{s+1}{s^2+2s}\right] = \text{Sum of the residues of } e^{st}\cdot\frac{s+1}{s^2+2s} \text{ at the poles.} \quad ...(2)$$

The poles of (1) are determined by equating the denominator to zero, *i.e.*

$$s^2+2s=0 \quad \text{or} \quad s(s+2)=0 \qquad i.e.\ s=0,-2$$

There are two simple poles at $s=0$ and $s=-2$.

$$\text{Residue of } e^{st}\cdot F(s)\ (\text{at } s=0) = \lim_{s\to 0}\left[(s-0)\frac{e^{st}\cdot(s+1)}{s^2+2s}\right] = \lim_{s\to 0}\left[\frac{e^{st}(s+1)}{(s+2)}\right] = \frac{1}{2}$$

$$\text{Residue of } e^{st}\cdot F(s)\ (\text{at } s=-2) = \lim_{s\to -2}\left[\frac{(s+2)\,e^{st}(s+1)}{s(s+2)}\right]$$
$$= \lim_{s\to -2}\left[\frac{e^{st}(s+1)}{s}\right] = \frac{e^{-2t}(-2+1)}{-2} = \frac{e^{-2t}}{2}$$

$$\text{Sum of the residue } [\text{at } s=0 \text{ and } s=-2] = \frac{1}{2}+\frac{e^{-2t}}{2}$$

Putting the value of residues in (2) we get

$$L^{-1}\left[\frac{s+1}{s^2+2s}\right]=\frac{1}{2}+\frac{e^{-2t}}{2}$$

HEAVISIDE S Inverse Formula of $\frac{F(s)}{G(s)}$

If $F(s)$and $G(s)$ be two polynomials in (s). The degree of $F(s)$ is less than that of

$G(s)$. Let $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n$ be n roots of the equation $G(s)=0$

Inverse Laplace formula of $\frac{F(s)}{G(s)}$ is given by

$$L^{-1}\left\{\frac{F(s)}{G(s)}\right\}=\sum_{i=1}^{n}\frac{F(\alpha_i)}{G'(\alpha_i)}e^{\alpha_i t}$$

Example *Find* $L^{-1}\left\{\frac{2s^2+5s-4}{s^3+s^2-2s}\right\}$.

Solution. Let $F(s) = 2s^2+5s-4$

and $G(s) = s^3+s^2-2s = s(s^2+s-2) = s(s+2)(s-1)$

$G'(s) = 3s^2+2s-2$

$G(s) = 0$ has three roots, 0, 1, –2.

or $\alpha_1 = 0,\ \alpha_2 = 1,\ \alpha_3 = -2$

By Heaviside's Inverse formula $L^{-1}\left\{\frac{F(s)}{G(s)}\right\}=\sum_{i=1}^{n}\frac{F(\alpha_1)}{G'(\alpha_i)}e^{\alpha_i t}$

$$=\frac{F(\alpha_1)}{G'(\alpha_1)}e^{t\alpha_1}+\frac{F(\alpha_2)}{G'(\alpha_2)}e^{t\alpha_2}+\frac{F(\alpha_3)}{G'(\alpha_3)}e^{t\alpha_3}=\frac{F(0)}{G'(0)}e^{0}+\frac{F(1)}{G'(1)}e^{t}+\frac{F(-2)}{G'(-2)}e^{-2t}$$

$$=\frac{-4}{-2}e^{0}+\frac{3}{3}e^{t}+\frac{(-6)}{(6)}e^{-2t}=2+e^{t}-e^{-2t}$$

Using Heaviside's expansion formula, find the inverse Laplace transform of the following:

1. $\dfrac{s-1}{s^2+3s+2}$ **Ans.** $-2e^{-t}+3e^{-2t}$

2. $\dfrac{s}{(s-1)(s-2)\ (s-3)}$ **Ans.** $\frac{1}{2}e^{t}-2\,e^{2t}+\frac{3}{2}e^{3t}$

Example

A spring-mass system with mass (2), damping (4), and spring constant (10) is subject to a hammer blow at time (t = 0): The blow imparts a total impulse of(1) to the system, which was initially at rest. Find the response of the system.

Solution: The situation is modeled by the initial value problem

$$2y'' + 4y' + 10y = \delta(t), \quad y(0) = 0, \quad y'(0) = 0.$$

Taking Laplace transform of both sides we find

$$2s^2Y(s) + 4sY(s) + 10Y(s) = 1.$$

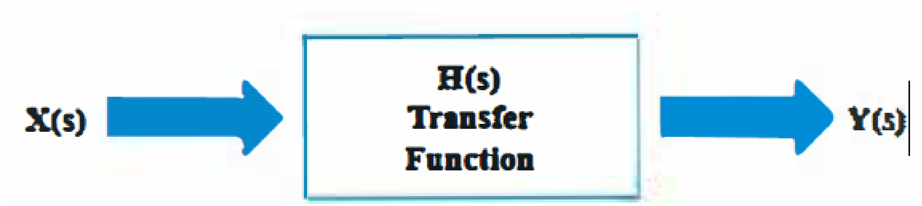
Solving for $Y(s)$ we find

$$Y(s) = \frac{1}{2s^2 + 4s + 10}.$$

The impulsive response is

$$y(t) = \mathcal{L}^{-1}\left(\frac{1}{2}\frac{1}{(s+1)^2 + 2^2}\right) = \frac{1}{4}e^{-2t}\sin 2t$$

The **Laplace transform** is designed to analyze a specific class of time domain **signals impulse responses** consisting of sinusoids and exponentials



$$Y(s) = H(s)X(s)$$

EXAMPLE **Find the transfer function and impulse response of the system described by the following differential equation:**

$$3\frac{dy}{dt} + 4y = f(t).$$

Solution **To find the transfer function replace *f (t)* by *δ(t)* and take the Laplace transform of the resulting equation assuming zero initial conditions:**

$$3\frac{dy}{dt} + 4y = \delta(t).$$

Taking the Laplace transform of both sides of the equation we get

$$3(sY - y(0)) + 4Y = 1$$

As $y(0) = 0$,

$$Y = \frac{1}{3s + 4} = H(s).$$

To find the impulse response function we take the inverse transform of the transfer function to find

$$h(t) = \mathcal{L}^{-1}\left\{\frac{1}{3s+4}\right\} = e^{\frac{-4t}{3}}.$$

EXAMPLE The impulse response of a system is known to be $h(t)= e^{3t}$
Find the response of the system to an input of $f(t) = 6\cos(2t)$ given zero initial conditions.
Solution Method 1. We can take Laplace transforms and use

Y(s) = $H(s)F(s)$. In this case

$$h(t) = e^{3t} \quad \Leftrightarrow \quad H(s) = \mathcal{L}\{e^{3t}\} = \frac{1}{s-3}$$

$$f(t) = 6\cos(2t) \quad \Leftrightarrow \quad F(s) = \mathcal{L}\{6\cos(2t)\} = \frac{6s}{4+s^2}$$

Hence

$$Y(s) = H(s)F(s) = \frac{6s}{(s-3)(4+s^2)}.$$

As we want to find *y(t)*, we use partial fractions:

$$\frac{6s}{(s-3)(4+s^2)} = \frac{6s}{(s-3)(s+j2)(s-j2)}$$

$$= \frac{18}{13(s-3)} + \frac{3}{(j2-3)(s-j2)}$$

$$- \frac{3}{(j2+3)(s+j2)} \quad \text{(using the `cover up' rule)}$$

$$= \frac{18}{13(s-3)} - \frac{3(6s-8)}{13(s^2+4)}$$

$$= \frac{18}{13(s-3)} - \frac{18s}{13(s^2+4)} + \frac{12}{13}\frac{2}{(s^2+4)}.$$

We can now take the inverse transform to find the system response:

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$= \mathcal{L}^{-1}\left\{\frac{18}{13(s-3)} - \frac{18s}{13(s^2+4)} + \frac{12}{13}\frac{2}{(s^2+4)}\right\}$$

$$= \frac{18}{13}e^{3t} - \frac{18}{13}\cos(2t) + \frac{12}{13}\sin(2t).$$

Alternative method. Find y(t) by taking the convolution of f (t) with the impulse response function

$$y(t) = f(t) * h(t) = (6\cos(2t)) * (e^{3t})$$

By definition of convolution

$$(6\cos(2t)) * (e^{3t}) = \int_0^t 6\cos(2\tau)\, e^{3(t-\tau)}\, d\tau.$$

As this is a real integral we can use the trick of writing $\cos(2\tau) = \mathrm{Re}(e^{j2\tau})$ to make the integration easier. So we find

$$\begin{aligned} I &= \int_0^t 6e^{j2\tau} e^{3(t-\tau)}\, d\tau \\ &= 6\,e^{3t} \int_0^t e^{\tau(j2-3)}\, d\tau \\ &= 6\,e^{3t} \left[\frac{e^{\tau(j2-3)}}{j2-3}\right]_0^t \\ &= 6\,e^{3t} \left(\frac{e^{t(j2-3)}}{j2-3} - \frac{1}{j2-3}\right) \end{aligned}$$

$$= \frac{6\,e^{3t}(-j2-3)(e^{-3t}(\cos(2t) + j\sin(2t)) - 1)}{4+9}.$$

Taking the real part of this result we get the system response as

$$\begin{aligned} \int_0^t 6\cos(2\tau)e^{3(t-\tau)}\, d\tau &= \frac{6}{13}(-3\cos(2t) + 2\sin(2t)) + \frac{18}{13}e^{3t} \\ &= -\frac{18}{13}\cos(2t) + \frac{12}{13}\sin(2t) + \frac{18}{13}e^{3t} \end{aligned}$$

which confirms the result of the first method.

Example A system transfer function is known to be

$$H(s) = \frac{1}{3s+1}$$

then find the steady state response to the following:

(a) $f(t) = e^{j2t}$;

(b) $f(t) = 3\cos(2t)$.

Solution (a) The steady state response to a single frequency $e^{j\omega t}$ is given $H(j\omega)e^{j\omega t}$. Here $f(t) = e^{j2t}$, so in this case $\omega = 2$ and $H(s)$ is given as $1/(3s + 1)$. Hence we get the steady state response as

$$H(j2)e^{j2t} = \frac{1}{3(j2)+1}e^{j2t} = \frac{e^{j2t}}{1+j6} = \frac{(1-j6)e^{j2t}}{37}$$

(b) Using $(1/2)(H(j\omega)e^{j\omega t}+H(-j\omega)e^{-j\omega t})$

as the response to $\cos(\omega t)$ and substituting for H and $\omega = 2$ gives

$$\frac{1}{2}\left(\frac{(1-6j)e^{j2t}}{37} + \frac{(1+j6)}{37}e^{-j2t}\right)$$
$$= \frac{1}{74}((1-j6)(\cos(2t)+j\sin(2t))$$
$$+ (1+j6)(\cos(2t)-j\sin(2t))$$
$$= \frac{1}{37}(\cos(2t)+6\sin(2t)).$$

Example

Find the **impulse response** of a system with a transfer function

$$H(s) = \frac{2}{(s+1)(s+2)}$$

Solution: The impulse response is the inverse Laplace transform of the transfer function H(s):

$$\begin{aligned} h(t) &= \mathcal{L}^{-1}\{H(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{2}{(s+1)(s+2)}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{2}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{2}{s+2}\right\} \\ &= 2e^{-t} - 2e^{-2t} \end{aligned}$$

APPLICATION TO CIRCUITS

EXAMPLE. A RESISTANCE $\mathbb{R}$ IN SERIES WITH INDUCTANCE $\mathbb{L}$ IS CONNECTED WITH E.M.F. $\mathbb{E}$ (t). THE CURRENT i IS GIVEN BY

$$L\frac{di}{dt} + Ri = E(t).$$

If the switch is connected at t = 0 and disconnected at t = a, find the current i in terms of t.

Solution. Conditions under which current i flows are $i = 0$ at $t = 0$,

$$E(t) = \begin{cases} E, & 0 < t < a \\ 0, & t > a \end{cases}$$

Given equation is $\quad L\frac{di}{dt} + Ri = E(t)$ ....(1)

Taking Laplace transform of (1), we get

$$L[s\bar{i} - i(0)] + R\bar{i} = \int_0^\infty e^{-st} E(t)\, dt$$

Note: Instead of $\bar{i}$ we can use **I(s)**

$$L s\bar{i} + R\bar{i} = \int_0^\infty e^{-st} E(t)\, dt$$

$[i(0) = 0]$

$$(Ls+R)\,\bar{i} = \int_0^\infty e^{-st}\,.\,E\,dt = \int_0^a e^{-st}E\,dt + \int_a^\infty e^{-st}E\,dt$$

$$= E\left[\frac{e^{-st}}{-s}\right]_0^a + 0 = \frac{E}{s}[1-e^{-as}] = \frac{E}{s} - \frac{E}{s}e^{-as}$$

$$\bar{i} = \frac{E}{s\,(Ls+R)} - \frac{Ee^{-as}}{s\,(Ls+R)}$$

$$i = L^{-1}\left[\frac{E}{s\,(Ls+R)}\right] - L^{-1}\left[\frac{E\,e^{-as}}{s\,(Ls+R)}\right] \qquad ...(2)$$

$$L^{-1}\left[\frac{E}{s\,(Ls+R)}\right] = \frac{E}{L}L^{-1}\left[\frac{1}{s\left(s+\frac{R}{L}\right)}\right] \quad \text{(Resolving into partial fractions)}$$

$$= \frac{E}{L}\frac{L}{R}L^{-1}\left|\frac{1}{s} - \frac{1}{s+\frac{R}{L}}\right| = \frac{E}{R}\left[1-e^{-\frac{R}{L}t}\right]$$

Now we have to find the value of $L^{-1}\left|\frac{E}{s\,(Ls+R)}\right|$

$$L^{-1}\left[\frac{E\,e^{-as}}{s\,(Ls+R)}\right] = \frac{E}{R}\left[1-e^{-\frac{R}{L}(t-a)}\right]u\,(t-a)$$

[By the second shifting theorem] On substituting the values of the inverse transforms in (2) we get

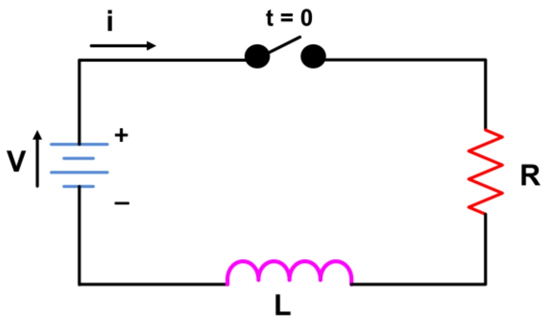
$$i = \frac{E}{R}\left[1 - e^{-\frac{R}{L}t}\right] - \frac{E}{R}\left[1 - e^{-\frac{R}{L}(t-a)}\right] u(t-a)$$

$$i = \frac{E}{R}\left[1 - e^{-\frac{R}{L}t}\right] \quad \text{for } 0 < t < a, \quad [u(t-a) = 0]$$

$$i = \frac{E}{R}\left[1 - e^{-\frac{R}{L}t}\right] - \frac{E}{R}\left\{1 - e^{-\frac{R}{L}(t-a)}\right\} \qquad \text{for } t > a \quad [u(t-a) = 1]$$

$$= \frac{E}{R}\left[e^{-\frac{R}{L}(t-a)} - e^{-\frac{R}{L}t}\right] = \frac{E}{R} e^{-\frac{R}{L}t}\left[e^{\frac{Ra}{L}} - 1\right] \qquad \textbf{Ans.}$$

EXAMPLE. A RESISTANCE R IN SERIES WITH INDUCTANCE L IS CONNECTED WITH E.M.F. E (t). THE CURRENT i IS GIVEN BY



$$L\frac{di}{dt} + Ri = E(t).$$

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Note: Instead of

$\bar{i}$ we can use **I(s)**