Engineering Analysis

Third Stage – Civil Engineering Department

2024 – 2025

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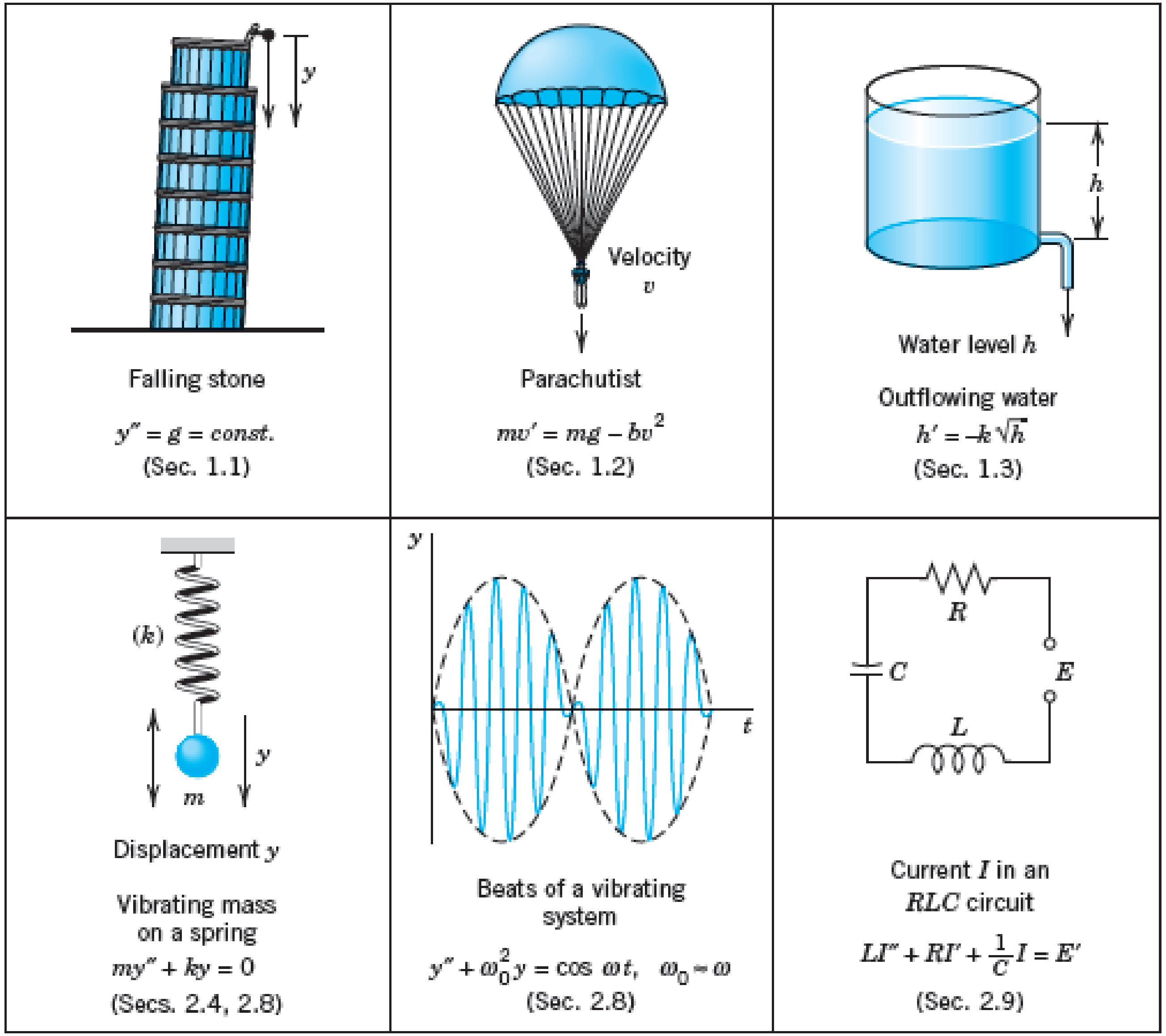
Assessment / Grading

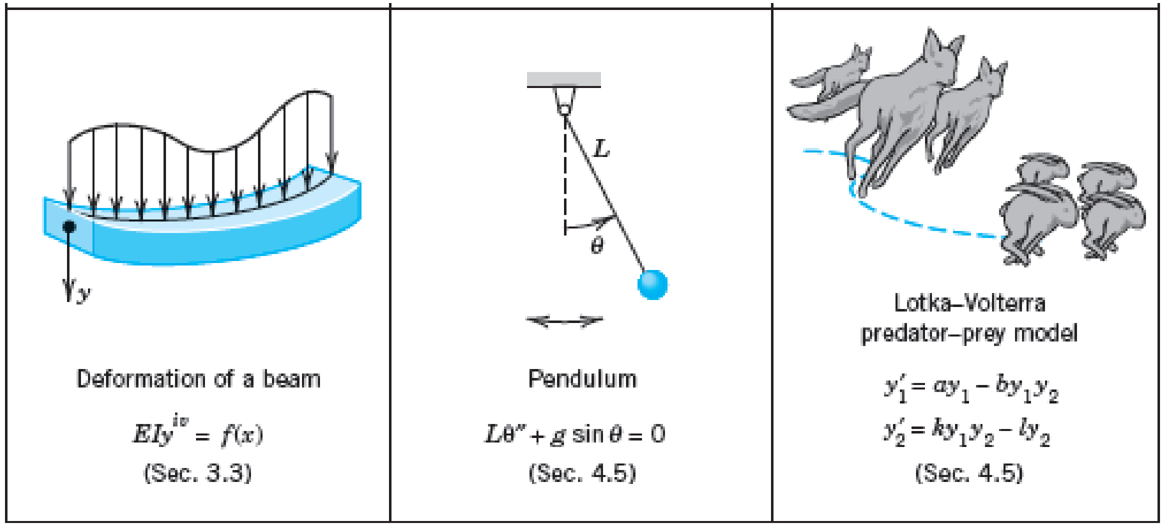
- *Homework*
- *Exams*

Part-1 Ordinary Differential Eqautions

Review
First-Order ODEs

- Understanding the basics of ODEs requires solving problems by hand.
- The process of setting up a model, solving it mathematically, and interpreting the result in physical or other terms is called *mathematical modeling* or, briefly, **modeling**.
- a model is very often an equation containing derivatives of an unknown function. Such a model is called a **differential equation**.





Some applications of differential equations

An **ordinary differential equation (ODE)** is an equation that contains one or several derivatives of an unknown function, which we usually call $y(x)$ (or sometimes $y(t)$ if the independent variable is time t). The equation may also contain y itself, known functions of x (or t), and constants. For example,

(1) $$y' = \cos x$$

(2) $$y'' + 9y = e^{-2x}$$

(3) $$y'y''' - \tfrac{3}{2}y'^2 = 0$$

are ordinary differential equations (ODEs). Here, as in calculus, y' denotes dy/dx, $y'' = d^2y/dx^2$, etc. The term *ordinary* distinguishes them from *partial differential equations* (PDEs), which involve partial derivatives of an unknown function of *two or more* variables. For instance, a PDE with unknown function u of two variables x and y is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

$$F(x, y, y') = 0 \qquad (4)$$

Concept of Solution

A function

$$y = h(x)$$

is called a **solution** of a given ODE (4) on some open interval $a < x < b$ if $h(x)$ is defined and differentiable throughout the interval and is such that the equation becomes an identity if y and y' are replaced with h and h', respectively. The curve (the graph) of h is called a **solution curve**.

Here, **open interval** $a < x < b$ means that the endpoints a and b are not regarded as points belonging to the interval. Also, $a < x < b$ includes *infinite intervals* $-\infty < x < b$, $a < x < \infty$, $-\infty < x < \infty$ (the real line) as special cases.

Geometric Meaning of $y' = f(x, y)$ Direction Fields, Euler's Method

A first-order ODE

$$y' = f(x, y) \tag{1}$$

has a simple geometric interpretation. From calculus you know that the derivative $y'(x)$ of $y(x)$ is the slope of $y(x)$. Hence a solution curve of (1) that passes through a point (x_0, y_0) must have, at that point, the slope $y'(x_0)$ equal to the value of f at that point; that is,

$$y'(x_0) = f(x_0, y_0).$$

Using this fact, we can develop graphic or numeric methods for obtaining approximate solutions of ODEs (1). This will lead to a better conceptual understanding of an ODE (1). Moreover, such methods are of practical importance since many ODEs have complicated solution formulas or no solution formulas at all, whereby numeric methods are needed.

Figure 7 shows a direction field for the ODE

$$y' = y + x \tag{2}$$

obtained by a CAS (Computer Algebra System) and some approximate solution curves fitted in.

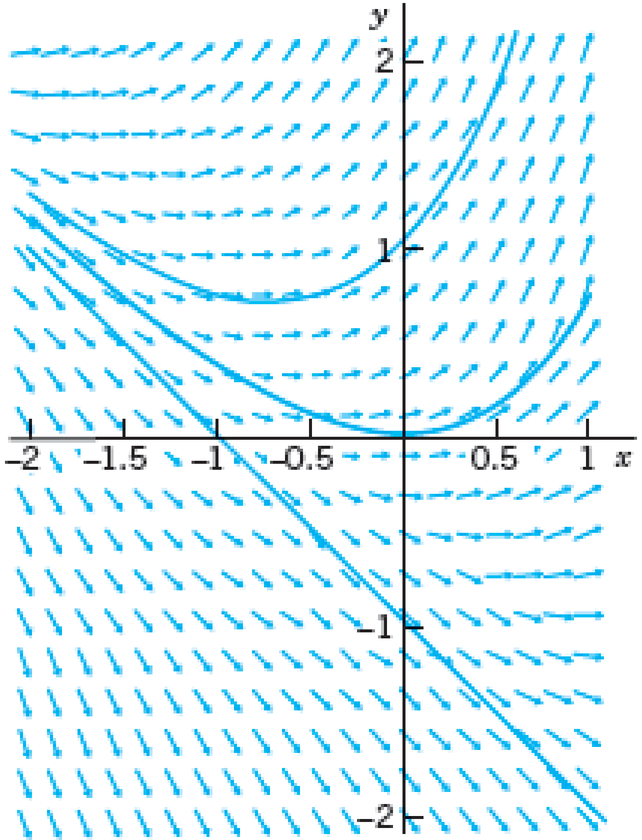


Fig. 7. Direction field of $y' = y + x$, with three approximate solution curves passing through (0, 1), (0, 0), (0, −1), respectively

Numeric Method by Euler

Given an ODE (1) and an initial value $y(x_0) = y_0$, **Euler's method** yields approximate solution values at equidistant x-values $x_0, x_1 = x_0 + h, x_2 = x_0 + 2h, \cdots$, namely,

$$y_1 = y_0 + hf(x_0, y_0) \qquad \text{(Fig. 8)}$$

$$y_2 = y_1 + hf(x_1, y_1), \qquad \text{etc.}$$

In general,

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$$

where the step h equals, e.g., 0.1 or 0.2 (as in Table 1.1) or a smaller value for greater accuracy.

Table 1.1 **Euler method for $y' = y + x, y(0) = 0$ for $x = 0, \cdots, 1.0$ with step $h = 0.2$**

n	x_n	y_n	$y(x_n)$	Error
0	0.0	0.000	0.000	0.000
1	0.2	0.000	0.021	0.021
2	0.4	0.04	0.092	0.052
3	0.6	0.128	0.222	0.094
4	0.8	0.274	0.426	0.152
5	1.0	0.488	0.718	0.230

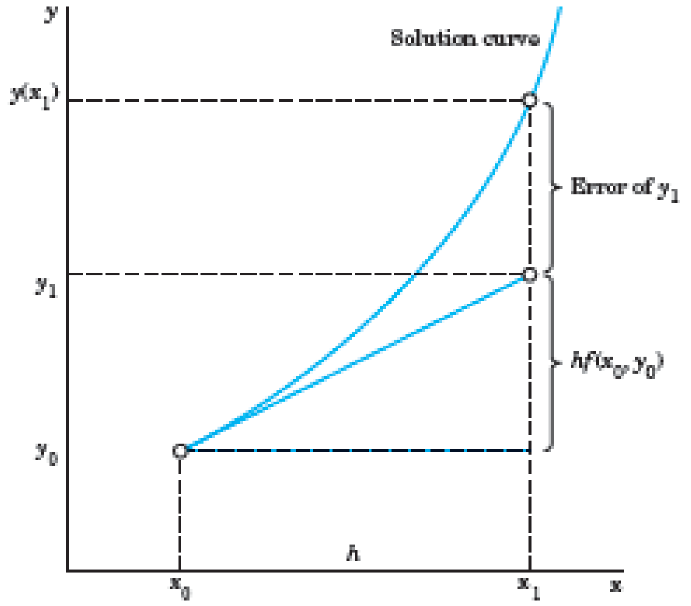


Fig. 8. First Euler step, showing a solution curve, its tangent at (x_0, y_0), step h and increment $hf(x_0, y_0)$ in the formula for y_1

Linear ODEs. Bernoulli Equation. Population Dynamics

- A first-order ODE is said to be **linear** if it can be brought into the form

(1) $$y' + p(x)y = r(x),$$

by algebra, and **nonlinear** if it cannot be brought into this form.

- by algebra, and **nonlinear** if it cannot be brought into this form.
- *p* and *r* may be ***any*** given functions of *x*.
- If in an application the independent variable is time, we write *t* instead of *x*.

Homogeneous Linear ODE. We want to solve (1) in some interval $a < x < b$, call it J, and we begin with the simpler special case that $r(x)$ is zero for all x in J. (This is sometimes written $r(x) \equiv 0$.) Then the ODE (1) becomes

$$y' + p(x)y = 0 \tag{2}$$

and is called **homogeneous**. By separating variables and integrating we then obtain

$$\frac{dy}{y} = -p(x)\,dx, \qquad \text{thus} \qquad \ln|y| = -\int p(x)\,dx + c^*.$$

Taking exponents on both sides, we obtain the general solution of the homogeneous ODE (2),

$$y(x) = ce^{-\int p(x)dx} \qquad (c = \pm e^{c^*} \quad \text{when} \quad y \gtrless 0); \tag{3}$$

here we may also choose $c = 0$ and obtain the **trivial solution** $y(x) = 0$ for all x in that interval.

Nonhomogeneous Linear ODE. We now solve (1) in the case that $r(x)$ in (1) is not everywhere zero in the interval J considered. Then the ODE (1) is called **nonhomogeneous.**

We multiply (1) by $F(x)$, obtaining

$$Fy' + pFy = rF. \tag{1*}$$

The left side is the derivative $(Fy)' = F'y + Fy'$ of the product Fy if

$$pFy = F'y, \qquad \text{thus} \qquad pF = F'.$$

By separating variables, $dF/F = p\,dx$. By integration, writing $h = \int p\,dx$,

$$\ln|F| = h = \int p\,dx, \qquad \text{thus} \qquad F = e^h.$$

With this F and $h' = p$, Eq. (1*) becomes

$$e^h y' + h' e^h y = e^h y' + (e^h)' y = (e^h y)' = re^h.$$

By integration,

$$e^h y = \int e^h r\,dx + c.$$

Dividing by e^h, we obtain the desired solution formula

$$y(x) = e^{-h}\left(\int e^h r\,dx + c\right), \qquad h = \int p(x)\,dx. \tag{4}$$

$$y(x) = e^{-h}\int e^{h} r\,dx + ce^{-h}, \tag{4*}$$

we see the following:

$$\text{Total Output} = \text{Response to the Input } r + \text{Response to the Initial Data.} \tag{5}$$

EXAMPLE 1 First-Order ODE, General Solution, Initial Value Problem

Solve the initial value problem

$$y' + y\tan x = \sin 2x, \qquad y(0) = 1.$$

Solution. Here $p = \tan x$, $r = \sin 2x = 2\sin x\cos x$, and

$$h = \int p\,dx = \int \tan x\,dx = \ln|\sec x|.$$

From this we see that in (4),

$$e^{h} = \sec x, \qquad e^{-h} = \cos x, \qquad e^{h}r = (\sec x)(2\sin x\cos x) = 2\sin x,$$

and the general solution of our equation is

$$y(x) = \cos x\left(2\int \sin x\,dx + c\right) = c\cos x - 2\cos^2 x.$$

From this and the initial condition, $1 = c \cdot 1 - 2 \cdot 1^2$; thus $c = 3$ and the solution of our initial value problem is $y = 3\cos x - 2\cos^2 x$. Here $3\cos x$ is the response to the initial data, and $-2\cos^2 x$ is the response to the input $\sin 2x$. ■

EXAMPLE 2 Electric Circuit

Model the *RL*-circuit in Fig. 19 and solve the resulting ODE for the current $I(t)$ A (amperes), where t is time. Assume that the circuit contains as an EMF $E(t)$ (electromotive force) a battery of $E = 48$ V (volts), which is constant, a *resistor* of $R = 11\ \Omega$ (ohms), and an *inductor* of $L = 0.1$ H (henrys), and that the current is initially zero.

Physical Laws. A current I in the circuit causes a voltage drop RI across the resistor (Ohm's law) and a voltage drop $LI' = L\,dI/dt$ across the conductor, and the sum of these two voltage drops equals the EMF (Kirchhoff's Voltage Law, KVL).

Remark. In general, KVL states that "The voltage (the electromotive force EMF) impressed on a closed loop is equal to the sum of the voltage drops across all the other elements of the loop." For Kirchoff's Current Law (KCL) and historical information, see footnote 7 in Sec. 2.9.

Solution. According to these laws the model of the *RL*-circuit is $LI' + RI = E(t)$, in standard form

$$I' + \frac{R}{L}I = \frac{E(t)}{L}. \tag{6}$$

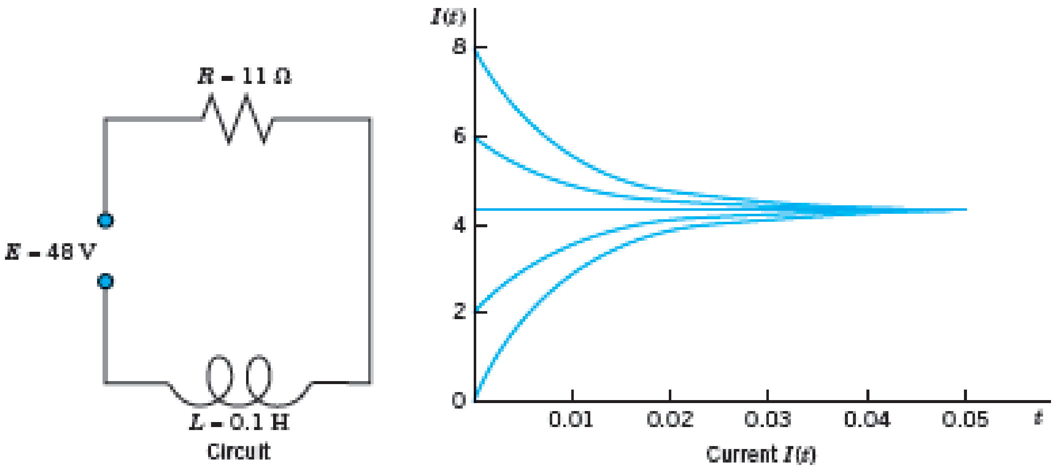


Fig. 19. *RL*-circuit

We can solve this linear ODE by (4) with $x = t, y = I, p = R/L, h = (R/L)t$, obtaining the general solution

$$I = e^{-(R/L)t}\left(\int e^{(R/L)t}\frac{E(t)}{L}dt + c\right).$$

By integration,

$$I = e^{-(R/L)t}\left(\frac{E}{L}\frac{e^{(R/L)t}}{R/L} + c\right) = \frac{E}{R} + ce^{-(R/L)t}. \tag{7}$$

In our case, $R/L = 11/0.1 = 110$ and $E(t) = 48/0.1 = 480 = \text{const}$; thus,

$$I = \tfrac{48}{11} + ce^{-110t}.$$

In modeling, one often gets better insight into the nature of a solution (and smaller roundoff errors) by inserting given numeric data only near the end. Here, the general solution (7) shows that the current approaches the limit $E/R = 48/11$ faster the larger R/L is, in our case, $R/L = 11/0.1 = 110$, and the approach is very fast, from below if $I(0) < 48/11$ or from above if $I(0) > 48/11$. If $I(0) = 48/11$, the solution is constant (48/11 A). See Fig. 19.

The initial value $I(0) = 0$ gives $I(0) = E/R + c = 0$, $c = -E/R$ and the particular solution

$$I = \frac{E}{R}(1 - e^{-(R/L)t}), \qquad \text{thus} \qquad I = \frac{48}{11}(1 - e^{-110t}). \tag{8}$$ ■

EXAMPLE 3 **Hormone Level**

Assume that the level of a certain hormone in the blood of a patient varies with time. Suppose that the time rate of change is the difference between a sinusoidal input of a 24-hour period from the thyroid gland and a continuous removal rate proportional to the level present. Set up a model for the hormone level in the blood and find its general solution. Find the particular solution satisfying a suitable initial condition.

Solution. *Step 1. Setting up a model.* Let $y(t)$ be the hormone level at time t. Then the removal rate is $Ky(t)$. The input rate is $A + B \cos \omega t$, where $\omega = 2\pi/24 = \pi/12$ and A is the average input rate; here $A \geqq B$ to make the input rate nonnegative. The constants A, B, K can be determined from measurements. Hence the model is the linear ODE

$$y'(t) = \text{In} - \text{Out} = A + B \cos \omega t - Ky(t), \qquad \text{thus} \qquad y' + Ky = A + B \cos \omega t.$$

The initial condition for a particular solution y_{part} is $y_{\text{part}}(0) = y_0$ with $t = 0$ suitably chosen, for example, 6:00 A.M.

Step 2. General solution. In (4) we have $p = K = \text{const}$, $h = Kt$, and $r = A + B \cos \omega t$. Hence (4) gives the general solution (evaluate $\int e^{Kt} \cos \omega t \, dt$ by integration by parts)

$$y(t) = e^{-Kt}\int e^{Kt}\left(A + B\cos\omega t\right)dt + ce^{-Kt}$$

$$= e^{-Kt}e^{Kt}\left[\frac{A}{K} + \frac{B}{K^2 + \omega^2}\left(K\cos\omega t + \omega\sin\omega t\right)\right] + ce^{-Kt}$$

$$= \frac{A}{K} + \frac{B}{K^2 + (\pi/12)^2}\left(K\cos\frac{\pi t}{12} + \frac{\pi}{12}\sin\frac{\pi t}{12}\right) + ce^{-Kt}.$$

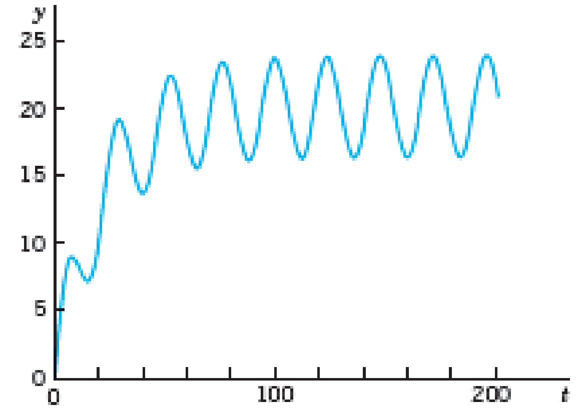


Fig. 20. Particular solution in Example 3

The last term decreases to 0 as t increases, practically after a short time and regardless of c (that is, of the initial condition). The other part of $y(t)$ is called the steady-state solution because it consists of constant and periodic terms. The entire solution is called the transient-state solution because it models the transition from rest to the steady state. These terms are used quite generally for physical and other systems whose behavior depends on time.

Step 3. Particular solution. Setting $t = 0$ in $y(t)$ and choosing $y_0 = 0$, we have

$$y(0) = \frac{A}{K} + \frac{B}{K^2 + (\pi/12)^2}\frac{u}{\pi}K + c = 0, \qquad \text{thus} \qquad c = -\frac{A}{K} - \frac{KB}{K^2 + (\pi/12)^2}.$$

Inserting this result into $y(t)$, we obtain the particular solution

$$y_{part}(t) = \frac{A}{K} + \frac{B}{K^2 + (\pi/12)^2}\left(K\cos\frac{\pi t}{12} + \frac{\pi}{12}\sin\frac{\pi t}{12}\right) - \left(\frac{A}{K} + \frac{KB}{K^2 + (\pi/12)^2}\right)e^{-K}$$

with the steady-state part as before. To plot y_{part} we must specify values for the constants, say, $A = B = 1$ and $K = 0.05$. Figure 20 shows this solution. Notice that the transition period is relatively short (although K is small), and the curve soon looks sinusoidal; this is the response to the input $A + B\cos(\frac{1}{12}\pi t) = 1 + \cos(\frac{1}{12}\pi t)$. ■

Homogeneous Linear ODEs of Second Order

A second-order ODE is called **linear** if it can be written

$$y'' + p(x)y' + q(x)y = r(x) \quad (1)$$

and **nonlinear** if it cannot be written in this form.

$$y'' + p(x)y' + q(x)y = 0 \quad (2)$$

and is called **homogeneous**. If $r(x) \not\equiv 0$, then (1) is called **nonhomogeneous**.

an example of a nonlinear ODE is

$$y''y + y'^2 = 0.$$

EXAMPLE 1 **Homogeneous Linear ODEs: Superposition of Solutions**

The functions $y = \cos x$ and $y = \sin x$ are solutions of the homogeneous linear ODE

$$y'' + y = 0$$

for all x. We verify this by differentiation and substitution. We obtain $(\cos x)'' = -\cos x$; hence

$$y'' + y = (\cos x)'' + \cos x = -\cos x + \cos x = 0.$$

Similarly for $y = \sin x$ (verify!). We can go an important step further. We multiply $\cos x$ by any constant, for instance, 4.7, and $\sin x$ by, say, -2, and take the sum of the results, claiming that it is a solution. Indeed, differentiation and substitution gives

$$(4.7 \cos x - 2 \sin x)'' + (4.7 \cos x - 2 \sin x) = -4.7 \cos x + 2 \sin x + 4.7 \cos x - 2 \sin x = 0.$$ ■

EXAMPLE 2 A Nonhomogeneous Linear ODE

Verify by substitution that the functions $y = 1 + \cos x$ and $y = 1 + \sin x$ are solutions of the nonhomogeneous linear ODE

$$y'' + y = 1,$$

but their sum is not a solution. Neither is, for instance, $2(1 + \cos x)$ or $5(1 + \sin x)$. ■

EXAMPLE 3 A Nonlinear ODE

Verify by substitution that the functions $y = x^2$ and $y = 1$ are solutions of the nonlinear ODE

$$y''y - xy' = 0,$$

but their sum is not a solution. Neither is $-x^2$, so you cannot even multiply by -1! ■

EXAMPLE 4 **Initial Value Problem**

Solve the initial value problem

$$y'' + y = 0, \qquad y(0) = 3.0, \qquad y'(0) = -0.5.$$

Solution. *Step 1. General solution.* The functions $\cos x$ and $\sin x$ are solutions of the ODE (by Example 1), and we take

$$y = c_1 \cos x + c_2 \sin x.$$

This will turn out to be a general solution as defined below.

Step 2. Particular solution. We need the derivative $y' = -c_1 \sin x + c_2 \cos x$. From this and the initial values we obtain, since $\cos 0 = 1$ and $\sin 0 = 0$,

$$y(0) = c_1 = 3.0 \qquad \text{and} \qquad y'(0) = c_2 = -0.5.$$

This gives as the solution of our initial value problem the particular solution

$$y = 3.0 \cos x - 0.5 \sin x.$$

Figure 29 shows that at $x = 0$ it has the value 3.0 and the slope -0.5, so that its tangent intersects the x-axis at $x = 3.0/0.5 = 6.0$. (The scales on the axes differ!) ■

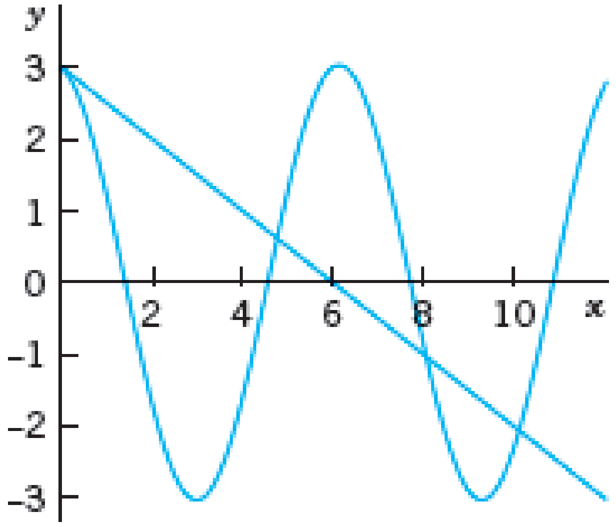


Fig. 29. Particular solution and initial tangent in Example 4

Homogeneous Linear ODEs with Constant Coefficients

$$y'' + ay' + by = 0. \qquad (1)$$

the solution of the first-order linear ODE with a constant coefficient *k* $\quad y' + ky = 0$

is an exponential function $y = ce^{-kx}$. This gives us the idea to try as a solution of (1) the function

$$y = e^{\lambda x}. \qquad (2)$$

Substituting (2) and its derivatives

$$y' = \lambda e^{\lambda x} \quad \text{and} \quad y'' = \lambda^2 e^{\lambda x}$$

into our equation (1), we obtain

$$(\lambda^2 + a\lambda + b)e^{\lambda x} = 0.$$

Hence if λ is a solution of the important **characteristic equation** (or *auxiliary equation*)

$$\lambda^2 + a\lambda + b = 0 \tag{3}$$

then the exponential function (2) is a solution of the ODE (1). Now from algebra we recall that the roots of this quadratic equation (3) are

$$\lambda_1 = \tfrac{1}{2}\left(-a + \sqrt{a^2 - 4b}\right), \qquad \lambda_2 = \tfrac{1}{2}\left(-a - \sqrt{a^2 - 4b}\right). \tag{4}$$

(3) and (4) will be basic because our derivation shows that the functions

$$y_1 = e^{\lambda_1 x} \qquad \text{and} \qquad y_2 = e^{\lambda_2 x} \tag{5}$$

are solutions of (1). Verify this by substituting (5) into (1).

depending on the sign of the discriminant $a^2 - 4b$, namely,

(Case I)	*Two real roots if* $a^2 - 4b > 0$,
(Case II)	*A real double root if* $a^2 - 4b = 0$,
(Case III)	*Complex conjugate roots if* $a^2 - 4b < 0$.

Case I. Two Distinct Real-Roots λ_1 and λ_2

In this case, a basis of solutions of (1) on any interval is

$$y_1 = e^{\lambda_1 x} \qquad \text{and} \qquad y_2 = e^{\lambda_2 x}$$

because y_1 and y_2 are defined (and real) for all x and their quotient is not constant. The corresponding general solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}. \tag{6}$$

EXAMPLE 2 Initial Value Problem in the Case of Distinct Real Roots

Solve the initial value problem

$$y'' + y' - 2y = 0, \qquad y(0) = 4, \qquad y'(0) = -5.$$

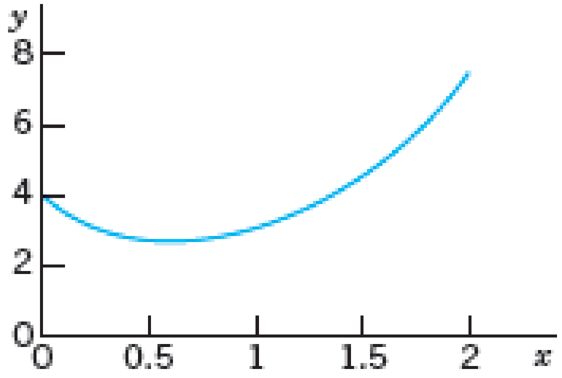


Fig. 30. Solution in Example 2

Solution. ***Step 1. General solution.*** The characteristic equation is

$$\lambda^2 + \lambda - 2 = 0.$$

Its roots are

$$\lambda_1 = \tfrac{1}{2}(-1 + \sqrt{9}) = 1 \qquad \text{and} \qquad \lambda_2 = \tfrac{1}{2}(-1 - \sqrt{9}) = -2$$

so that we obtain the general solution

$$y = c_1 e^x + c_2 e^{-2x}.$$

Step 2. Particular solution. Since $y'(x) = c_1 e^x - 2c_2 e^{-2x}$, we obtain from the general solution and the initial conditions

$$y(0) = c_1 + c_2 = 4,$$
$$y'(0) = c_1 - 2c_2 = -5.$$

Hence $c_1 = 1$ and $c_2 = 3$. This gives the *answer* $y = e^x + 3e^{-2x}$. Figure 30 shows that the curve begins at $y = 4$ with a negative slope (-5, but note that the axes have different scales!), in agreement with the initial conditions. ■

Case II. Real Double Root $\lambda = -a/2$

If the discriminant $a^2 - 4b$ is zero, we see directly from (4) that we get only one root, $\lambda = \lambda_1 = \lambda_2 = -a/2$, hence only one solution,

$$y_1 = e^{-(a/2)x}.$$

To obtain a second independent solution y_2 (needed for a basis), we use the method of reduction of order discussed in the last section, setting $y_2 = uy_1$. Substituting this and its derivatives $y_2' = u'y_1 + uy_1'$ and y_2'' into (1), we first have

$$(u''y_1 + 2u'y_1' + uy_1'') + a(u'y_1 + uy_1') + buy_1 = 0.$$

Collecting terms in u'', u', and u, as in the last section, we obtain

$$u''y_1 + u'(2y_1' + ay_1) + u(y_1'' + ay_1' + by_1) = 0.$$

The expression in the last parentheses is zero, since y_1 is a solution of (1). The expression in the first parentheses is zero, too, since

$$2y_1' = -ae^{-ax/2} = -ay_1.$$

We are thus left with $u''y_1 = 0$. Hence $u'' = 0$. By two integrations, $u = c_1x + c_2$. To get a second independent solution $y_2 = uy_1$, we can simply choose $c_1 = 1$, $c_2 = 0$ and take $u = x$. Then $y_2 = xy_1$. Since these solutions are not proportional, they form a basis. Hence in the case of a double root of (3) a basis of solutions of (1) on any interval is

$$e^{-ax/2}, \qquad xe^{-ax/2}.$$

The corresponding general solution is

$$y = (c_1 + c_2x)e^{-ax/2}. \tag{7}$$

EXAMPLE 4 Initial Value Problem in the Case of a Double Root

Solve the initial value problem

$$y'' + y' + 0.25y = 0, \qquad y(0) = 3.0, \quad y'(0) = -3.5.$$

Solution. The characteristic equation is $\lambda^2 + \lambda + 0.25 = (\lambda + 0.5)^2 = 0$. It has the double root $\lambda = -0.5$. This gives the general solution

$$y = (c_1 + c_2 x)e^{-0.5x}.$$

We need its derivative

$$y' = c_2 e^{-0.5x} - 0.5(c_1 + c_2 x)e^{-0.5x}.$$

From this and the initial conditions we obtain

$$y(0) = c_1 = 3.0, \qquad y'(0) = c_2 - 0.5c_1 = 3.5; \qquad \text{hence} \qquad c_2 = -2.$$

The particular solution of the initial value problem is $y = (3 - 2x)e^{-0.5x}$. See Fig. 31. ■

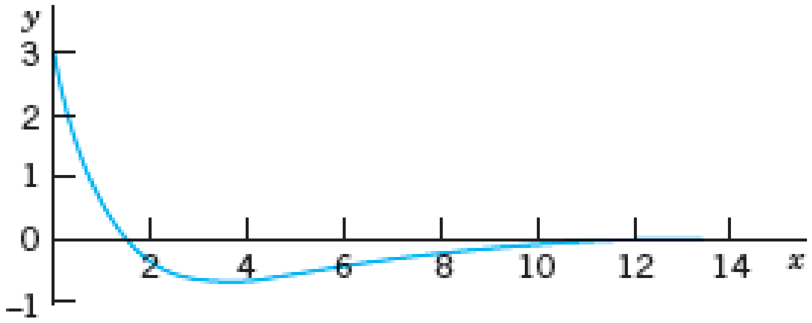


Fig. 31. Solution in Example 4

Case III. Complex Roots $-\frac{1}{2}a + i\omega$ and $-\frac{1}{2}a - i\omega$

This case occurs if the discriminant $a^2 - 4b$ of the characteristic equation (3) is negative. In this case, the roots of (3) are the complex $\lambda = -\frac{1}{2}a \pm i\omega$ that give the complex solutions of the ODE (1). However, we will show that we can obtain a basis of *real* solutions

$$y_1 = e^{-ax/2} \cos \omega x, \qquad y_2 = e^{-ax/2} \sin \omega x \qquad (\omega > 0) \tag{8}$$

where $\omega^2 = b - \frac{1}{4}a^2$. It can be verified by substitution that these are solutions in the present case. We shall derive them systematically after the two examples by using the complex exponential function. They form a basis on any interval since their quotient $\cot \omega x$ is not constant. Hence a real general solution in Case III is

$$y = e^{-ax/2}(A \cos \omega x + B \sin \omega x) \qquad (A, B \text{ arbitrary}). \tag{9}$$

EXAMPLE 5 Complex Roots. Initial Value Problem

Solve the initial value problem

$$y'' + 0.4y' + 9.04y = 0, \qquad y(0) = 0, \qquad y'(0) = 3.$$

Solution. *Step 1. General solution.* The characteristic equation is $\lambda^2 + 0.4\lambda + 9.04 = 0$. It has the roots $-0.2 \pm 3i$. Hence $\omega = 3$, and a general solution (9) is

$$y = e^{-0.2x}(A \cos 3x + B \sin 3x).$$

Step 2. Particular solution. The first initial condition gives $y(0) = A = 0$. The remaining expression is $y = Be^{-0.2x} \sin 3x$. We need the derivative (chain rule!)

$$y' = B(-0.2e^{-0.2x} \sin 3x + 3e^{-0.2x} \cos 3x).$$

From this and the second initial condition we obtain $y'(0) = 3B = 3$. Hence $B = 1$. Our solution is

$$y = e^{-0.2x} \sin 3x.$$

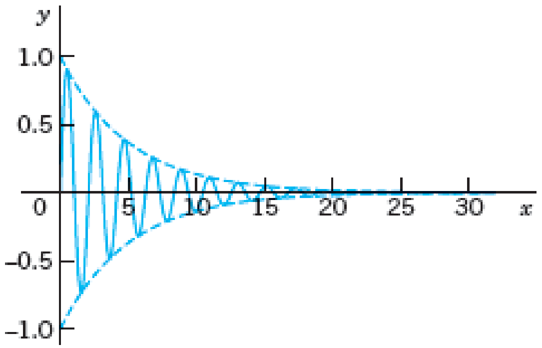


Fig. 32. Solution in Example 5

Thank you

Homogeneous Linear Higher Order Constant Coefficient Equations

Engineering Analysis

2016/2017

Lecture 2

Undetermined Coefficients: Particular Integrals

- Like the nonhomogeneous second order constant coefficient differential equation, a **particular integra**l *yp*(*x*) of the nonhomogeneous linear higher order constant coefficient differential equation

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = f(x) \qquad (53)$$

is a solution of the equation that does not contain arbitrary constants, so

$$\frac{d^n y_p}{dx^n} + a_1 \frac{d^{n-1} y_p}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy_p}{dx} + a_n y_p = f(x).$$

- The **complementary function** *yc*(*x*) associated with (53) is the general solution of the homogeneous form of the equation

$$\frac{d^n y_c}{dx^n} + a_1 \frac{d^{n-1} y_c}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy_c}{dx} + a_n y_c = 0,$$

- It follows from the definitions of *yc*(*x*) and *yp*(*x*) and the linearity of the equation that the general solution *y*(*x*) of (53) can be written

$$y(x) = y_c(x) + y_p(x). \qquad (54)$$

Example: Find the general solution of

Find the general solution of

$$y'' + 5y' + 6y = 4e^{-x} + 5\sin x.$$

Solution The general solution is

$$y(x) = y_c(x) + y_p(x),$$

where $y_c(x)$ is the complementary function satisfying the homogeneous form of the equation

$$y_c'' + 5y_c' + 6y_c = 0,$$

and $y_p(x)$ is a particular integral that corresponds to the nonhomogeneous term $4e^{-x} + 5\sin x$.

The characteristic equation is

$$\lambda^2 + 5\lambda + 6 = 0,$$

with the roots $\lambda_1 = -2$ and $\lambda_2 = -3$ corresponding to the linearly independent solutions e^{-2x} and e^{-3x}, so the complementary function is

$$y_c(x) = C_1 e^{-2x} + C_2 e^{-3x},$$

where C_1 and C_2 are arbitrary constants.

To find a particular integral, we notice first that neither the term e^{-x} nor the term $\sin x$ is contained in the complementary function. This means that the only form of particular integral $y_p(x)$ that can produce the nonhomogeneous term $4e^{-x} + 5\sin x$ is

$$y_p(x) = Ae^{-x} + B\sin x + C\cos x,$$

where A, B, and C are the *undetermined coefficients* that must be found.

Substituting this expression for $y_p(x)$ into the differential equation leads to the result

$$(Ae^{-x} - B\sin x - C\cos x) + 5(-Ae^{-x} + B\cos x - C\sin x) + 6(Ae^{-x} + B\sin x + C\cos x) = 4e^{-x} + 5\sin x.$$

When we collect terms involving e^{-x}, $\sin x$, and $\cos x$ this becomes

$$2Ae^{-x} + 5(B - C)\sin x + 5(B + C)\cos x = 4e^{-x} + 5\sin x.$$

If $y_p(x)$ is a particular integral, this expression must be an identity (true for all x), but this is only possible if the coefficients of corresponding functions of x on either side of the equation are identical. Equating corresponding coefficients gives

$$(\text{coefficients of } e^{-x}) \quad 2A = 4, \quad \text{so } A = 2$$
$$(\text{coefficient of } \sin x) \quad 5(B - C) = 5$$
$$(\text{coefficient of } \cos x) \quad 5(B + C) = 0.$$

Solving the last two equations for B and C gives $B = 1/2$, $C = -1/2$, so the particular integral is

$$y_p(x) = 2e^{-x} + (1/2)\sin x - (1/2)\cos x.$$

Substituting $y_c(x)$ and $y_p(x)$ into $y(x) = y_c(x) + y_p(x)$ shows that the general solution is

$$y(x) = C_1 e^{-2x} + C_2 e^{-3x} + 2e^{-x} + (1/2)\sin x - (1/2)\cos x.$$ ■

A complication arises if a term in the nonhomogeneous term $f(x)$ is contained in the complementary function, as illustrated in the next example.

Example:

Find a particular integral of the equation

$$y'' + y' - 12y = e^{3x}.$$

Solution This equation has the complementary function

$$y_c(x) = C_1e^{3x} + C_2e^{-4x},$$

so e^{3x} is contained in both the nonhomogeneous term and the complementary function.

An attempt to find a particular integral of the form $y_p(x) = Ae^{3x}$ will fail, because e^{3x} is a solution of the homogeneous form of the equation, so its substitution into the left-hand side of the differential equation will lead to the contradiction $0 = e^{3x}$. To overcome this difficulty we need to seek a more general particular integral that, when substituted into the differential equation, produces a multiple of e^{3x} whose scale factor can be equated to the coefficient of the nonhomogeneous term and other terms that cancel. As exponentials are involved, a natural choice is $y_p(x) = Axe^{3x}$.

Differentiation of $y_p(x)$ gives

$$y_p'(x) = Ae^{3x} + 3Axe^{3x} \quad \text{and} \quad y_p''(x) = 6Ae^{3x} + 9Axe^{3x}.$$

Substituting these results into the differential equation gives

$$6Ae^{3x} + 9Axe^{3x} + Ae^{3x} + 3Axe^{3x} - 12Axe^{3x} = e^{3x},$$

so after cancellation of the terms in Axe^{3x} this reduces to

$$7Ae^{3x} = e^{3x},$$

showing that $A = 1/7$. So the required particular integral is

$$y_p(x) = \frac{1}{7}xe^{3x}.$$

■

TABLE 6.2 Particular Integrals by the Method of Undetermined Coefficients

The method applies to the linear constant coefficient differential equation

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = f(x),$$

which has the characteristic equation

$$\lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0,$$

with the roots $\lambda_1, \lambda_2, \ldots, \lambda_n$, and the complementary function

$$y_c(x) = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x),$$

where $y_1(x), y_2(x), \ldots, y_n(x)$ are the linearly independent solutions of the homogeneous equation appropriate to the nature of the roots.

1. $f(x) = \text{constant.}$ $(\lambda \neq 0)$

Include in $y_p(x)$ the constant term K.

2. $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m.$

(a) If the left-hand side of the differential equation contains an undifferentiated term $y(x)$, include in $y_p(x)$ the polynomial

$$A_0 x^m + A_1 x^{m-1} + \cdots + A_m.$$

(b) If the left-hand side of the differential equation contains no undifferentiated function of $y(x)$, and the lowest order derivative is $d^s y/dx^s$, include in $y_p(x)$ the polynomial

$$A_0 x^{m+s} + A_1 x^{m+s-1} + \cdots + A_m x^s.$$

3. $f(x) = Pe^{ax}.$

(a) If e^{ax} is not contained in the complementary function, include in $y_p(x)$ the term

$$Be^{ax}.$$

(b) If the complementary function contains the terms $e^{ax}, xe^{ax}, \ldots, x^m e^{ax}$, include in $y_p(x)$ the term

$$Bx^{m+1}e^{ax}.$$

4. $f(x)$ contains terms in $\cos px$ and/or $\sin px$.

(a) If $\cos px$ and/or $\sin px$ are not contained in the complementary function, include in $y_p(x)$ the terms

$$P\cos px + Q\sin px.$$

(b) If the complementary function contains the terms $x\cos px$ and/or $x\sin px$, include in $y_p(x)$ terms of the form

$$x^2(P\cos px + Q\sin px).$$

(c) If the complementary function contains the terms $x^2 \cos px$ and/or $x^2 \sin px$, include in $y_p(x)$ terms of the form

$$x^3(P\cos px + Q\sin px).$$

5. $f(x)$ contains terms in $e^{px} \cos qx$ and/or $e^{px} \sin qx$.

(a) If $e^{px} \cos qx$ and/or $e^{px} \sin qx$ are not contained in the complementary function, include in $y_p(x)$ terms of the form

$$e^{px}(R\cos qx + S\sin qx).$$

(b) If the complementary function contains $xe^{px} \cos qx$ and/or $xe^{px} \sin qx$, include in $y_p(x)$ terms of the form

$$x^2 e^{px}(R\cos qx + S\sin qx).$$

6. The required particular integral $y_p(x)$ is the sum of all the terms produced by identifying each term belonging to $f(x)$ with one of the types of term listed above.
7. The values of the undetermined coefficients K, A_0, A_1, …, A_m, B, P, Q, R, and S are found by substituting $y_p(x)$ into the differential equation, equating the coefficients of corresponding functions on either side of the equation to make the result an identity, and then solving the resulting simultaneous equations for the undetermined coefficients.

Table 6.2 lists the form of particular integral that correspond to the most common nonhomogeneous terms. Each of its entries can be constructed by using arguments similar to the one just given. When the nonhomogeneous term is a linear combination of terms in the table, the form of $y_p(x)$ is found by adding the forms of the corresponding particular integrals.

Example:

Find the general solution of

$$y''' - 5y'' + 6y' = x^2 + \sin x.$$

Solution The characteristic equation is

$$\lambda^3 - 5\lambda^2 + 6\lambda = 0, \quad \text{or} \quad \lambda(\lambda^2 - 5\lambda + 6) = 0,$$

with the roots $\lambda_1 = 0$, $\lambda_2 = 2$, and $\lambda_3 = 3$, so the complementary function is

$$y_c(x) = C_1 + C_2 e^{2x} + C_3 e^{3x}.$$

The function x^2 on the right-hand side is not contained in the complementary function, but there is no undifferentiated term involving $y(x)$ in the equation, so from Step 2(b) in Table 6.2 the appropriate form of particular integral corresponding to this term is

$$Ax + Bx^2 + Cx^3.$$

The function $\sin x$ is not contained in the complementary function, so the form of particular integral appropriate to this term is seen from Step 4(a) to be

$$D\sin x + E\cos x.$$

Combining these two forms shows that the general form of $y_p(x)$ is

$$y_p(x) = Ax + Bx^2 + Cx^3 + D\sin x + E\cos x.$$

Substituting $y_p(x)$ into the differential equation gives

$$(6C - D\cos x + E\sin x) - 5(2B + 6Cx - D\sin x - E\cos x)$$
$$+ 6(A + 2Bx + 3Cx^2 + D\cos x - E\sin x) = x^2 + \sin x.$$

Equating coefficients of corresponding functions on each side of this expression to make it an identity, we have

$$\begin{aligned}
&\text{(constant terms)} && 6C - 10B + 6A = 0,\\
&\text{(terms in } x) && -30C + 12B = 0,\\
&\text{(terms in } x^2) && 18C = 1,\\
&\text{(terms in } \sin x) && 5D - 5E = 1,\\
&\text{(terms in } \cos x) && 5D + 5E = 0.
\end{aligned}$$

Solving these simultaneous equations gives $A = 19/108$, $B = 5/36$, $C = 1/18$, $D = 1/10$, and $E = -1/10$, so the particular integral is

$$y_p(x) = \frac{19}{108}x + \frac{5}{36}x^2 + \frac{1}{18}x^3 + \frac{1}{10}\sin x - \frac{1}{10}\cos x.$$

Combining this with the complementary function shows the general solution to be

$$y(x) = C_1 + C_2 e^{2x} + C_3 e^{3x} + \frac{19}{108}x + \frac{5}{36}x^2 + \frac{1}{18}x^3 + \frac{1}{10}\sin x - \frac{1}{10}\cos x. \quad ■$$

Theorem:

Existence and uniqueness of solutions of nonhomogeneous linear equations Let the coefficients and nonhomogeneous term of differential equation (53) be continuous functions over an interval $a < x < b$ that contains the point x_0. Then a unique solution exists on this interval that satisfies the initial conditions

$$y(x_0) = k_0, \quad y^{(1)}(x_0) = k_1, \ldots, y^{(n-1)}(x_0) = k_{n-1}.$$

Example:

Solve the initial value problem

$$y'' + 4y' + 3y = e^{-x}, \quad \text{with } y(0) = 2, \quad y'(0) = 1.$$

Solution The characteristic equation is

$$\lambda^2 + 4\lambda + 3 = 0,$$

with the roots $\lambda_1 = -1$ and $\lambda_2 = -3$, so the complementary function is

$$y_c(x) = C_1 e^{-x} + C_2 e^{-3x}.$$

The nonhomogeneous term e^{-x} is contained in the complementary function, so by Step 3(b) in Table 6.2 we must seek a particular integral of the form

$$y_p(x) = Axe^{-x}.$$

Substituting the expression for $y_p(x)$ into the differential equation gives

$$(-2Ae^{-x} + Axe^{-x}) + 4(Ae^{-x} - Axe^{-x}) + 3Axe^{-x} = e^{-x}, \quad \text{or} \quad 2Ae^{-x} = e^{-x},$$

showing that $A = 1/2$. So, in this case, the particular integral is $y_p(x) = (1/2)xe^{-x}$ and the general solution is

$$y(x) = C_1 e^{-x} + C_2 e^{-3x} + (1/2)xe^{-x}.$$

The initial condition $y(0) = 2$ will be satisfied if

$$2 = C_1 + C_2,$$

and the initial condition $y'(0) = 1$ will be satisfied if

$$1/2 = -C_1 - 3C_2,$$

so $C_1 = 13/4$ and $C_2 = -5/4$. Substituting these values for C_1 and C_2 in the general solution gives the solution of the initial value problem

$$y(x) = \left(\frac{13}{4} + \frac{1}{2}x\right)e^{-x} - \frac{5}{4}e^{-3x}.$$

■

Engineering analysis

Control of Medical System Engineering

Third year

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Dr Safanah Mudheher

Exact, Least-Squares, and Cubic Spline Curve-Fits

- Engineers conduct experiments and collect data in the laboratories. To make use of the collected data, these data often need to be fitted with some particularly selected curves.
- For example, one may want to find a parabolic equation $y = c_1 + c_2 x + c_3 x^2$.
- which passes three given points (x_i, y_i) for $i = 1,2,3$.
- This is a problem of *exact curvefit*.

- In case that we may want express this straight line by the equation $y = c_1 + c_2x$ for the stress and strain data collected for a stretching test of a metal bar in the elastic range, then the question of how to determine the two coefficients c_1 and c_2 is a matter of deciding on which criterion to adopt.
- The *Least-Squares* method is one of the criteria which is most popularly used. The two cases cited are the consideration of adopting the two and three lowest *polynomial terms*, x(0), x(1), and x(2),and linearly combining them.

If the collected data are supposed to represent a sinusoidal function of time, the curve to be determined may have to be assumed as $x(t) = c_1\sin t + c_2\sin 3t + c_3\sin 5t + c_4\sin 7t$ by linearly combining 4 odd sine terms. This is the case of selecting four particular functions, namely, $f_i(t) = \sin(2i–1)t$ for $i = 1,2,3,4.$, and to determine the coefficients $c_{1–4}$ by application of the least-squares method.

EXACT CURVE FIT

let us consider the problem of finding a parabolic equation
$y = c_1 + c_2x + c_3x^2$ which passes three given points

(x_i,y_i) for $i = 1,2,3$. This is a problem of *exact curve-fit*. By simple substitutions of the three points into the parabolic equation, we can obtain:

$$c_1 + c_2x_i + c_3x_i^2 = y_i \qquad \text{for } i = 1,2,3 \qquad (1)$$

In matrix form, we write these equations as:

$$[A]\{C\} = \{Y\} \qquad (2)$$

where $\{C\} = [c_1\ c_2\ c_3]^T$, $\{Y\} = [y_1\ y_2\ y_3]^T$, and $[A]$ is a three-by-three coefficient matrix whose elements if denoted as $a_{i,j}$ are to be calculated using the formula:

$$a_{i,j} = x_i^{j-1} \qquad \text{for } i, j = 1, 2, 3 \qquad (3)$$

GENERALIZED LEAST-SQUARES CURVEFIT

Let us consider N points whose coordinates are (X_k, Y_k) for k = 1 to N and let the M selected function be $f_1(X)$ to $f_M(X)$ and the equation determined by the least-squares curve-fit be:

$$Y(X) = a_1 f_1(X) + a_2 f_2(X) + \ldots + a_M f_M(X) = \sum_{j=1}^{M} a_j f_j(X) \qquad (1)$$

Series Solutions of Differential Equations

$$y' + p(x)y = r(x) \quad \text{with } y(x_0) = y_0, \tag{1}$$

$$y'' + P(x)y' + Q(x)y = R(x) \quad \text{with } y(x_0) = y_0,\ y'(x_0) = y_1, \tag{2}$$

where the functions $p(x)$, $r(x)$, $P(x)$, $Q(x)$, and $R(x)$ can all be expanded as Taylor series about the point x_0.

Functions with this property are said to be **analytic** in a **neighborhood** of the point x_0 or, more simply, to be **analytic** at x_0. The method to be developed will be seen to be capable of extension to a higher order linear differential equation in an obvious manner, provided only that the coefficients of y and its derivatives that are involved and the nonhomogeneous term are analytic at x_0.

The approach is best illustrated by considering equation (1), and seeking a solution about x_0 of the form

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)y'(x_0) + \frac{(x - x_0)^2}{2!}y''(x_0) + \frac{(x - x_0)^3}{3!}y'''(x_0) + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(x - x_0)^n}{n!} y^{(n)}(x_0), \quad \text{with } y^{(n)}(x) = d^n y/dx^n. \end{aligned} \tag{3}$$

Setting $x = x_0$ in (1) gives

$$y^{(1)}(x_0) + p(x_0)y(x_0) = r(x_0),$$

but $y(x_0) = y_0$, so

$$\begin{aligned} y^{(1)}(x_0) &= r(x_0) - p(x_0)y(x_0) \\ &= r(x_0) - p(x_0)y_0. \end{aligned}$$

To determine $y^{(2)}(x)$ we differentiate equation (1) once with respect to x to obtain

$$y^{(2)}(x) + p^{(1)}(x)y(x) + p(x)y^{(1)}(x) = r^{(1)}(x),$$

where $p^{(1)}(x) = p'(x)$ and $r^{(1)}(x) = r'(x)$. Then, after setting $x = x_0$ and using the fact that $y^{(1)}(x_0) = r(x_0) - p(x_0)y_0$, we find that

$$y^{(2)}(x_0) = r^{(1)}(x_0) - p^{(1)}(x_0)y_0 - p(x_0)[r(x_0) - p(x_0)y_0].$$

Find the first five terms in the series solution of

$$y' + (1 + x^2)y = \sin x, \quad \text{with } y(0) = a.$$

Solution As the initial condition is specified at $x = 0$, the power series solution is an expansion about the origin and so is, in fact, a Maclaurin series. The functions $1 + x^2$ and $\cos x$ are analytic for all x, so the series expansion can certainly be found about the origin.

Setting $x = 0$ in the equation and substituting for the initial conditions shows that $y'(0) = y^{(1)}(0) = -a$. Differentiation of the differential equation gives

$$y^{(2)} + 2xy + (1 + x^2)y^{(1)} = \cos x,$$

where $y^{(2)} = y''$, so setting $x = 0$ this becomes

$$y^{(2)}(0) + y^{(1)}(0) = 1,$$

but $y^{(1)}(0) = -a$ and so $y^{(2)}(0) = 1 + a$. Repeating this process to find higher order derivatives leads to the results $y^{(3)}(0) = -(1 + 3a)$, $y^{(4)}(0) = 9a, \ldots$. Substituting these results into series (3) shows that, to terms of order x^4, the required solution takes the form

$$y(x) = a - ax + (1 + a)\frac{x^2}{2!} - (1 + 3a)\frac{x^3}{3!} + 9a\frac{x^4}{4!} + \cdots.$$ ■

Find the terms up to x^5 in the series solution of

$$y'' + xy' + (1 - x^2)y = x \quad \text{with } y(0) = a,\ y'(0) = b.$$

Solution The coefficients x and $(1 - x^2)$ and the nonhomogeneous term x are analytic for all x, so as the initial data is given at $x = 0$, a Maclaurin series solution can be found.

Setting $x = 0$ in the equation and using the initial conditions $y(0) = a$ and $y'(0) = b$ gives $y^{(2)}(0) = -a$. Differentiating the differential equation we have

$$y^{(3)} + y^{(1)} + xy^{(2)} - 2xy + (1 - x^2)y^{(1)} = 1,$$

so setting $x = 0$ and using the results $y^{(2)}(0) = -a$ and $y^{(1)}(0) = b$ shows that $y^{(3)}(0) = 1 - 2b$. A repetition of this process leads to the results $y^{(4)}(0) = 5a$, $y^{(5)}(0) = 14b - 4, \ldots$, so substituting into (3) shows that to terms of order x^5 the Maclaurin series expansion of the solution is

$$y(x) = a + bx - \frac{1}{2}ax^2 + \left(\frac{1 - 2b}{6}\right)x^3 + \frac{5a}{24}x^4 + \left(\frac{7b - 2}{60}\right)x^5 + \cdots. \quad \blacksquare$$

A General Approach to Power Series Solutions of Homogeneous Equations

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{n=0}^{\infty} a_n x^n. \tag{7}$$

$$y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots = \sum_{n=1}^{\infty} n a_n x^{n-1}, \tag{8}$$

$$y''(x) = 2a_2 + 2 \cdot 3a_3 x + 3 \cdot 4a_4 x^2 + \cdots = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}. \tag{9}$$

Example

Find the recurrence relation that must be satisfied by coefficients in the series solution of the differential equation

$$y'' + 2xy' + (1 + x^2)y = 0$$

when the expansion is about the origin. Solve the initial value problem for this differential equation given that $y(0) = 3$ and $y'(0) = -1$.

Solution Substituting $y(x) = \sum_{n=0}^{\infty} a_n x^n$ into the differential equation and using (8) and (9) gives

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 2x \sum_{n=1}^{\infty} n a_n x^{n-1} + (1 + x^2) \sum_{n=0}^{\infty} a_n x^n = 0.$$

Taking the factor $2x$ in the second term and the factor x^2 in the third term under their respective summation signs allows the equation to be written in the form

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+2} = 0.$$

The powers of x in the first and last summations are different from those in the middle two summations, so before combining the summations in order to find the coefficient of each power of x, it will first be necessary to change the power of x in the first and last terms from $n-2$ and $n+2$ to n.

In the first summation we set $m = n-2$, causing the summation to become

$$\sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} x^m.$$

However, m is simply a summation index that can be replaced by any other symbol, so we will replace it by n to obtain the equivalent expression

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n.$$

Similarly, by setting $m = n+2$ in the last summation, and then replacing m by n, we find that

$$\sum_{n=0}^{\infty}a_n x^{n+2} \quad \text{becomes} \quad \sum_{n=2}^{\infty}a_{n-2}x^n.$$

We now substitute these last two results into the series solution of the differential equation to obtain

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty}2na_n x^n + \sum_{n=0}^{\infty}a_n x^n + \sum_{n=2}^{\infty}a_{n-2}x^n = 0,$$

where now each summation involves x^n, though not all summations start from $n = 0$.

Separating out the terms corresponding to $n = 0$ and $n = 1$, and collecting all the remaining terms under a single summation sign in which the summation starts from $n = 2$, this becomes

$$2a_2 + a_0 + (6a_3 + 3a_1)x + \sum_{n=2}^{\infty}[(n+2)(n+1)a_{n+2} + 3a_n + a_{n-2}]x^n = 0.$$

As already remarked, if this power series is to be a solution of the differential equation it must satisfy the equation identically for all x, but this will only be possible if in the foregoing expression the coefficient of each power of x vanishes. Applying this condition to the preceding series we find that for it to vanish identically for all x,

(coefficient of x^0) $\quad 2a_2 + a_0 = 0$

(coefficient of x) $\quad 6a_3 + 3a_1 = 0$

and

$$(\text{coefficient of } x^n) \quad (n+2)(n+1)a_{n+2} + 3a_n + a_{n-2} = 0, \quad \text{for } n \geq 2.$$

The first condition shows that

$$a_2 = -\frac{1}{2}a_0,$$

while the second condition shows that

$$a_3 = -\frac{1}{2}a_1,$$

where a_0 and a_1 are arbitrary constants.

Legendre's equation

- An important application of the power series method of solution is to the **Legendre differential equation**

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0, \quad (10)$$

in which $\alpha \geq 0$ is a real parameter. The equation arises in a variety of applications, but mainly in connection with physical problems in which spherical symmetry is present. It will be seen later that the equation finds its origin in the study of Laplace's equation when expressed in spherical coordinates. Solutions of (10) are called **Legendre functions**, and they are examples of **special functions**, or so-called **higher transcendental functions**, as distinct from elementary functions such as sine, cosine, exponential, and logarithm. We first develop the series solutions for arbitrary $\alpha \geq 0$, and then consider the cases $\alpha = n = 0, 1, 2, \ldots,$ which lead to a special class of polynomial solutions $P_n(x)$ called **Legendre polynomials** in which n is the degree of the polynomial. The important properties of Legendre polynomials will be examined later when the topic of orthogonal functions is introduced.

Singular Points of Linear Differential Equations

- Previously, the power series method was used to find a solution of a homogeneous variable coefficient differential equation of the form

$$a(x)y'' + b(x)y' + c(x)y = 0. \qquad (19)$$

- Expressed differently, when (19) is written in the standard form

$$y'' + P(x)y' + Q(x)y = 0, \qquad (20)$$

with

$$P(x) = \frac{b(x)}{a(x)} \quad \text{and} \quad Q(x) = \frac{c(x)}{a(x)}, \qquad (21)$$

- the power series method can be applied to develop a solution about any point *x*0 at which the functions *P*(*x*) and *Q*(*x*) are analytic.
- Points **regular and singular** where *P*(*x*) and *Q*(*x*) are analytic are called **regular points** of the differential equation, and points where at least one is not analytic are called **singular points**.
- Equation (20) will be said to have a **regular singular point** at *x*0 if the functions

$$(x - x_0)P(x) \quad \text{and} \quad (x - x_0)^2 Q(x)$$

- are analytic at *x*0, and so have Taylor series expansions about *x*0. If at least one of these functions is not analytic at *x*0, the point will be said to be an **irregular singular point**.

Eigenvalues, Eigenvectors

Control and System Engineering Dept./Branch of

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Let $\mathbf{A} = [a_{jk}]$ be an $n \times n$ matrix. Consider the equation

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x} \tag{12}$$

where λ is a scalar (a real or complex number) to be determined and $\mathbf{x}$ is a vector to be determined. Now, for every λ, a solution is $\mathbf{x} = \mathbf{0}$. A scalar λ such that (12) holds for some vector $\mathbf{x} \neq \mathbf{0}$ is called an **eigenvalue** of $\mathbf{A}$, and this vector is called an **eigenvector** of $\mathbf{A}$ corresponding to this eigenvalue λ.

We can write (12) as

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = \mathbf{0}. \tag{13}$$

These are n linear algebraic equations in the n unknowns $x_1, \cdots, x_n$ (the components of $\mathbf{x}$). For these equations to have a solution $\mathbf{x} \neq \mathbf{0}$, the determinant of the coefficient matrix $\mathbf{A} - \lambda\mathbf{I}$ must be zero. This is proved as a basic fact in linear algebra (Theorem 4

For n=2

$$\begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \tag{14}$$

in components,

$$\begin{aligned} (a_{11} - \lambda)x_1 + \quad a_{12}x_2 \quad &= 0 \\ a_{21}x_1 \quad + (a_{22} - \lambda)x_2 &= 0. \end{aligned} \tag{14*}$$

Now $\mathbf{A} - \lambda\mathbf{I}$ is singular if and only if its determinant det $(\mathbf{A} - \lambda\mathbf{I})$, called the **characteristic determinant** of $\mathbf{A}$ (also for general n), is zero. This gives

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} \\ &= (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} \\ &= \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0. \end{aligned} \tag{15}$$

This quadratic equation in λ is called the **characteristic equation** of $\mathbf{A}$. Its solutions are the eigenvalues λ_1 and λ_2 of $\mathbf{A}$. First determine these. Then use (14*) with $\lambda = \lambda_1$ to determine an eigenvector $\mathbf{x}^{(1)}$ of $\mathbf{A}$ corresponding to λ_1. Finally use (14*) with $\lambda = \lambda_2$ to find an eigenvector $\mathbf{x}^{(2)}$ of $\mathbf{A}$ corresponding to λ_2. Note that if $\mathbf{x}$ is an eigenvector of $\mathbf{A}$, so is $k\mathbf{x}$ with any $k \neq 0$.

Example

Eigenvalue Problem

Find the eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} -4.0 & 4.0 \\ -1.6 & 1.2 \end{bmatrix} \tag{16}$$

Solution. The characteristic equation is the quadratic equation

$$\det|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} -4-\lambda & 4 \\ -1.6 & 1.2-\lambda \end{vmatrix} = \lambda^2 + 2.8\lambda + 1.6 = 0.$$

It has the solutions $\lambda_1 = -2$ and $\lambda_2 = -0.8$. These are the eigenvalues of $\mathbf{A}$.

Eigenvectors are obtained from (14*). For $\lambda = \lambda_1 = -2$ we have from (14*)

$$\begin{aligned} (-4.0 + 2.0)x_1 + \quad 4.0x_2 \quad &= 0 \\ -1.6x_1 \quad + (1.2 + 2.0)x_2 &= 0. \end{aligned}$$

A solution of the first equation is $x_1 = 2, x_2 = 1$. This also satisfies the second equation. (Why?) Hence an eigenvector of $\mathbf{A}$ corresponding to $\lambda_1 = -2.0$ is

$$\text{(17)} \qquad \mathbf{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \qquad \text{Similarly,} \qquad \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ 0.8 \end{bmatrix}$$

is an eigenvector of $\mathbf{A}$ corresponding to $\lambda_2 = -0.8$, as obtained from (14*) with $\lambda = \lambda_2$. Verify this. ■

Separable ODEs. Modeling

Many practically useful ODEs can be reduced to the form

(1) $$g(y)\, y' = f(x)$$

by purely algebraic manipulations. Then we can integrate on both sides with respect to x, obtaining

(2) $$\int g(y)\, y' dx = \int f(x)\, dx + c.$$

On the left we can switch to y as the variable of integration. By calculus, $y'dx = dy$, so that

(3) $$\int g(y)\, dy = \int f(x)\, dx + c.$$

If f and g are continuous functions, the integrals in (3) exist, and by evaluating them we obtain a general solution of (1). This method of solving ODEs is called the **method of separating variables**, and (1) is called a **separable equation**, because in (3) the variables are now separated: x appears only on the right and y only on the left.

Mixing Problem

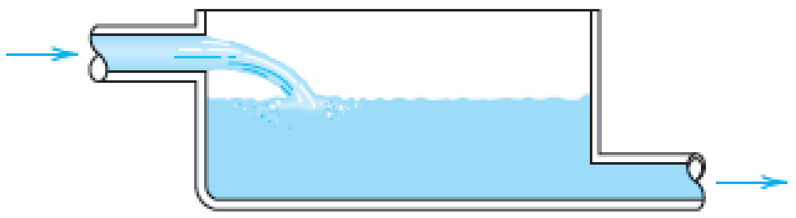
Mixing problems occur quite frequently in chemical industry. We explain here how to solve the basic model involving a single tank. The tank in Fig. 11 contains 1000 gal of water in which initially 100 lb of salt is dissolved. Brine runs in at a rate of 10 gal/min, and each gallon contains 5 lb of dissoved salt. The mixture in the tank is kept uniform by stirring. Brine runs out at 10 gal/min. Find the amount of salt in the tank at any time t.

Solution. ***Step 1. Setting up a model.*** Let $y(t)$ denote the amount of salt in the tank at time t. Its time rate of change is

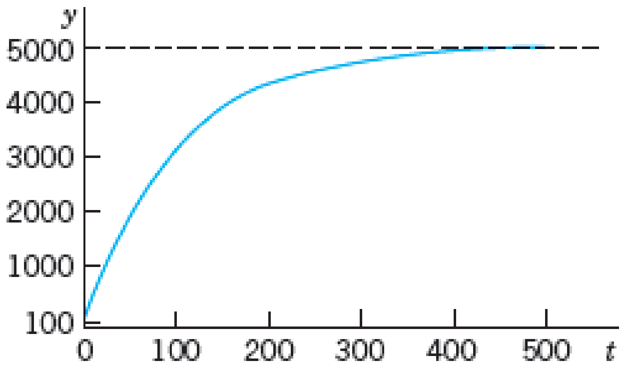
$$y' = \text{Salt inflow rate} - \text{Salt outflow rate} \qquad \textbf{Balance law.}$$

5 lb times 10 gal gives an inflow of 50 lb of salt. Now, the outflow is 10 gal of brine. This is $10/1000 = 0.01$ ($= 1\%$) of the total brine content in the tank, hence 0.01 of the salt content $y(t)$, that is, $0.01\,y(t)$. Thus the model is the ODE

$$y' = 50 - 0.01y = -0.01(y - 5000). \tag{4}$$



Tank



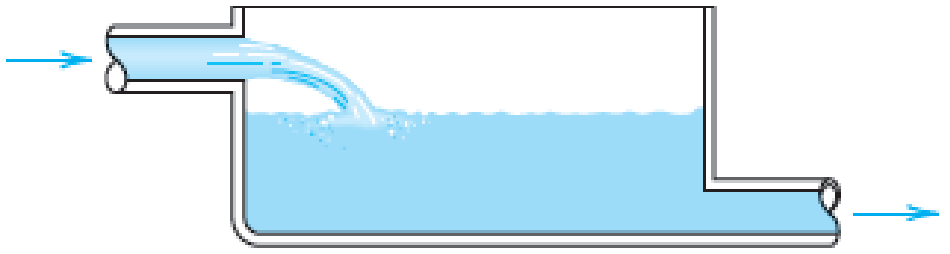
Salt content $y(t)$

Step 2. Solution of the model. The ODE (4) is separable. Separation, integration, and taking exponents on both sides gives

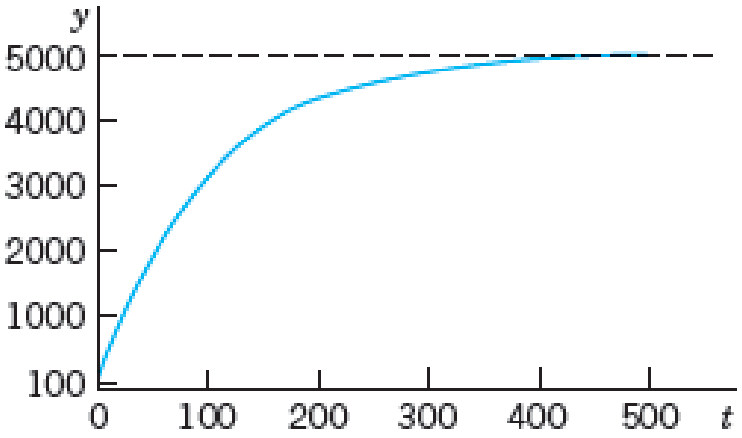
$$\frac{dy}{y - 5000} = -0.01\,dt, \qquad \ln|y - 5000| = -0.01t + c^*, \qquad y - 5000 = ce^{-0.01t}.$$

Initially the tank contains 100 lb of salt. Hence $y(0) = 100$ is the initial condition that will give the unique solution. Substituting $y = 100$ and $t = 0$ in the last equation gives $100 - 5000 = ce^0 = c$. Hence $c = -4900$. Hence the amount of salt in the tank at time t is

$$y(t) = 5000 - 4900e^{-0.01t}. \tag{5}$$

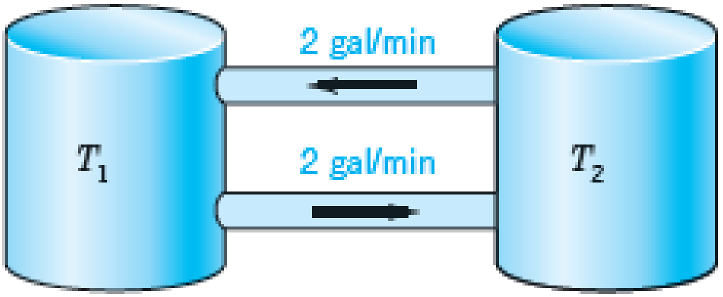


Tank



Salt content $y(t)$

Systems of ODEs as Models in Engineering Applications



System of tanks

- **Mixing Problem Involving Two Tanks**
- A mixing problem involving a single tank is modeled by a single ODE.
- The model will be a system of two first-order ODEs.
- Tank and in Fig. contain initially 100 gal of water each. In the water is pure, whereas 150 lb of fertilizer are dissolved in . By circulating liquid at a rate of and stirring (to keep the mixture uniform) the amounts of fertilizer in and in change with time *t*.
- How long should we let the liquid circulate so that *T*1 will contain at least half as much fertilizer as there will be left in *T*2?

Solution. ***Step 1.*** **Setting up the model.** As for a single tank, the time rate of change $y_1'(t)$ of $y_1(t)$ equals inflow minus outflow. Similarly for tank T_2. From Fig. 78 we see that

$$y_1' = \text{Inflow/min} - \text{Outflow/min} = \frac{2}{100}y_2 - \frac{2}{100}y_1 \qquad \text{(Tank } T_1\text{)}$$

$$y_2' = \text{Inflow/min} - \text{Outflow/min} = \frac{2}{100}y_1 - \frac{2}{100}y_2 \qquad \text{(Tank } T_2\text{).}$$

Hence the mathematical model of our mixture problem is the system of first-order ODEs

$$\begin{aligned} y_1' &= -0.02y_1 + 0.02y_2 && \text{(Tank } T_1\text{)} \\ y_2' &= 0.02y_1 - 0.02y_2 && \text{(Tank } T_2\text{).} \end{aligned}$$

As a vector equation with column vector $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ and matrix $\mathbf{A}$ this becomes

$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \qquad \text{where} \qquad \mathbf{A} = \begin{bmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{bmatrix}.$$

Step 2. **General solution.** As for a single equation, we try an exponential function of t,

$$\text{(1)} \qquad \mathbf{y} = \mathbf{x}e^{\lambda t}. \qquad \text{Then} \qquad \mathbf{y}' = \lambda \mathbf{x}e^{\lambda t} = \mathbf{A}\mathbf{x}e^{\lambda t}.$$

Dividing the last equation $\lambda \mathbf{x}e^{\lambda t} = \mathbf{A}\mathbf{x}e^{\lambda t}$ by $e^{\lambda t}$ and interchanging the left and right sides, we obtain

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x}.$$

We need nontrivial solutions (solutions that are not identically zero). Hence we have to look for eigenvalues and eigenvectors of $\mathbf{A}$. The eigenvalues are the solutions of the characteristic equation

$$\text{(2)} \qquad \det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -0.02 - \lambda & 0.02 \\ 0.02 & -0.02 - \lambda \end{vmatrix} = (-0.02 - \lambda)^2 - 0.02^2 = \lambda(\lambda + 0.04) = 0.$$

We see that $\lambda_1 = 0$ (which can very well happen—don't get mixed up—it is eigen*vectors* that must not be zero) and $\lambda_2 = -0.04$. Eigenvectors are obtained from (14*) in Sec. 4.0 with $\lambda = 0$ and $\lambda = -0.04$. For our present $\mathbf{A}$ this gives [we need only the first equation in (14*)]

$$-0.02x_1 + 0.02x_2 = 0 \qquad \text{and} \qquad (-0.02 + 0.04)x_1 + 0.02x_2 = 0,$$

respectively. Hence $x_1 = x_2$ and $x_1 = -x_2$, respectively, and we can take $x_1 = x_2 = 1$ and $x_1 = -x_2 = 1$. This gives two eigenvectors corresponding to $\lambda_1 = 0$ and $\lambda_2 = -0.04$, respectively, namely,

$$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

we thus obtain a solution

$$(3) \qquad \mathbf{y} = c_1\mathbf{x}^{(1)}e^{\lambda_1 t} + c_2\mathbf{x}^{(2)}e^{\lambda_2 t} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-0.04t}$$

where c_1 and c_2 are arbitrary constants. Later we shall call this a **general solution.**

Step 3. **Use of initial conditions.** The initial conditions are $y_1(0) = 0$ (no fertilizer in tank T_1) and $y_2(0) = 150$. From this and (3) with $t = 0$ we obtain

$$\mathbf{y}(0) = c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}c_1 + c_2\\c_1 - c_2\end{bmatrix} = \begin{bmatrix}0\\150\end{bmatrix}.$$

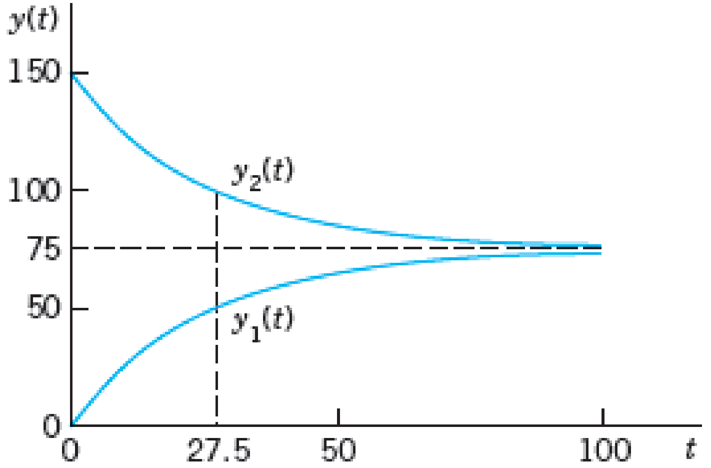
In components this is $c_1 + c_2 = 0$, $c_1 - c_2 = 150$. The solution is $c_1 = 75$, $c_2 = -75$. This gives the answer

$$\mathbf{y} = 75\mathbf{x}^{(1)} - 75\mathbf{x}^{(2)}e^{-0.04t} = 75\begin{bmatrix}1\\1\end{bmatrix} - 75\begin{bmatrix}1\\-1\end{bmatrix}e^{-0.04t}.$$

In components,

$$y_1 = 75 - 75e^{-0.04t} \qquad \text{(Tank } T_1\text{, lower curve)}$$

$$y_2 = 75 + 75e^{-0.04t} \qquad \text{(Tank } T_2\text{, upper curve).}$$



Electrical Network

- Find the currents I1 and I2 in the network. Assume all currents and charges to be zero at t=0, the instant when the switch is closed.

- ***Solution. Step 1.*** **Setting up the mathematical model.** The model of this network is obtained from

Kirchhoff's Voltage Law,

- Let *I*1(*t*) and *I*2(*t*) be the currents